MAT 342
Applied Complex Analysis
2020 Summer II

Final Examination

Question	1	2	3	4	5	6	7	Total
Grade								

- Instructor: Willie Rush Lim
- Due date: August 14th 2020, 11.59pm EST
- This test has 7 questions, each carrying different weights. The total number of points is 200.
- The use of calculator or other similar aids such as Matlab and WolframAlpha is prohibited during the test.
- Credit will be given for all questions attempted with clear explanation.
- Submit your answers as one pdf on blackboard.
- In case of technical difficulties, email lim.willie@stonybrook.edu.

Plagiarism Statement[1]

I certify that my answers are my own work, based on my personal study and/or material from lectures. I also certify that I have not copied in part or whole, or otherwise plagiarised the work of other students and/or persons. I acknowledge that students who plagiarize or otherwise engage in academic dishonesty will face serious consequences, including grade reduction or course failure.

____________________ ____________________ [10]

Signature Date

[1] If you plan on submitting handwritten answers, please copy the plagiarism statement above on your answer sheet.

1. Let $f(z)$ be the rational function

$$f(z) = \frac{3i}{1 - 2z - 8z^2}.$$

   (a) Find the Laurent series representation of f defined on the annulus $\{1 < |4z| < 2\}$. [10]

   (b) Compute the residue of f at 0. [4]

   Suppose γ is the closed curve parametrised by $\gamma(t) = -e^{it}\cos t$ for $\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$.

   (b) Sketch the curve γ and find the winding number of $f \circ \gamma$ without drawing the curve $f(\gamma(t))$. [10]

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2. Explain whether or not any of the following statements are true.

   (a) The imaginary part of an entire function cannot be an entire function. [6]

   (b) The integral of $1/z$ along the curve $\gamma(t) = e^{it}$, $-2\pi \le t \le \pi$ is πi because [6]

   $$\int_\gamma \frac{1}{z} dz = \mathrm{Log}(\gamma(\pi)) - \mathrm{Log}(\gamma(-2\pi)) = \mathrm{Log}(-1) - \mathrm{Log}(1) = \pi i.$$

   (c) There is a non-constant entire function f such that $f(n) = 0$ for all integers n. [6]

   (d) The product of the real and imaginary parts of a holomorphic function is always harmonic. [8]

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3. The function

$$f(z) = \frac{2\sinh\frac{z}{2}}{e^{2z}-1}$$

has a Laurent series representation $\sum_{n=k}^{\infty} a_n z^n$, where $a_k \neq 0$, convergent on the punctured disk $\{0 < |z| < R\}$ for some integer k and positive real number R.

(a) Find and classify all the isolated singularities of f. For each pole, state its order as well. [8]

(b) Find the integer k, the value a_k and the maximum possible value of R, if exists. [8]

Consider the polynomial $p(z) = z^{2020} - z^{10} + 5iz + 2$.

(c) What is the number of zeros the polynomial p inside the annular domain $\{1 < |z| < \pi\}$? [12]

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4. Let's prove the following integral identity[2]

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i a x}}{\cosh \pi x} dx = \frac{1}{\cosh \pi a}, \qquad (\circledast)$$

for any real number $a \in \mathbb{R}$ using the method of residues.

(a) Let $R > 0$ be an arbitrarily large positive number and consider the positively oriented simple closed contour γ, a concatenation of the following four smooth curves:

$$\gamma_1 = \{t \mid -R \leq t \leq R\}, \qquad \gamma_2 = \{R + it \mid 0 \leq t \leq 2\},$$
$$\gamma_3 = \{t + 2i \mid -R \leq t \leq R\}, \qquad \gamma_4 = \{-R + it \mid 0 \leq t \leq 2\}.$$

Sketch the curve γ together with all the singularities of the function

$$f(z) = \frac{e^{-2\pi i a z}}{\cosh \pi z}$$

enclosed by γ. [6]

(b) Show that the integral of $f(z)$ along γ is $2(e^{\pi a} - e^{3\pi a})$. [8]

(c) By taking the limit as $R \to \infty$, prove the integral identity ($\circledast$) above. [14]

[2]This identity says that the Fourier transform of $\cosh \pi x$ is itself.

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5. Let γ the positively oriented unit circle $\{|z| = 1\}$, U be the set $\{z \in \mathbb{C} \mid |z| \neq 1\}$, and $f : U \to \mathbb{C}$ be the function defined by

$$f(z) = \oint_\gamma \frac{iw - 1}{(w - z)^2} dw.$$

(a) State whether U is open, closed, bounded and/or connected. [6]

(b) Show that f is constant on each connected component of U and find its image. [10]

(c) Evaluate the real integral

$$\int_0^{2\pi} e^{\sin x} \cos(\cos x) dx$$

using the method of residues. [12]

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6. Consider the function $u(x, y) = e^{2x} \sin 2y + x$ on $\mathbb{R}^2$.

   (a) Verify that the function is harmonic. [6]

   (b) Find the unique solution $f(z)$ to the equation $\mathrm{Re} f(x + iy) = u(x, y)$ such that $f(\pi) = \pi$. Express your answer as a function of $z \in \mathbb{C}$ (no x or y). [10]

   We are given that a harmonic function $g(r, \theta)$ on the domain

$$S = \{(r, \theta) \mid 0 \leq r < \pi, -\pi < \theta \leq \pi\} \subset \mathbb{R}^2,$$

   written in polar coordinates, which extends continuously to the boundary ∂S. It is known that g satisfies the following inequalities:

$$|g(\pi, \theta)| \leq 1, \text{ if } 0 \leq \theta \leq \pi, \qquad |g(\pi, \theta)| \leq 3, \text{ if } -\pi < \theta \leq 0.$$

   (c) Use the mean value property to show that that $|g(0, \theta)| \leq 2$. [8]

   (d) Suppose $h(r, \theta)$ is another harmonic function on S satisfying $h(\pi, \theta) = g(\pi, \theta)$ for all angles θ. Show that $h \equiv g$ on S. [8]

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7. Consider the following complex functions

$$A(z) = e^{e^z}, \qquad B(z) = \mathrm{Log}(3z+4).$$

(a) Find the extremal values of $|A(z)|$ on the closed rectangular region

$$R = \{x + iy \mid 0 \le x \le \ln \pi, -\pi \le y \le \pi\}.$$

State the corresponding points on R at which extremal values are attained. [8]

(b) Find the derivative $A'(z)$ of A and show that each maximum point of $|A(z)|$ on R is also a maximum point of $|A'(z)|$ on R. What is the maximum value of $|A'(z)|$ on R? [8]

(c) Find a primitive $C(z)$ of $B(z)$ and show that it can be written in the form of

$$C(z) = (z + c)\mathrm{Log}(3z+4) - z$$

for some constant $c \in \mathbb{C}$. Suggest a branch cut for $C(z)$. [8]

(d) Evaluate the contour integral of B along the contour γ representing the polar graph

$$r = \sin\left(\frac{\theta}{1262}\right), \qquad 0 \le \theta \le 631\pi,$$

with 0 as its starting point. [10]

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