# Applied Complex Analysis

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# Introduction

Denote the set of complex numbers by

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$$

where $i = \sqrt{-1}$ is defined such that $i^2 = -1$.

Complex analysis is the study of functions of a complex variable. In the first few chapters, we shall explore some introductory concepts, such as basic properties of complex numbers and continuity of complex-valued functions. The main emphasis is the concept of *holomorphic* functions, i.e. complex-valued functions which are differentiable in a complex sense, and the many applications of their somewhat magical properties. I used the word 'magical' because holomorphicity is such a rigid condition that many of the results you will see are somewhat unintuitive yet true.

We will start with some motivation. Basic algebra tells us that the number of roots of a polynomial with real coefficients is at most its degree. For example, $x^2 + c$ has two real roots if $c < 0$, one root if $c = 0$, and no roots if $c > 0$. Introducing the imaginary number $i$ provides us with a more elegant way of formulating this idea.

**Theorem** (Fundamental Theorem of Algebra)**.** *The field $\mathbb{C}$ is algebraically closed, that is, any polynomial with coefficients in $\mathbb{C}$ of degree $d > 1$ has exactly $d$ roots in $\mathbb{C}$, counting multiplicity.*

Many initial attempts of proving the theorem by prominent mathematicians D'Alembert, Euler, Gauss, Lagrange, and Laplace in 1700s were incomplete. In 1806, a Swiss accountant, Parisian bookstore manager and 'amateur' mathematician Jean-Robert Argand completed D'Alembert's ideas and hence became the first person to rigorously prove the fundamental theorem of algebra. We will in fact use properties of holomorphic functions to give 3 different proofs of the theorem, including D'Alembert and Argand's approach.

It is difficult to list the many applications of the fundamental theorem of algebra. The main idea is that the field of complex numbers is the perfect setting to solve equations!

A direct consequence in linear algebra is that every square matrix with entries in $\mathbb{C}$ admits an eigenvalue. When a $2 \times 2$ matrix has imaginary eigenvalues, it acts as a rotation of the plane rather than expansion or contraction in certain directions. In the study of continuous dynamics arising from mechanical systems, it is common to use complex numbers in order to capture oscillations in the system.

One of the direct applications of the study of holomorphic functions is contour integration. The integral of a complex function along a closed path is not dependent on the path itself but rather on certain values called *residues* of the function's singularities. This means that it is often easier to integrate a real function of a real variable by converting it into a problem involving a contour integral in the complex plane.

Fourier series and Fourier transforms are useful in decomposing functions into its frequency components. (Think of decomposing nice functions as a sum or an integral of different sine and cosine waves.) Fourier analysis can be easily formulated via complex analysis, and it comes up everywhere: in differential equations, probability, quantum mechanics, signal processing, etc.

Mechanical and electrical engineers as well as computer musicians also encounter complex variables in electrical circuits with alternating current. Digital filters are designed by looking at the locations of *zeros* and *poles* of rational functions called *transfer functions*, which essentially model a device's inputs and outputs.

Iterations of holomorphic functions have long been known to have many applications. Complex polynomials, for example, can be used to model the population of rabbits over time. Powerful basic results in complex analysis, many of which do not apply to generic real differentiable functions, make up one of the many reasons why the study of iterations of holomorphic functions (holomorphic dynamics) is very well developed compared to the other branches of the field of dynamical systems.

Conformal functions are holomorphic functions with strictly non-zero derivative. Such functions have an amazing geometric property of angle preservation at every point and are useful in transforming regions with complicated boundary to those of a much nicer shape (square, disk, etc). You may, for example, want to transform a mechanical problem on a complicated domain into an equivalent problem on a circular disk. In cartography, conformal maps are useful in creating a world map as well as local nautical charts using Mercator and stereographic projections. More recently, conformal functions are applied to the surface of the human brain for brain development study and diagnosis of Alzheimer's disease and schizophrenia.

# Chapter 1

# Complex Numbers

In this chapter, we will go through the basic algebraic and geometric properties of complex numbers.

## 1.1 The Algebra of $\mathbb{C}$

The set $\mathbb{C}$ is equipped with the usual arithmetic operators, namely:

- addition $+$: $(x + iy) + (a + ib) = (x + a) + i(y + b)$,
- multiplication $\times$: $(x + iy) \times (a + ib) = (xa - yb) + i(xb + ya)$.

Let's denote by $\mathbb{C}^*$ the set of non-zero complex numbers $\mathbb{C}\backslash\{0\}$. This set is equipped with an additional operator:

- inversion of a non-zero number: $(x + iy)^{-1} = \frac{x-iy}{x^2+y^2}$.

Similar to $\mathbb{R}$, the set of complex numbers $\mathbb{C}$ is a *field*; it satisfies the following axioms:

1. $(\mathbb{C}, +)$ is an abelian group:
   - $+$ is associative and commutative,
   - $0$ is the identity element of $+$, i.e. $z + 0 = z$ for all $z \in \mathbb{C}$,
   - Additive inverses exist, i.e. $z + (-z) = 0$ for all $z \in \mathbb{C}$;
2. $(\mathbb{C}^*, \times)$ is an abelian group:
   - $\times$ is associative and commutative,
   - $1$ is the identity element of $\times$, i.e. $z \times 1 = z$ for all $z \in \mathbb{C}^*$,

- Multiplicative inverses exist, i.e. $z \times z^{-1} = 1$ for all $z \in \mathbb{C}^*$;

3. $\times$ is distributive over $+$.

The set $\mathbb{C}$ of complex numbers can be identified with the real vector space $\mathbb{R}^2$ by the vector space isomorphism:

$$\begin{aligned} \mathbb{C} &\to \mathbb{R}^2, \\ z &\mapsto (\operatorname{Re} z, \operatorname{Im} z), \\ x + iy &\mapsto (x, y). \end{aligned}$$

Unlike $\mathbb{C}$, the real plane $\mathbb{R}^2$ is only equipped with addition operator $+$ but not a natural multiplication operator $\times$. Nonetheless, the mapping above allows us to geometrically represent complex numbers as points on the plane. This is typically known as *Argand diagram*.

## 1.2 The Geometry of $\mathbb{C}$

Every complex number $z = x + iy$ comes with a unique *real part* $x$ and *imaginary part* $y$. We shall denote them as follows:

$$\operatorname{Re} z = x, \qquad \operatorname{Im} z = y.$$

Geometrically, Re and Im can be thought as functions $\mathbb{C} \to \mathbb{R}$ acting as projections onto the real and imaginary axes respectively.

The *complex conjugate* $\bar{z}$ of a complex number $z = x + iy$ is $\bar{z} = x - iy$. Geometrically, the operation $z \mapsto \bar{z}$ is a reflection over the real axis. The following identity can be thought of as a change of basis from $(x, y)$ to $(z, \bar{z})$.

**Proposition 1.1.** *For any $z \in \mathbb{C}$, $\operatorname{Re} z = \frac{z+\bar{z}}{2}$ and $\operatorname{Im} z = \frac{z-\bar{z}}{2i}$.*

Another straightforward algebraic exercise also gives us the following basic properties.

**Proposition 1.2.** *For any $z, w \in \mathbb{C}$, $\overline{z+w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z}\bar{w}$. If $z \neq 0$, $\overline{z^{-1}} = \bar{z}^{-1}$.*

The *absolute value / modulus* of a complex number $z = x + iy$ is

$$|z| = \sqrt{x^2 + y^2}.$$

Phytagoras' theorem indicates that geometrically the modulus $|z|$ of $z$ is equal to the distance between 0 and $z$.

**Proposition 1.3.** *For any $z, w \in \mathbb{C}$,*

- $|zw| = |z||w|$,
- $z\bar{z} = |z|^2$,
- $|z+w| \leq |z| + |w|$ *(Triangle inequality).*

The *argument* of $z$, $\arg(z)$, is defined to be the counterclockwise angle (measured in radians) subtended by the positive real axis $\mathbb{R}^+$ and the line segment joining 0 and $z$. See figure 1.1.

Notice that arg is a multivalued function. For example, both $\pi$ and $3\pi$ are arguments of $i$. We can refine this by defining the *principal argument* of $z$, $\operatorname{Arg}(z)$, to be the unique argument of $z$ lying in $(-\pi, \pi]$.

*Remark.* The interval $[0, 2\pi)$ is also often chosen to be the codomain of the principal argument.

**Proposition 1.4.** *For any $z, w \in \mathbb{C}^*$,*

- $\arg(zw) = \arg(z) + \arg(w)$,
- $\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w)$ *mod* $2\pi$.

**Example 1.** Let $z = 1 + i$ and $w = -1 + \sqrt{3}i$. The modulus and arguments of $z$ and $w$ are:

$$|z| = \sqrt{2}, \quad |w| = 2, \quad \arg(z) = \frac{\pi}{4}, \quad \arg w = \frac{2\pi}{3}.$$

Then, the modulus and argument of $(1+i)(-1+\sqrt{3}i)$ are $2\sqrt{2}$ and $\frac{11\pi}{12}$ respectively.

For any non-zero complex number $z = x + iy$, if $r = |z|$ and $\theta = \operatorname{Arg}(z)$, then basic trigonometry gives us the following change of variables:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

The expression $z = r(\cos\theta + i\sin\theta)$ from above is the *polar form* of $z$.

**Theorem 1.5** (Euler's formula)**.** *For any $\theta$, $e^{i\theta} = \cos\theta + i\sin\theta$.*

*Proof.* We will give two different proofs of the result - one with differential equations, and another with Maclaurin series. The expression $e^{i\theta}$ is a non-zero complex number, so there is a unique $r > 0$ and $\hat{\theta}$ such that

$$e^{i\theta} = r(\cos\hat{\theta} + i\sin\hat{\theta}). \tag{1.1}$$

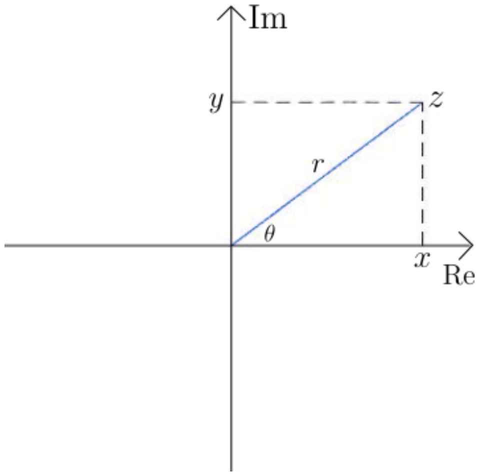



Figure 1.1: A point $z = x + iy = re^{i\theta}$ on the Argand diagram

Here, $r$ and $\hat{\theta}$ are functions of $\theta$. When $\theta = 0$, $r(0) = 1$ and $\hat{\theta}(0) = 0$. Differentiating (1.1) with respect to $\theta$, we obtain

$$\begin{aligned} ie^{i\theta} &= \frac{dr}{d\theta}(\cos\hat{\theta} + i\sin\hat{\theta}) + r\frac{d\hat{\theta}}{d\theta}(-\sin\hat{\theta} + i\cos\hat{\theta}) \\ &= \frac{dr}{d\theta}\frac{e^{i\theta}}{r} + i\frac{d\hat{\theta}}{d\theta}e^{i\theta} \\ &= \left(\frac{dr}{d\theta} + i\frac{d\hat{\theta}}{d\theta}\right)e^{i\theta}, \end{aligned}$$

where the second equality above is obtained from (1.1). From above, we see that $\frac{dr}{d\theta} = 0$ and $\frac{d\hat{\theta}}{d\theta} = 1$. By our initial conditions, we obtain $r(\theta) \equiv 1$ and $\hat{\theta} \equiv \theta$.

Alternatively, recall the following Maclaurin series: $e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$. Using the fact that $i^n = (-1)^{n/2}$ if $n$ is even, and $i^n = (-1)^{n-1/2}i$ if $n$ is odd,

$$\begin{aligned} e^{i\theta} &= \sum_{\text{even } n} \frac{(-1)^n\theta^n}{n!} + \sum_{\text{odd } n} \frac{(-1)^{n-1/2}\theta^n}{n!} \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right) = \cos\theta + i\sin\theta. \end{aligned}$$

□

**Example 2.** When $\theta = \pi$, we have Euler's identity: $e^{i\pi} = -1$.

The polar form of a complex number $z$ can alternatively be written in the form of $z = re^{i\theta}$. This expression is particularly useful when performing multiplication of complex numbers as we can use the laws of exponent. One particular instance is the following.

**Theorem 1.6** (De Moivre's Theorem)**.** *For any $\theta$ and integer $n \in \mathbb{Z}$,*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

**Example 3.** To compute and simplify $\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^{10}$, we can use De Moivre's theorem. The term inside the bracket is essentially $\cos\theta + i\sin\theta$ where $\theta = \frac{2\pi}{3}$. Then,

$$\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^{10} = \cos\left(\frac{20\pi}{3}\right) + i\sin\left(\frac{20\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}i}{2}.$$

## 1.3 Complex Roots

Consider a complex number $z_0$ and a positive integer $n$. A complex number $w$ satisfying $w^n = z_0$ is called an $n^{th}$ *root* of $z_0$.

Suppose $z_0 = 0$. Regardless of $n$, there is only one root of 0, which is 0 itself. This is due to the fact that $\mathbb{C}$ is an integral domain, i.e. for any two complex numbers $z_1$ and $z_2$, if $z_1 z_2 = 0$ then either $z_1 = 0$ or $z_2 = 0$.

Suppose $z \neq 0$ now, then surely any root $w$ is also non-zero. Using their polar forms $z = re^{i\theta}$ and $w = se^{it}$, then the equation becomes:

$$s^n e^{int} = re^{i\theta}$$

Considering the modulus and the argument independently, we obtain two real equations $s^n = r$ and $nt = \theta \bmod 2\pi$. There are therefore $n$ different solutions to $w$:

$$w_k = r^{1/n} e^{i(\theta + 2\pi k)/n}, \qquad k \in \{0, 1, \ldots n-1\}.$$

In the expression above, $w_0$ is called the *principal root* of $z_0$. On the complex plane, these roots are evenly spaced on the circle $\{z \in \mathbb{C} \mid |z| = r^{1/n}\}$ of radius $r^{1/n}$ centered at the origin.

When $z_0 = 1$, the $n^{\text{th}}$ roots of 1 are called the $n^{th}$ *roots of unity.* They all lie on the unit circle and are of the form $e^{2\pi ik/n}$, where $k \in \{0, 1, \ldots n-1\}$.

**Example 4.** The 3rd roots of unity are 1, $e^{2\pi i/3}$ and $e^{4\pi i/3}$. The Cartesian forms of these roots are 1, $\frac{-1+i\sqrt{3}}{2}$, and $\frac{-1-i\sqrt{3}}{2}$.

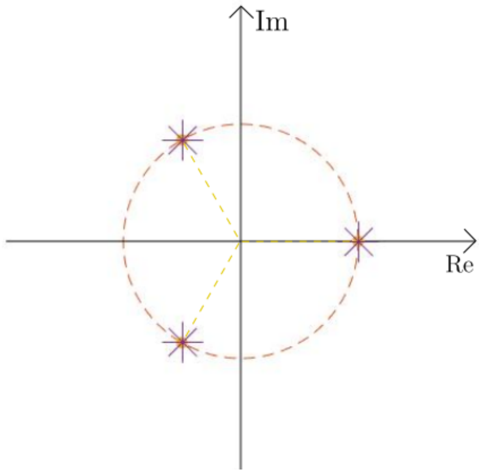



Figure 1.2: $3^{\text{rd}}$ roots of unity

## 1.4 The Topology of $\mathbb{C}$

An *open disk* of radius $r > 0$ centred at a complex number $a \in \mathbb{C}$ is a subset of $\mathbb{C}$ of the form:

$$\mathbb{D}(z, r) = \{z \in \mathbb{C} \mid |z - a| < r\}.$$

The boundary of this disk is a circle of of radius $r > 0$ centred at $a$, denoted with a partial sign in front:

$$C(z, r) = \partial\mathbb{D}(z, r) = \{z \in \mathbb{C} \mid |z - a| = r\}.$$

If we include the boundary, we obtain a *closed disk* typically denoted with an overline:

$$\overline{\mathbb{D}(z, r)} = \{z \in \mathbb{C} \mid |z - a| \leq r\}.$$

**Example 5.** Let's consider the sets

$$A = \{re^{i\theta} \mid r = \sin\theta, \theta \in \mathbb{R}\}, \qquad B = \{re^{i\theta} \mid 0 < r < \sin\theta, \theta \in \mathbb{R}\}.$$

If $z = x + iy$ lies in $A$, then $x = \sin\theta\cos\theta$ and $y = \sin^2\theta$ for some $\theta$. By double angle formulas,

$$\sin^2(2\theta) + \cos^2(2\theta) = (2x)^2 + (1 - 2y)^2 = 1.$$

This equation represents a circle of radius $\frac{1}{2}$ centered at $\frac{i}{2}$. Therefore, $A$ is the circle $C(\frac{i}{2}, \frac{1}{2})$. For points $z$ on the set $B$, we only need to consider the case when $\sin\theta > 0$, or principally when $0 < \theta < \pi$. The set $B$ is the open disk $\mathbb{D}(\frac{i}{2}, \frac{1}{2})$.

The geometric and topological properties of the complex plane $\mathbb{C}$ are essentially the same as those of the real plane $\mathbb{R}^2$ since we have the obvious identification $x + iy \mapsto (x, y)$. We will give a brief introduction of necessary topological terminology that we will use in the next few chapters.

**Definition 1.** A subset $S \subset \mathbb{C}$ is:

- *open* if for every $s \in S$, there is some $r > 0$ such that $\mathbb{D}(s, r) \subset S$,
- *closed* if its complement $\mathbb{C} \backslash S$ is open,
- *bounded* if there is some $r > 0$ where $S \subset \mathbb{D}(0, r)$,
- *compact* if $S$ is closed and bounded.

**Example 6.** Below are some subsets of $\mathbb{C}$ which we will commonly encounter.

1. The empty set $\emptyset$ is trivially open and compact.
2. The complex plane $\mathbb{C}$ is both open and closed, but not bounded.
3. The punctured plane $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$ is open, but not closed nor bounded.
4. The unit disk $\mathbb{D} := \mathbb{D}(0, 1)$ is open and bounded, but not closed.
5. The closed unit disk $\overline{\mathbb{D}}$ and its boundary $\partial\mathbb{D}$ are compact.
6. The real axis $\mathbb{R}$ is closed and unbounded.

**Definition 2.** An open/closed set $S \subset \mathbb{C}$ is:

- *connected* if $S$ cannot be expressed as a disjoint union of two open/closed non-empty subsets of $\mathbb{C}$,
- *simply connected* if it is connected and it has no "holes", i.e. the complement $\mathbb{C} \backslash S$ has no bounded connected component,
- *multiply connected* if it is connected but not simply connected.

We say that $S$ is a *domain* if it is a non-empty open connected subset of $\mathbb{C}$.

Figure 1.3: Four connected subsets of $\mathbb{C}$. Solid boundary lines are included in the colored set, whereas dashed boundary lines are not included. The first (from the left) is a simply connected domain. The second is closed and simply connected. The third is a punctured disk, which is a multiply connected domain. The last is closed and multiply connected.

**Example 7.**

1. $\emptyset$, $\mathbb{C}$, $\mathbb{D}$, $\overline{\mathbb{D}}$ and $\mathbb{R}$ are simply connected.

2. The punctured plane $\mathbb{C}^*$, the punctured unit disk $\mathbb{D}^* := \mathbb{D}\backslash\{0\}$, and the unit circle $\partial\mathbb{D}$ are multiply connected.

3. The annulus $\{z \in \mathbb{C} \mid r < |z| < R\}$ of inner radius $r$ and outer radius $R$ is multiply connected.

**Example 8.** Consider the set $S = \{z \in \mathbb{C} \mid |\mathrm{Im}\left(\frac{1}{z}\right)| < 1\}$. In polar form $z = re^{i\theta}$, the inequality becomes

$$|\mathrm{Im}(r^{-1}e^{-i\theta})| = |r^{-1}\sin(-\theta)| = r^{-1}|\sin\theta| < 1$$

Therefore, $|\sin\theta| < r$. Similar to Example 5, this represents all the complex numbers lying outside two closed disks $\overline{\mathbb{D}(\pm\frac{i}{2}, \frac{1}{2})}$. The set $S$ is illustrated in Figure 1.4; it is open, unbounded, and multiply connected.

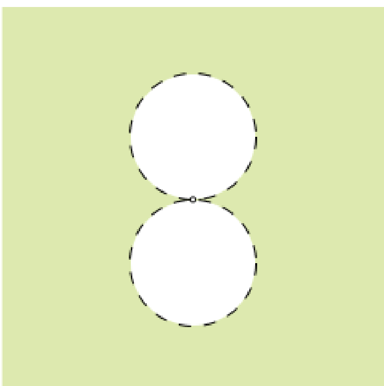


Figure 1.4: The set $S$.

**Short Quiz 1**

1. Simplify $\frac{1+i}{i-1}$.
2. Find the modulus of $(3+4i)(-4+3i)$.
3. Find the argument(s) of $\arg(-1+i)$.
4. Express $2e^{-2\pi i/3}$ in the form of $x+iy$.
5. What are the $3^{\text{th}}$ roots of $8i$?
6. Find the value of $(1+i)^6$.
7. If $z \neq 0$, express $\operatorname{Im}\left(\frac{z}{z+\bar{z}}\right)$ in terms of $\theta = \operatorname{Arg}(z)$.

Consider the following subsets:

$$A = \mathbb{D}(2,2) \cup \mathbb{D}(-2,2), \qquad B = \overline{\mathbb{D}(i,1)} \cup \overline{\mathbb{D}(-i,1)},$$
$$C = \mathbb{D}(2,1) \cup \mathbb{D}(-2,1), \qquad D = C(i,1) \cup \overline{\mathbb{D}(-i,1)}.$$

8. Identify subsets which are open.
9. Identify subsets which are connected.
10. Identify subsets which are simply connected.

Answers: 1. $-i$, 2. 25, 3. $\frac{3\pi}{4} + 2\pi k$, 4. $-1 - i\sqrt{3}$, 5. $\pm\sqrt{3} + i$ & $-2i$, 6. $-8i$, 7. $\frac{1}{2}\tan\theta$, 8. $A$ and $C$, 9. $B$ and $D$, 10. $B$.

# Chapter 2

# Complex Functions

## 2.1 Convergence and Continuity

**Definition 3.** A sequence of complex numbers $\{z_n\}_{n\in\mathbb{N}}$ *converges* to a *limit* $z$ if and only if:

$$\text{for all } \epsilon > 0, \text{ there exists } N > 0 \text{ such that for all } n \geq N, |z_n - z| < \epsilon.$$

Convergence of a sequence $z_n$ to $z$ can be denoted by $z_n \to z$, $|z_n - z| \to 0$, or $\lim_{n\to\infty} z_n = z$.

**Proposition 2.1.** *The limit of a convergent sequence is unique.*

*Proof.* Suppose for a contradiction that there are distinct limits $z \neq w$ of a sequence $z_n$. Let $\epsilon = \frac{1}{2}|z - w| > 0$, then for all sufficiently high $n$, $|z_n - z| < \epsilon$ and $|z_n - w| < \epsilon$. However, by triangle inequality,

$$|z - w| \leq |z - z_n| + |z_n - w| < 2\epsilon = |z - w|.$$

We then have a contradiction. □

**Theorem 2.2.** *If $z_n \to z$ and $w_n \to w$, then*

- $z_n + w_n \to z + w$,
- $z_n w_n \to zw$.

*Proof.* Let's pick $\epsilon > 0$. There are some high $N_1, N_2 \in \mathbb{N}$ such that $|z_n - z| < \epsilon/2$ if $n \geq N_1$, and $|w_n - w| < \epsilon/2$ if $n \geq N_2$. By triangle inequality, when $n \geq \max\{N_1, N_2\}$,

$$|z_n + w_n - z - w| \leq |z_n - z| + |w_n - w| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $z_n + w_n \to z + w$.

Set $M = \max\{|w_1|, \dots |w_{N_2}|, |w| + \epsilon\}$. The sequence $\{w_n\}_{n\in\mathbb{N}}$ is bounded because we have the following inclusions:

$$\{w_n\}_{n\in\mathbb{N}} \subset \{w_1, \dots w_{N_2}\} \cup \mathbb{D}(w, \epsilon/2) \subset \mathbb{D}(0, M).$$

There are some $N_3, N_4 \in \mathbb{N}$ such that $|z_n - z| < \epsilon/2M$ if $n \geq N_3$, and $|w_n - w| < \epsilon/2\max\{1, |z|\}$ if $n \geq N_4$. Then, when $n \geq \max\{N_3, N_4\}$,

$$\begin{aligned}|z_n w_n - zw| &\leq |z_n w_n - z w_n| + |z w_n - zw| = |w_n||z_n - z| + |z||w_n - w| \\ &< M \cdot \frac{\epsilon}{2M} + |z| \cdot \frac{\epsilon}{2\max\{1, |z|\}} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

This shows that $z_n w_n \to zw$. □

In particular, a sequence of complex numbers converges exactly when the real parts and the imaginary parts converge respectively.

**Corollary 2.3.** *$x_n + iy_n \to x + iy$ if and only if $x_n \to x$ and $y_n \to y$.*

*Proof.* The $\Leftarrow$ direction is immediate from the previous proposition. The $\Rightarrow$ direction comes from the following inequality:

$$\max\{|x_n - x|, |y_n - y|\} \leq \sqrt{|x_n - x|^2 + |y_n - y|^2} = |x_n + iy_n - x - iy|.$$

As $|x_n + iy_n - x - iy| \to 0$, sandwich rule forces both $|x_n - x|$ and $|y_n - y|$ to converge to 0 too. □

**Definition 4.** Let $U$ and $V$ be non-empty subsets of $\mathbb{C}$. A function $f : U \to V$ is *continuous at* $a \in U$ if and only if:

$$\begin{aligned}&\text{for all } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that} \\ &\text{if } z \in U \cap \mathbb{D}(a, \delta), \text{ then } f(z) \in \mathbb{D}(f(a), \epsilon).\end{aligned}$$

We say that $f$ is *continuous* if it is continuous at every point in $U$.

**Proposition 2.4.** *A function $f : U \to V$ is continuous at $a \in U$ if and only if for any sequence $z_n$ in $U$, if $z_n \to a$, then $f(z_n) \to f(a)$.*

*Proof.* Let $f$ be continuous at $a$ and pick the pair $(\epsilon, \delta)$ in the definition of continuity. Suppose $z_n \to a$, then there is some high $N \in \mathbb{N}$ such that if $n \geq N$, $|z_n - a| < \delta$. By continuity, if $n \geq N$, $|f(z_n) - f(a)| < \epsilon$. Therefore, $f(z_n) \to f(a)$.

Suppose for any sequence $z_n$ converging to $a$, $f(z_n) \to f(a)$. Suppose for a contradiction that $f$ is not continuous at $a$, then there is some $\epsilon > 0$ and sequence of points $z_n \in U \cap \mathbb{D}(a, \frac{1}{n})$ for $n \in \mathbb{N}$ such that $|f(z_n) - f(a)| \geq \epsilon$. Since $|z_n - a| < \frac{1}{n} \to 0$ as $n \to \infty$, then $z_n \to a$. The assumption implies that $f(z_n) \to f(a)$, but this cannot happen because $f(z_n)$ is always at least $\epsilon$ away from $f(a)$. This gives the contradiction. □

The statement above gives an equivalent way of defining continuity at a point. A shorter way of saying this is

$$\lim_{z \to a} f(z) = f(a).$$

The following is a direct consequence of 2.2.

**Proposition 2.5.** *Let $f, g : U \to V$ and $h : V \to W$ be continuous functions, then the sum $f+g$, the product $f \cdot g$ and the composition $h \circ f$ are continuous.*

**Example 9.** Constant functions $f(z) = a$ are trivially continuous. The identity function $\mathrm{Id}(z) = z$ is also continuous. By taking products and sums, we can inductively obtain that every complex polynomial $a_d z^d + a_{d-1} z^{d-1} + \ldots a_1 z + a_0$ is continuous.

**Example 10.** The modulus function $m(z) = |z|$ is a continuous function on $\mathbb{C}$. Indeed, for any $a \in \mathbb{C}$, if $z \to a$, then by triangle inequality,

$$\big|m(z) - m(a)\big| \leq \big||z| - |a|\big| \leq |z - a| \to 0.$$

By sandwich rule, $m(z) \to m(a)$ too.

By the previous proposition, if $f(z)$ is a continuous function on a subset of $\mathbb{C}$, then so is the composition $|f(z)|$.

**Example 11.** The functions Re and Im are continuous. (Refer to Corollary 2.3.) Since $\bar{z} = \mathrm{Re}(z) - i\,\mathrm{Im}(z)$, complex conjugation is also continuous.

Continuous functions behave nicely on compact subsets of $\mathbb{C}$.

**Theorem 2.6.** *Let $f : K \to V$ be a continuous function and $K$ be a compact subset of $\mathbb{C}$. Then, $f$ attains a maximum and a minimum on $K$, i.e. there are points $a, b \in K$ where $|f(a)| \leq |f(z)| \leq |f(b)|$ for all $z \in K$.*

The theorem above is a consequence of a result from topology. In particular, the image $|f(K)|$ of a compact set $K$ under a continuous function $|f|$ is compact. Any compact subset of $\mathbb{R}$ contains maximum and minimum points because it must be a finite union of closed finite intervals.

## 2.2 Holomorphic Functions

We can define differentiability of complex-valued functions the same way as we define that of functions of one real variable. However, we will emphasise in the next few sections that complex differentiability is actually a much more rigid notion than the usual multivariable real differentiability.

**Definition 5.** Let $U, V \subset \mathbb{C}$ be open and non-empty. A complex function $f$ is *(complex) differentiable* at a point $a$ if and only if the following limit exists:

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

If so, then $f'(a)$ is called the *(complex) derivative* of $f$ at $a$. The function $f$ is said to be *holomorphic* on $U$ if it is holomorphic at every point in $U$, and *entire* if additionally $U = \mathbb{C}$.

*Remark.* The term "analytic" and "complex differentiable" are often used interchangeably with "holomorphic".

**Example 12.**

1. Constant functions $f(z) = a$ are entire with derivative 0 everywhere.

2. The identity function $\mathrm{Id}(z) = z$ is an entire function and its derivative is 1 everyhere.

3. The inversion function $\tau(z) = 1/z$ is is holomorphic on $\mathbb{C}^*$ and its derivative is $-z^{-2}$. Indeed, if we choose any angle $\theta$, then if $z = re^{i\theta} + a$,

$$\begin{aligned} \lim_{z \to a} \frac{\tau(z) - \tau(a)}{z - a} &= \lim_{r \to 0} \frac{\frac{1}{re^{i\theta}+a} - \frac{1}{a}}{(re^{i\theta} + a) - a} = \lim_{r \to 0} \frac{\frac{-re^{i\theta}}{a(re^{i\theta}+a)}}{re^{i\theta}} \\ &= \lim_{r \to 0} -\frac{1}{a(re^{i\theta} + a)} = -\frac{1}{a^2}. \end{aligned}$$

4. Complex conjugation $f(z) = \bar{z}$ has no derivative at any point. If we choose any angle $\theta$, then using $z = re^{i\theta} + a$,

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{r \to 0} \frac{(re^{-i\theta} + \bar{a}) - \bar{a}}{(re^{i\theta} + a) - a} = e^{-2i\theta},$$

but the value of this limit is not the same if we choose different values of $\theta$. For example, the limit is 1 when $\theta = 0$, but it is $-1$ if $\theta = \frac{\pi}{2}$.

**Proposition 2.7.** *Every holomorphic function is continuous.*

*Proof.* Let $f : U \to V$ be holomorphic. If $a \in U$,

$$\begin{aligned} \lim_{z \to a} f(z) - f(a) &= \lim_{z \to a} (z - a) \frac{f(z) - f(a)}{z - a} \\ &= \lim_{z \to a} (z - a) \cdot \lim_{z \to a} \frac{f(z) - f(a)}{z - a} \\ &= 0 \cdot f'(a) = 0, \end{aligned}$$

where the second equality follows from Theorem 2.2. Therefore, $f$ is continuous at $a$. As $a$ is arbitrary, $f$ is continuous on $U$. $\square$

The rules for differentiation of complex-valued functions is more or less the same as those of functions of one real variable.

**Proposition 2.8.** *Let $f, g : U \to V$ and $h : V \to W$ be holomorphic. Then,*

*(a) the sum $f + g$ is holomorphic and $(f + g)'(z) = f'(z) + g'(z)$,*

1. *the product $f \cdot g$ is holomorphic and $(f \cdot g)'(z) = f'(z)g(z) + f(z)g'(z)$,*

2. *the composition $h \circ f$ is holomorphic and $(h \circ f)'(z) = h'(f(z))f'(z)$.*

*Proof.* $(a)$ follows immediately from Proposition 2.5. For $(b)$,

$$\begin{aligned}(f \cdot g)'(a) &= \lim_{z \to a} \frac{f(z)g(z) - f(a)g(a)}{z - a} \\ &= \lim_{z \to a} \frac{f(z)\big(g(z) - g(a)\big)}{z - a} + \frac{g(a)\big(f(z) - f(a)\big)}{z - a} \\ &= f(a)g'(a) + g(a)f'(a).\end{aligned}$$

For $(c)$, we use the fact that $f$ is continuous:

$$\begin{aligned}(h \circ f)'(a) &= \lim_{z \to a} \frac{h(f(z)) - h(f(a))}{z - a} \\ &= \lim_{z \to a} \frac{h(f(z)) - h(f(a))}{f(z) - f(a)} \cdot \frac{f(z) - f(a)}{z - a} \\ &= \lim_{w \to f(a)} \frac{h(w) - h(f(a))}{w - f(a)} \cdot \lim_{z \to a} \frac{f(z) - f(a)}{z - a} \\ &= h'(f(a))f'(a).\end{aligned}$$

□

**Example 13.**

1. Every polynomial $p(z) = \sum_{n=0}^{d} a_n z^n$ with complex coefficients $a_n \in \mathbb{C}$ is an entire function. We can show this inductively by product rule above that every monomial $a_n z^n$ is holomorphic with derivative $a_n n z^{n-1}$ and by the addition rule, $p$ is holomorphic.

2. Every rational function, i.e. a function of the form $f(z) = p(z)/q(z)$ where $p$ and $q$ are polynomials, is holomorphic on $\mathbb{C} \backslash \{z \in \mathbb{C} \,|\, q(z) = 0\}$.

Every complex function $f(z)$ admits a unique real-imaginary splitting $f(x+iy) = u(x, y)+iv(x, y)$, where $u$ and $v$ are real-valued functions defined on an open subset of $\mathbb{R}^2$ given by:

$$u(x, y) = \mathrm{Re} f(x + iy), \qquad v(x, y) = \mathrm{Im} f(x + iy).$$

We say that $f$ is *(real) differentiable* if the partial derivatives of $u$ and $v$ with respect to $x$ and $y$ exist. When we are given $u$ and $v$, we will see that holomorphic functions are precisely solutions of a system of partial differential equations.

**Theorem 2.9** (Cauchy-Riemann Equations)**.** *Let $f = u + iv$ be a complex function on an open set $U \subset \mathbb{C}$. Then, $f$ is holomorphic if and only if $u$ and $v$ are continuously differentiable and $u_x = v_y$ and $v_x = -u_y$.*

*Proof.* Let $f$ be holomorphic at a point $a = a_1 + ia_2 \in U$, then

$$f'(a) = \lim_{h\to 0} \frac{u(a_1 + h_1, a_2 + h_2) - u(a_1, a_2)}{h} + i\frac{v(a_1 + h_1, a_2 + h_2) - v(a_1, a_2)}{h}.$$

The dummy variable $h = h_1 + ih_2$ can converge to zero in various directions, but we will only consider two cases. Suppose $h_2 = 0$, then

$$\begin{aligned} f'(a) &= \lim_{h_1\to 0} \frac{u(a_1 + h_1, a_2) - u(a)}{h_1} + i\frac{v(a_1 + h_1, a_2) - u(a)}{h_1} \\ &= u_x + iv_x. \end{aligned} \tag{2.1}$$

Similarly, if we consider the limit in the direction satisfying $h_1 = 0$,

$$\begin{aligned} f'(a) &= \lim_{ih_2\to 0} \frac{u(a_1, a_2 + h_2) - u(a)}{ih_2} + i\frac{v(a_1, a_2 + ih_2) - v(a)}{ih_2} \\ &= -iu_y + v_y. \end{aligned} \tag{2.2}$$

Comparing (2.1) and (2.2), it is clear by taking the real and imaginary parts of $f'(a)$ that the partial derivatives of $u$ and $v$ at $a$ exist and satisfy $u_x = v_y$ and $v_x = -u_y$. To show that these partial derivatives are continuous, we need continuity of $f'$. We will obtain this for granted in Corollary 4.4.

Conversely, suppose $u$ and $v$ are continuously differentiable satisfying $u_x = v_y$ and $v_x = -u_y$. The Taylor series of $u$ and $v$ at $a$ can be expressed as:

$$\begin{aligned} u(a_1 + h_1, a_2 + h_2) &= u(a_1, a_2) + u_x h_1 + u_y h_2 + |h|\psi(h), \\ v(a_1 + h_1, a_2 + h_2) &= v(a_1, a_2) + v_x h_1 + v_y h_2 + |h|\phi(h), \end{aligned}$$

for some functions $\psi$ and $\phi$ where $\psi(h)$, $\phi(h) \to 0$ as $h \to 0$. All partial derivatives of $u$ and $v$ are evaluated at $(a_1, a_2)$. Then,

$$\begin{aligned}
&\lim_{h\to 0} \frac{u(a_1+h_1, a_2+h_2) - u(a_1, a_2)}{h} + i\frac{v(a_1+h_1, a_2+h_2) - v(a_1, a_2)}{h} \\
&= \lim_{h\to 0} \frac{[u_x h_1 + u_y h_2 + |h|\psi(h)] + i[v_x h_1 + v_y h_2 + |h|\phi(h)]}{h} \\
&= \lim_{h\to 0} \frac{(u_x + iv_x)(h_1 + ih_2) + |h|\big(\psi(h) + i\phi(h)\big)}{h} \\
&= u_x + iv_x.
\end{aligned}$$

As the limit converges to $f' = u_x + iv_x$, $f$ is indeed holomorphic. $\square$

It is natural to consider a change of variables from $(x, y)$ to $(z, \bar{z}) = (x+iy, x-iy)$. By using multivariable chain rule, we can obtain an expression for partial derivatives $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

**Definition 6.** The *Wirtinger derivatives* are defined as follows:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

It is straightforward to check that the following identities hold:

$$\frac{\partial z}{\partial z} = \frac{\partial \bar{z}}{\partial \bar{z}} = 1, \qquad \frac{\partial \bar{z}}{\partial z} = \frac{\partial z}{\partial \bar{z}} = 0.$$

Given a complex function $f = u+iv$ with differentiable real and imaginary parts $u$ and $v$,

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(f_x + if_y) = \frac{1}{2}\big((u_x - v_y) + i(v_x + u_y)\big)$$

Clearly, the Cauchy-Riemann equations hold if and only if the expression above is 0. The equation

$$\frac{\partial f}{\partial \bar{z}} = 0$$

is the complex form of the Cauchy-Riemann equations, as it allows us to deduce holomorphicity without having to know the real and imaginary parts $u$ and $v$, but rather by the absence of the variable $\bar{z}$. Roughly speaking, if $f$ is not a function of $\bar{z}$, then it is holomorphic!

If $f$ is holomorphic, the notation for the complex derivative of $f$ is also consistent since

$$\frac{\partial f}{\partial z} = \frac{1}{2}(f_x - if_y) = \frac{1}{2}\big((u_x + v_y) + i(v_x - u_y)\big) = u_x + iv_x = f',$$

where the last inequality follows from (2.1).

**Example 14.** Re and Im are differentiable but not holomorphic. (Refer to Proposition 1.1.)

## 2.3 Exponential and Logarithmic Functions

One of the many elementary functions we will commonly encounter is the exponential function

$$\exp(z) := e^z = e^x e^{iy}.$$

The real and imaginary parts are $u = e^x \cos(y)$ and $v = e^x \sin(y)$, and we can easily check that the Cauchy-Riemann equations are satisfied. Thus, this is an entire function and its derivative is itself. Below are some of its properties:

1. $|\exp(z)| = e^x$,
2. $\arg(\exp(z)) = y + 2\pi ik$ where $k \in \mathbb{Z}$,
3. $\exp(z + 2\pi ik) = \exp(z)$ for any $k \in \mathbb{Z}$,
4. The image of exp on $\mathbb{C}$ is $\mathbb{C}^*$.

We would now like to find the inverse of exp. If we denote the inverse by log, then

$$\log(z) := \ln|z| + i\arg(z)$$

for any $z \neq 0$. However, as arg is multivalued, log is multi-valued and therefore it is not a well-defined function. This is consistent with the fact that exp is not injective due to property 3.

This problem can be fixed by using the principle argument Arg. Doing so will introduce a ray of discontinuity $(-\infty, 0]$ corresponding to the points in $\mathbb{C}$ with argument $\pi$. We can replace arg with Arg and define the *principal value* of $\log(z)$, denoted by pv $\log(z)$. If we choose the codomain to be $\mathrm{Arg}(z) \in (-\pi, \pi]$, we then have the *principal branch* of the logarithmic function

$$\mathrm{Log} : \mathbb{C}\backslash(-\infty, 0] \to \mathbb{C}, \quad \mathrm{Log}(z) := \ln|z| + i\mathrm{Arg}(z).$$

The ray $(-\infty, 0]$ is often called a *branch cut.* Using this choice of branch cut, Log is holomorphic on $\mathbb{C}\backslash(-\infty, 0]$, with image $\{x + iy \mid x \in \mathbb{R}, -\pi < y < \pi\}$ and derivative $\frac{1}{z}$. (To verify this, compute the Wirtinger derivatives in polar coordinates.)

**Example 15.** $\log(i)$ is purely imaginary and multivalued since

$$\log(i) = \ln(1) + i\arg(i) = \frac{\pi i}{2} + 2\pi ik, \text{ where } k \in \mathbb{Z}.$$

Its principal value is $\mathrm{Log}(i) = \frac{\pi i}{2}$.

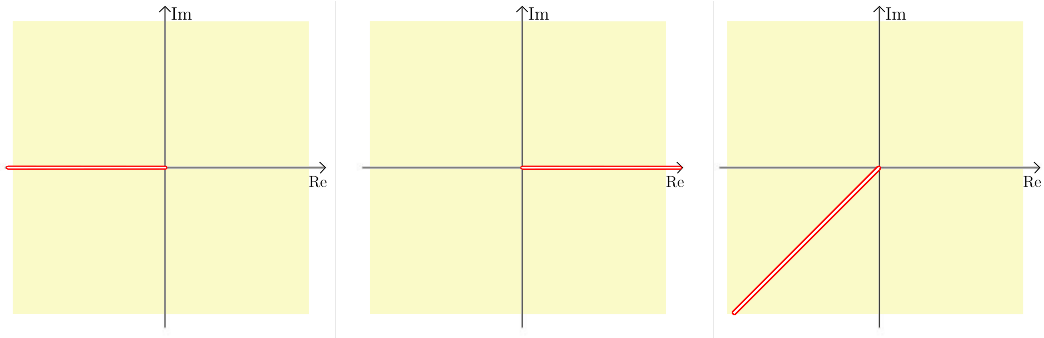



Figure 2.1: Various branch cuts of the logarithm

*Remark.* In general, the branch cut can be taken to be any unbounded curve from 0 which does not intersect itself. Straight rays of different angles are often used if necessary.

For a non-integer $c \in \mathbb{C}$, we define the *power function* $z^c$ to be the multivalued function on $\mathbb{C}^*$ given by

$$z^c := \exp(c \log(z)).$$

Similar to the logarithm, we can take the principal value of $z^c$ to be

$$\operatorname{pv} z^c := \exp(c \operatorname{Log}(z)).$$

Again, this becomes a holomorphic function outside a chosen branch cut $(-\infty, 0]$.

**Example 16.** $i^i$ is (perhaps surprisingly) real and multivalued since

$$i^i = e^{-\pi/2 - 2\pi k}, \text{ where } k \in \mathbb{Z}.$$

Its principal value is $\operatorname{pv} i^i = e^{-\pi/2}$.

## 2.4 Trigonometric Functions

Euler's formula allows us to express $\sin\theta$ and $\cos\theta$ in terms of $e^{\pm i\theta}$ as follows:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

It is then natural to define trigonometric functions of a complex variable $z$ using the exponential function:

$$\cos(z) := \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin(z) := \frac{e^{iz} - e^{-iz}}{2i}, \qquad \tan(z) = \frac{\sin(z)}{\cos(z)}.$$

As exp is entire, so are cos and sin. However, the function tan is holomorphic everywhere except at points $z$ such that $\cos(z) = 0$. You may check that the usual trigonometric identities still hold.

The generalisation of hyperbolic functions are also clear:

$$\cosh(z) := \frac{e^z + e^{-z}}{2}, \qquad \sinh(z) := \frac{e^z - e^{-z}}{2}, \qquad \tanh(z) = \frac{\sinh(z)}{\cosh(z)}.$$

The functions cosh and sinh can be viewed as the even part and the odd part of the exponential function respectively. Both are entire functions, but tanh is only holomorphic everywhere except at points $z$ such that $\cosh(z) = 0$. The following property can be easily deduced by definition.

**Proposition 2.10.** *For any $z \in \mathbb{C}$, $\cos(iz) = \cosh(z)$ and $\sin(iz) = i\sinh(z)$.*

A point $w$ is a *zero* of a function $f$ if $f(w) = 0$. It turns out that the zeros of trigonometric functions in $\mathbb{C}$ are the same as the zeros of trigonometric functions in $\mathbb{R}$.

**Proposition 2.11.** *The zeros of trigonometric and hyperbolic functions are as follows:*

$$\begin{aligned}
\{z \in \mathbb{C} \mid \sin(z) = 0\} &= \{n\pi\}_{n\in\mathbb{Z}}, \\
\{z \in \mathbb{C} \mid \cos(z) = 0\} &= \left\{\frac{\pi}{2} + n\pi\right\}_{n\in\mathbb{Z}}, \\
\{z \in \mathbb{C} \mid \sinh(z) = 0\} &= \{in\pi\}_{n\in\mathbb{Z}}, \\
\{z \in \mathbb{C} \mid \cosh(z) = 0\} &= \left\{i\left(\frac{\pi}{2} + n\pi\right)\right\}_{n\in\mathbb{Z}}.
\end{aligned}$$

*Proof.* By addition rule and Proposition 2.10, $\sin(x + iy) = \sin x \cosh y + i \sinh y \cos x$. Thus, if $z = x + iy$,

$$\begin{aligned}
|\sin(z)|^2 &= \sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x \\
&= \sin^2 x(1 + \sinh^2 y) + \sinh^2 y(1 - \sin^2 x) \\
&= \sin^2 x + \sinh^2 y. \qquad (2.3)
\end{aligned}$$

As such, $\sin(z) = 0$ if and only if $\sin x = 0$ and $\sinh y = 0$, and the latter occurs exactly when $x \in \{n\pi\}_{n\in\mathbb{Z}}$, and $y = 0$.

The zeros of cos can be obtained from those of sin using the identity $\cos(z) = \sin(\pi/2 - z)$, and the zeros of hyperbolic functions can be obtained from those of trigonometric functions by Proposition 2.10. □

Tgonometric and hyperbolic functions are definitely not surjective, but we are still able to find their local inverses on some restricted domains. Since they are made up of the exponential function, their inverses can be described in terms of logarithms.

**Proposition 2.12.** *The inverses of trigonometric and hyperbolic functions are multivalued functions:*

$$\begin{aligned}
\sin^{-1}(z) &= -i\log\left(iz + [1-z^2]^{1/2}\right), \\
\cos^{-1}(z) &= -i\log\left(z + i[1-z^2]^{1/2}\right), \\
\tan^{-1}(z) &= \frac{i}{2}\log\frac{i+z}{i-z}, \\
\sinh^{-1}(z) &= \log\left(z + [z^2+1]^{1/2}\right), \\
\cosh^{-1}(z) &= \log\left(z + [z^2-1]^{1/2}\right), \\
\tanh^{-1}(z) &= \frac{1}{2}\log\frac{1+z}{1-z}.
\end{aligned}$$

*Proof.* Let $\sin^{-1}(z) = w$, then $z = \frac{e^{iw}-e^{-iw}}{2i}$. This can be written in quadratic form:

$$(e^{iw})^2 - 2ize^{iw} - 1 = 0.$$

The quadratic formula gives us $e^{iw} = iz + [1-z^2]^{1/2}$, and using logarithm,

$$w = -i\log\left(iz + [1-z^2]^{1/2}\right).$$

The resulting function is multivalued since the square root and the logarithm are multivalued. Similar algebraic methods can be applied to obtain the inverses of other functions and will be left to the reader as an exercise. □

For each of the functions above, we can pick a branch cut of the square root and the logarithm in order to obtain a holomorphic branch of the function. However, finding a nice branch cut for the inverse of the function requires a more involved argument and we shall not attempt to find it.

**Example 17.** Let's find the solution for the equation $\sin(2\pi z) = 2$. By the proposition above,

$$\begin{aligned}
z &= -\frac{i}{2\pi}\log\left(2i + (-3)^{1/2}\right) = -\frac{i}{2\pi}\log\left(i(2\pm\sqrt{3})\right) \\
&= -\frac{i}{2\pi}\left[\ln(2\pm\sqrt{3}) + \frac{\pi i}{2} + 2\pi i k\right], \\
&= -\frac{i\ln(2\pm\sqrt{3})}{2\pi} + \frac{1}{4} + k, \qquad k\in\mathbb{Z},
\end{aligned}$$

## Short Quiz 2

1. Does the sequence $i^n$ converge as $n \to \infty$? If so, what is the limit?

2. Does the sequence $\left(\frac{i}{n}\right)^n$ converge as $n \to \infty$? If so, what is the limit?

3. Find the limit of the sequence $\frac{1}{n} + \left(1 + \frac{i}{n}\right)^n$.

4. Which of the following functions are continuous on $\mathbb{C}$?

$$z^3, \quad 1/z, \quad |z-2| + |z+2|, \quad \arg(z)$$

5. At which of the following points is $Arg(z)$ continuous?

$$1, \quad i, \quad -1, \quad -i, \quad 0$$

6. What are the solutions of the equation $e^z = -\pi$?

7. What is the value of $(i^i)^i$?

8. What are the solutions of the equation $\tanh(iz) = 0$?

9. Which of these functions are holomorphic on a domain containing $2\pi i$?

$$\sin z, \quad \sinh z, \quad \csc z \quad \mathrm{csch} z \quad \cot z \quad \coth z$$

10. Which of these complex numbers correspond to $\log(1)$?

$$0, \quad 1, \quad 2\pi i, \quad 1 + 2\pi i, \quad -2\pi i, \quad 1 - 2\pi i.$$

Answers: 1. No, 2. Yes, to 0, 3. $e^i$, 4. $z^3$ and $|z-2| + |z+2|$, 5. $1, i, -i$. 6. $\ln \pi + i(\pi + 2\pi i k)$. 7. $-i$. 8. $k\pi$. 9. $\sin z, \sinh z, \csc z, \cot z$. 10. 0 and $\pm 2\pi i$.

# Chapter 3

# Contour Integration

## 3.1 Curves in $\mathbb{C}$

**Definition 7.** An *arc / curve / path* in $\mathbb{C}$ is subset of $\mathbb{C}$ parametrized by a continuous function $\gamma : [a, b] \to \mathbb{C}$ defined on a closed interval $[a, b] \subset \mathbb{R}$.

A curve $\gamma : [a, b] \to \mathbb{C}$, can be expressed as $\gamma(t) = u(t) + iv(t)$ where $u$ and $v$ are real-valued functions. We say that $\gamma$ is *differentiable* when both $u$ and $v$ are differentiable on $[a, b]$ as real functions. The derivative of $\gamma$ at $t$ is

$$\gamma'(t) = u'(t) + iv'(t).$$

For each $t$, when $\gamma'(t) \neq 0$, $\gamma'(t)$ represents a tangent vector of the curve at the point $\gamma(t)$ of magnitude $|\gamma'(t)| = \sqrt{u'(t)^2 + v'(t)^2}$.

**Definition 8.** A curve $\gamma : [a, b] \to \mathbb{C}$ is:

- *closed* if $\gamma(a) = \gamma(b)$,
- *simple* if $\gamma$ is injective on the open interval $(a, b)$,
- *smooth* if $\gamma$ is differentiable and $\gamma'(t) \neq 0$ for all $t \in (a, b)$,
- a *contour* if $\gamma$ is piecewise smooth, i.e. $\gamma$ can be partitioned into finitely many smooth curves.

**Example 18.** A circle $C(z, r)$ of radius $r > 0$ centered at $w \in \mathbb{C}$ is a simple closed smooth curve. It can be parametrised by $\gamma(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$.

**Example 19.** The function $\sigma(t) = e^{it} \sin(2t)$, $0 \leq t \leq 2\pi$, parametrises the locus of the equation $r = \sin(2\theta)$. This curve is a closed and smooth but clearly not simple. See the leftmost curve in Figure 3.1.

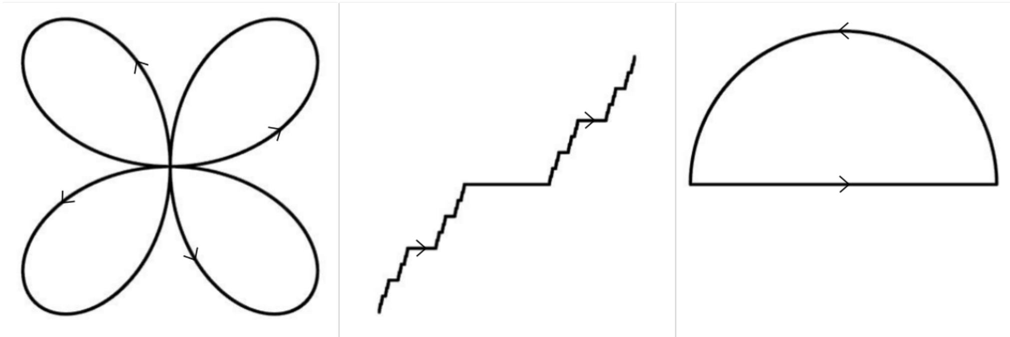


Figure 3.1: A closed non-simple smooth curve, a non-closed non-piecewise-smooth curve (Devil's Staircase), and a closed piecewise smooth curve.

We will state without proof two results from topology.

**Proposition 3.1.** *Any domain $U \subset \mathbb{C}$ is path-connected. That is, for any two points $z$ and $w$ in a domain $U \subset \mathbb{C}$, there is a smooth curve $\gamma : [0,1] \to U$ such that $\gamma(0) = z$ and $\gamma(1) = w$.*

**Theorem 3.2** (Jordan Curve Theorem)**.** *The complement of any simple closed curve in $\mathbb{C}$ has exactly two connected components, exactly one of which is bounded.*

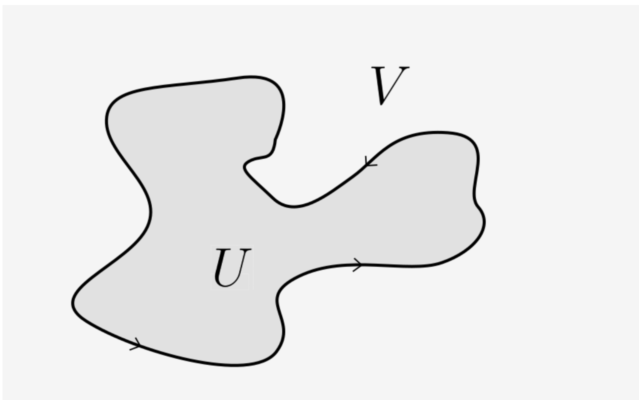



Figure 3.2: A simple closed curve splits its complement into a bounded domain $U$ and an unbounded domain $V$.

*Remark.* Due to this theorem, simple closed curves are often called *Jordan curves*. You may think that the theorem seems very intuitive, but the proof is rather involved. The very first proof was out in 1910s (not by Camille Jordan), relying on heavy machineries such as the theory of homology groups in algebraic topology. The shortest proof I know is by Maehara (1984), relying only on basic knowledge of topology, including the Brouwer fixed point theorem.

**Definition 9.** A point $w$ or a subset $S$ of $\mathbb{C}$ is said to be *enclosed* by a simple closed curve $\gamma$ if $\{w\}$ or $S$ is contained in the bounded connected component of the complement of $\gamma$ in $\mathbb{C}$.

Every curve $\gamma$ has two possible orientations. Given a curve $\gamma : [a, b] \to \mathbb{C}$, we can reverse its orientation to obtain $\gamma^- : [a, b] \to \mathbb{C}$ defined by $\gamma^-(t) = \gamma(a + b - t)$.

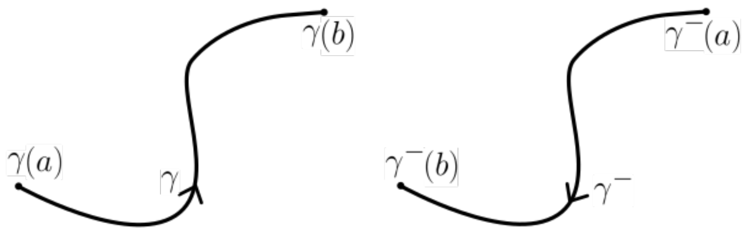



Figure 3.3: Reversing the orientation of $\gamma$.

When $\gamma$ is a simple closed curve, the orientation of $\gamma$ is *positive* if for any point $w$ enclosed by $\gamma$, $\gamma(t)$ runs anticlockwise with respect to the basepoint $w$ as $t$ increases. The orientation is *negative* if $\gamma(t)$ runs clockwise. Unless stated otherwise, we always assume that every simple closed curve $\gamma$ is positively oriented, and $\gamma^-$ is negatively oriented.

## 3.2 Integration Along a Contour

Any closed interval $[a, b]$ can be parametrized by a trivial curve $\gamma(t) = t$, $t \in [a, b]$. The integral of a continuous function $f$ along $\gamma$ is taken to be $\int_a^b f(z)dz$. We shall introduce a way to generalise line integrals along an arbitrary contour.

**Definition 10.** Let $f$ be a complex-valued continuous function defined on a smooth curve parametrized by $\gamma : [a, b] \to \mathbb{C}$. The *integral* of $f$ along $\gamma$ is

$$\int_\gamma f(z)dz := \int_a^b f\big(\gamma(t)\big)\gamma'(t)dt.$$

If $\gamma$ is a contour which is smooth on $[a_{j-1}, a_j]$ for $j = 1, \dots N$ where $a_0 = a$ and $a_N = b$, then the *integral* of $f$ along $\gamma$ is a finite sum of the integral over the smooth parts:

$$\int_\gamma f(z)dz := \sum_{j=1}^{N} \int_{a_{j-1}}^{a_j} f\big(\gamma(t)\big)\gamma'(t)dt.$$

*Remark.* When $\gamma$ is a simple closed curve, it is common to denote the integral of $f$ along $\gamma$ by the notation

$$\oint_\gamma f(z)dz.$$

Two parametrizations $\gamma : [a, b] \to \mathbb{C}$ and $\sigma : [c, d] \to \mathbb{C}$ of a smooth curve are *equivalent* if there is a continuously differentiable bijection $h : [a, b] \to [c, d]$ such that $h(a) = c$, $h(b) = d$, and $\gamma = \sigma \circ h$.

Our definition of contour integral is robust because it is independent of our choice of parametrization of the curve. If two curves $\gamma$ and $\sigma$ are equivalent, convince yourself by change of variables $t \mapsto h(t)$ defined above that the integral of $f$ along $\gamma$ and $\sigma$ must be equal:

$$\int_\gamma f(z)dz = \int_\sigma f(z)dz.$$

If we reverse the orientation of $\gamma$, we can again apply change of variables $t \mapsto a + b - t$ and check that the integral of a continuous function $f$ on $\gamma^-$ is

$$\int_{\gamma^-} f(z)dz = -\int_\gamma f(z)dz. \tag{3.1}$$

**Example 20.** Let's compute the integral $I = \oint_\gamma f(z)dz$ of the function $f(z) = 1/(z - z_0)$ along the curve $\gamma(t) = z_0 + re^{it}$ where $0 \le t \le 2\pi$.

$$I = \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(z) - z_0} dt = \int_0^{2\pi} \frac{ire^{it}}{(z_0 + re^{it}) - z_0} dt = \int_0^{2\pi} idt = 2\pi i.$$

Notice that the value $I$ is independent of the radius $r$ and the center $z_0$.

The *length* $L(\gamma)$ of a smooth curve $\gamma : [a,b] \to \mathbb{C}$ can be computed by the following integral:

$$L(\gamma) = \int_a^b |\gamma'(t)| dt.$$

We can think of $L(\gamma)$ as the integral of the continuous function $f$ along $\gamma$ defined by $f(z) = |\gamma'(t)|/\gamma'(t)$ where $z = \gamma(t)$. As such, the length is invariant under parametrization and change of orientation.

**Example 21.** The length of any circle of radius $r$ is $2\pi r$. Indeed, using the parametrisation $\gamma(t) = z_0 + re^{it}$ where $0 \le t \le 2\pi$,

$$L(\gamma) = \int_0^{2\pi} |ire^{it}| dt = \int_0^{2\pi} r dt = 2\pi r.$$

An equivalent parametrization that is commonly used is $\sigma(t) = z_0 + re^{2\pi it}$, $0 \le t \le 1$. Can you show that the length computed using this parametrization is the same?

**Proposition 3.3.** *Let $f$ and $g$ be continuous functions on a contour $\gamma$ and let $\alpha, \beta \in \mathbb{C}$. Then,*

- $\int_\gamma \alpha f(z) + \beta g(z) dz = \alpha \int_\gamma f(z) dz + \beta \int_\gamma g(z) dz$, *(linearity)*
- $\left| \int_\gamma f(z) dz \right| \le L(\gamma) \cdot \max_{z\in\gamma} |f(z)|$. *(ML inequality)*

*Proof.* Linearity is trivial. The curve parametrized by $\gamma$ is a compact subset of $\mathbb{C}$ and therefore, by Theorem 2.6, $|f|$ always attains its maximum along $\gamma$. Let $M := \max_{z\in\gamma} |f(z)|$ and pick a parametrization $\gamma : [a,b] \to \mathbb{C}$. Then,

$$\begin{aligned} \left| \int_\gamma f(z) dz \right| &\le \left| \int_a^b f(\gamma(t))\gamma'(t) dt \right| \le \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)| dt \\ &\le \int_a^b M|\gamma'(t)| dt = M \int_a^b |\gamma'(t)| dt = ML(\gamma). \end{aligned}$$

The second inequality follows from the fact that for any continuous function $h : [a,b] \to \mathbb{C}$, we always have the inequality $|\int_a^b h(t) dt| \le \int_a^b |h(t)| dt$. We shall prove this below. Let $re^{i\theta}$ be the polar form of $\int_a^b h(t) dt$. Then,

$$\begin{aligned} \left| \int_a^b h(t) dt \right| &= e^{-i\theta} \int_a^b h(t) dt = \mathrm{Re} \int_a^b e^{-i\theta} h(t) dt \\ &= \int_a^b \mathrm{Re}[e^{-i\theta} h(t)] dt \le \int_a^b |e^{-i\theta} h(t)| dt = \int_a^b |h(t)| dt, \end{aligned}$$

and we are done. □

## 3.3 Primitives

**Definition 11.** An *antiderivative / primitive* of a continuous function $f$ on a domain $U \subset \mathbb{C}$ is a holomorphic function $F : U \to \mathbb{C}$ such that $F' = f$.

The existence of primitives makes computation of contour integrals much easier. Regardless of the shape of the contour, it turns out that the integral only depends on the endpoints of the contour.

**Lemma 3.4.** *Suppose $f$ is continuous on a domain $U$ and has a primitive $F$. Then, for any contour $\gamma : [a, b] \to U$,*

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)),$$

*Proof.* Let $\gamma : [a, b] \to U$ be a smooth curve, then

$$\int_\gamma f(z)dz = \int_a^b F'(\gamma(t))\gamma'(t)dt = \int_a^b \frac{d}{dt}F(\gamma(t))dt = F(\gamma(b)) - F(\gamma(a)),$$

If $\gamma$ is piecewise smooth, we can sum up the integral over all smooth parts to obtain a similar result. □

**Corollary 3.5.** *If $F$ is holomorphic on a domain $U$ and $F' \equiv 0$, then $F$ is a constant function.*

*Proof.* For any two points $u$ and $v$ in $U$, we can apply the lemma above by setting $f = 0$ to obtain $f(u) = f(v)$ because the integral over $f$ over any curve is 0. □

**Theorem 3.6.** *Suppose $f : U \to \mathbb{C}$ is continuous on a domain $U$. The following are equivalent:*

*(a) $f$ has a primitive $F : U \to \mathbb{C}$,*

*(b) For any closed contour $\gamma$ in $U$, $\oint_\sigma f(z)dz = 0$.*

*Proof.* Suppose $(a)$ is true. Let $\gamma : [a, c] \to U$ be a closed contour, then $F(\gamma(b)) = F(\gamma(b))$ because $\gamma(b) = \gamma(a)$. Then, Lemma 3.4 immediately gives us $(a) \Rightarrow (b)$.

Suppose $(b)$ is true. Pick a basepoint $z_0 \in U$ and for each $z \in U$, let $\gamma_z : [0, 1] \to U$ be a smooth curve from $\gamma_z(0) = z_0$ to $\gamma_z(1) = z$. Let's define $F$ by

$$F(z) = \int_{\gamma_z} f(w)dw.$$

To prove that $F$ is a well-defined function, we must show that the value $F(z)$ is independent of the choice of the smooth curve $\gamma_z$. Let $\gamma_z$ and $\sigma_z$ be two such curves, then reverse the orientation $\sigma_z$ to obtain $\sigma_z^-$, a smooth curve from $z$ to $z_0$. The two curves $\gamma_z$ and $\sigma_z^-$ glue together to form a closed contour $\Gamma_z$. By (3.1) and $(b)$,

$$\int_{\gamma_z} f(w)dw - \int_{\sigma_z} f(w)dw = \int_{\gamma_z} f(w)dw + \int_{\sigma_z^-} f(w)dw = \oint_{\Gamma_z} f(z)dw = 0.$$

Therefore, the integrals of $f$ along $\gamma_z$ and $\sigma_z$ coincide, proving that $F$ is a function. The proof is complete once we show that $F$ is holomorphic and $F' = f$. Pick any point $z \in U$, then $U$ contains a small ball $\mathbb{D}(z, \epsilon)$. Pick $h \in \mathbb{C}$ such that $|h| < \epsilon$, and define a line segment $\alpha_h(t) = z + th$, $0 \leq t \leq 1$. By a similar argument as above, $(b)$ implies

$$F(z+h) - F(z) = \int_{\alpha_h} f(w)dw.$$

Then, using the fact that $\int_{\alpha_h} dz = h$ and $L(\alpha_h) = |h|$, by ML inequality,

$$\begin{aligned}
&\left| f(z) - \frac{F(z+h) - F(z)}{h} \right| \\
&\quad = \left| f(z) - \frac{1}{h} \int_{\alpha_h} f(w)dw \right| = \left| \frac{1}{h} \int_{\alpha_h} \big(f(z) - f(w)\big)dw \right| \\
&\quad \leq \frac{1}{|h|} L(\alpha_h) \max_{w \in \alpha_h} |f(z) - f(w)| = \max_{w \in \alpha_h} |f(z) - f(w)|.
\end{aligned}$$

By continuity of $f$, as $h \to 0$, $\max_{w \in \alpha_h} |f(z) - f(w)| \to 0$. The limit of the left hand side of the inequality above is also 0, thus proving holomorphicity of $F$ near $z$. □

**Example 22.** The function $f(z) = (z - z_0)^{-n}$ on $\mathbb{C}\backslash\{z_0\}$ admits a primitive $\frac{(z-z_0)^{-n+1}}{-n+1}$ when $n \neq 1$. As such, if $n \neq 1$ and $\gamma$ is any closed contour not passing through $z_0$,

$$\oint_\gamma \frac{dz}{(z - z_0)^n} = 0.$$

**Example 23.** The derivative of $\log : \mathbb{C}\backslash(-\infty, 0] \to \mathbb{C}$ is the function $\tau(z) = 1/z$, so then $\tau$ has primitive log in the domain $\mathbb{C}\backslash(-\infty, 0]$. This domain cannot be extended any further since log becomes discontinuous on the branch cut. Alternatively, we can say that $\tau$ has no primitive in $\mathbb{C}^*$ because, from Example 20, the integral of $\tau$, along any circle centered at 0 is non-zero.

## 3.4 Cauchy-Goursat Theorem

Theorem 3.6 does not give a nice criterion for the existence of primitives because it can be rather troublesome to compute integrals over all possible closed contours.

**Theorem 3.7** (Cauchy-Goursat). *Let $f$ be holomorphic on a simply connected domain $U \subset \mathbb{C}$ and let $\gamma$ be a closed contour in $U$, then*

$$\oint_\gamma f(z)dz = 0.$$

Cauchy proved the theorem using Green's theorem from multivariable calculus, whereas Goursat's proof, albeit rather lengthly, only uses continuity of partial derivatives. We will only present the proof using Green's theorem.

*Proof.* Let's assume first that $\gamma$ is a simple closed contour and let $V$ be the bounded domain enclosed by $\gamma$. Let $f(x+iy) = u(x,y) + iv(x,y)$. Assume that $\gamma$ is positively oriented. By the usual change of variables $z = x+iy$ and by Green's theorem,

$$\begin{aligned}\oint_\gamma f(z)dz &= \oint_\gamma f(x+iy)dx + if(x+iy)dy \\ &= \iint_V if_x - f_y dxdy = \iint_V (iu_x - v_x) - (u_y + iv_y)dxdy = 0,\end{aligned}$$

where the last equality follows from Cauchy-Riemann equations.

When $\gamma$ is negatively oriented (clockwise), we can replace it by $\gamma^-$ and the minus sign will not change the value zero. When $\gamma$ is not a simple curve, there is a way to partition $\gamma$ into a finite number of components consisting of simple closed curves and a degenerate closed curves, i.e. those which enclose a region of zero area. Taking the sum of the integrals over each of these components, we will still obtain 0. □

*Remark.* Simply connectedness is an essential criterion. If the domain $U$ is multiply connected, there is always a contour $\gamma$ such that the domain $V$ enclosed by $U$ is not contained in $U$ and thus the integral of $f$ over the region $V \backslash U$ will not make sense. Example 20 shows that this theorem fails when the domain is multiply connected.

Combining this theorem with Theorem 3.6, we obtain the following.

**Corollary 3.8.** *Any holomorphic function on a simply connected domain has a primitive.*

**Theorem 3.9** (Deformation Theorem)**.** *Let $f$ be holomorphic on a domain $U \subset \mathbb{C}$, and let $\gamma$ and $\sigma$ be two simple closed contours such that $\sigma$ lies in the domain $V$ enclosed by $\gamma$ and that both contours have the same orientation. If the annulus $A$ enclosed by $\gamma$ and $\sigma$ is contained in $U$, then*

$$\oint_\gamma f(z)dz = \oint_\sigma f(z)dz.$$

*Proof.* Pick a simple contour $\alpha$ in $A$ joining a point on the outer boundary to another on the inner boundary. (See Figure 3.4.) Removing $\alpha$ from $A$ gives us a region $A'$ and its boundary can be parametrized by the closed contour $\Gamma$ obtained by gluing in order the curves $\gamma$, $\alpha$, $\sigma^-$, and $\alpha^-$. Then, by Cauchy-Goursat,

$$\begin{aligned}\oint_\gamma f(z)dz - \oint_\sigma f(z)dz &= \oint_\gamma f(z)dz + \oint_\alpha f(z)dz + \oint_{\sigma^-} f(z)dz + \oint_{\alpha^-} f(z)dz \\ &= \oint_\Gamma f(z)dz = 0.\end{aligned}$$

□

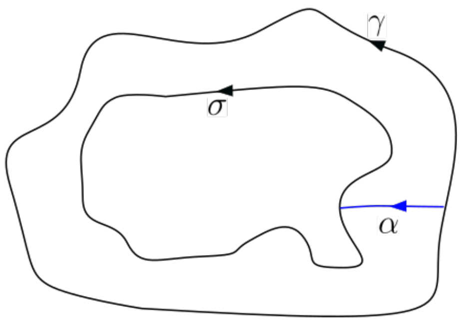



Figure 3.4: Curve $\alpha$ joining $\gamma$ and $\sigma$.

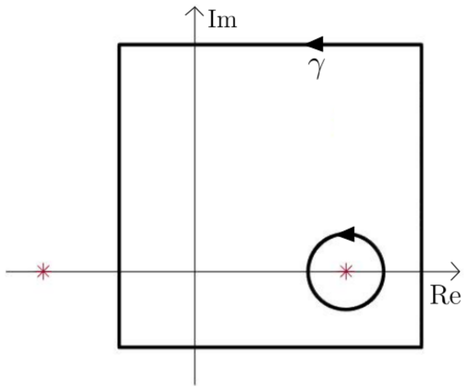



Figure 3.5: The square $\gamma$ can be replaced by the smaller circle.

**Example 24.** Let's compute the integral $I$ of $(z^2-4)^{-1}$ along $\gamma$, a simple closed contour parametrising the square of side length 4 centered at $1+i$. The partial fraction decomposition of $I$ is

$$I = \oint_\gamma \frac{1}{z^2-4}dz = \frac{1}{4}\oint_\gamma \frac{1}{z-2}dz - \frac{1}{4}\oint_\gamma \frac{1}{z+2}dz$$

Since the function $1/(z+2)$ is holomorphic on and inside $\gamma$, the second integral is zero by Cauchy-Goursat. The function $\frac{1}{z-2}$ is holomorphic everywhere except at 2, so then by deformation theorem, we can replace $\gamma$ on the first integral with any circle centered at 2, e.g. $C(2, 0.5)$. (See Figure 3.5.) Example 20 immediately tells us that the first integral is equal to $\frac{1}{4} \cdot 2\pi i$. Therefore, $I = \frac{1}{2}\pi i$.

In fact, the deformation theorem can be applied to any pair of arbitrary closed contours in $U$ which are *homotopic*. This means that one curve can be continuously deformed in $U$ to the other curve. If the two contours are not closed, then they must be *homotopic relative to their endpoints*, i.e. we need the additional assumption that both have the same endpoints. (Lemma 3.4 hints at why fixing endpoints are important.) Below is a simple pictorial explanation of homotopic curves.

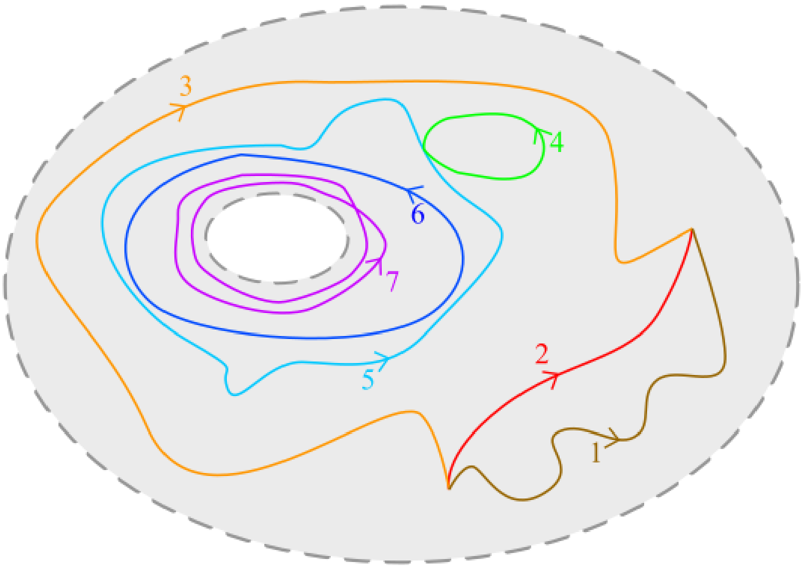



Figure 3.6: On the gray annular domain above, curves 1, 2 and 3 have the same endpoints but only 1 and 2 are homotopic relative to their endpoints. Among closed curves 4 - 7, the only pair of homotopic curves is 5 and 6.

**Short Quiz 3**

1. What is the integral of $1/z$ along the circle $C(0, 2)$?
2. Compute the length of the curve $\gamma(t) = cos(t)e^(it)$ where $0 \leq t \leq \pi$.
3. Compute the contour integral of the function $8z^3$ along an L-shaped contour which starts from 1 to $2i$ and passes through 0.

Answers: 1. $2\pi i$, 2. $\pi$, 3. 30.

# Chapter 4

# Integration Formulas

## 4.1 Cauchy's Formulas

**Theorem 4.1** (Cauchy's Integral Formula). *Let $f : U \to V$ be a holomorphic function, $\gamma$ be a simple closed contour in $U$, and $W$ be the domain enclosed by $\gamma$ such that $U \subset W$. For any point $z_0$ in $W$,*

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} dz.$$

*Proof.* We can assume that $z_0 = 0$ without loss of generality because when $z_0 \neq 0$, we can replace $f$ with the function $f(z + z_0)$ on the domain $\{z \in \mathbb{C} : z + z_0 \in U\}$ and the contour $\gamma(t)$ with $\gamma(t) - z_0$.

By the deformation theorem, we can replace $\gamma$ with $\gamma_r$, a contour parametrizing the circle $C(0, r)$ for arbitrarily small radius $r > 0$. Recall that

$$\frac{1}{2\pi i} \oint_{\gamma_r} \frac{1}{z} dz = 1.$$

Then, by taking the limit as $r \to 0$,

$$\begin{aligned}
\left| \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z} dz - f(0) \right| &= \left| \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(z)}{z} dz - f(0) \right| \\
&= \left| \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(z) - f(0)}{z} dz \right| \\
&\leq \frac{1}{2\pi} \cdot L(\gamma_r) \cdot \max_{z \in \gamma_r} \left| \frac{f(z) - f(0)}{z} \right| \\
&= r \cdot \max_{z \in \gamma_r} \left| \frac{f(z) - f(0)}{z} \right| \longrightarrow 0 \cdot f'(0) = 0.
\end{aligned}$$

As the term of the left hand side is independent of $r$, it is identically 0. $\square$

The case where the closed contour is chosen to be a circle yields an interesting property of holomorphic functions.

**Corollary 4.2** (Mean Value Property). *Let $f$ be holomorphic on a domain $U$. For any closed disk $\overline{\mathbb{D}(z_0, r)}$ in $U$,*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt.$$

*Proof.* Let $\gamma(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$ parametrize the circle $C(z_0, r)$. Since $f$ is holomorphic on a simply connected open neighbourhood of $\mathbb{D}(z_0, r)$, by Cauchy's integral formula,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} dz \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ire^{it} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt. \end{aligned}$$ □

The reason why the corollary above deserves its name is clear if you view the integral as the average value of $f$ along the circle $\gamma$ centered at $z_0$. Recall that the length element $ds$ in polar coordinates is $ds^2 = dr^2 + r^2 dt^2$. When $r$ is constant, $ds = \epsilon \, dt$ and the equation can be rewritten as:

$$f(z_0) = \frac{1}{L(\gamma)} \int_\gamma f(z) ds.$$

**Example 25.** Let's evaluate the integral $I$ along $\gamma$ parametrizing $C(0, 2)$, given by

$$I = \oint_\gamma \frac{e^z}{z^2 - 1} dz.$$

By partial fractions decomposition, we can split $I$ into $I_1 + I_2$ where

$$I_1 = \frac{1}{2} \oint_\gamma \frac{e^z}{z - 1} dz, \qquad I_2 = \frac{1}{2} \oint_\gamma \frac{e^z}{z + 1} dz.$$

By Cauchy's formula, we immediately obtain $I_1 = \pi i e$ and $I_2 = \pi i e^{-1}$. Thus, $I = 2\pi i \cosh(1)$.

**Theorem 4.3** (Cauchy's Differentiation Formula). *Let $f : U \to V$ be a holomorphic function, $\gamma$ be a simple closed contour in $U$, and $W$ be the domain enclosed by $\gamma$ such that $U \subset W$. Then, the $n^{th}$ derivative $f^{(n)}(z_0)$ of $f$ at a point $z_0$ in $W$ satisfies the following formula:*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

*Proof.* We can again assume without loss of generality that $z_0 = 0$. The base case where $n = 0$ is exactly the previous theorem. Suppose the formula holds for some natural number $n$. Then, for some small non-zero $a$,

$$\begin{aligned}\frac{f^{(n)}(a) - f^{(n)}(0)}{a} &= \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{a(z-a)^{n+1}}dz - \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{az^{n+1}}dz \\ &= \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)\cdot[z^{n+1} - (z-a)^{n+1}]}{az^{n+1}(z-a)^{n+1}}dz\end{aligned}$$

Using the algebraic identity $A^{n+1} - B^{n+1} = (A-B)\sum_{k=0}^{n} A^k B^{n-k}$, the term inside the square brackets simplifies to

$$a \cdot \sum_{k=0}^{n} z^k (z-a)^{n-k}.$$

Taking the limit as $a \to 0$, this simplifies to the desired formula:

$$\begin{aligned}f^{(n+1)}(0) &= \lim_{a\to 0} \frac{f^{(n)}(a) - f^{(n)}(0)}{a} \\ &= \lim_{a\to 0} \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)\sum_{k=0}^{n} z^k (z-a)^{n-k}}{z^{n+1}(z-a)^{n+1}}dz \\ &= \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)(n+1)z^n}{z^{2n+2}}dz \\ &= \frac{(n+1)!}{2\pi i}\oint_\gamma \frac{f(z)}{z^{n+2}}dz.\end{aligned}$$

By induction over $n$, the formula works for all $n$. $\square$

**Example 26.** We know that the function $f(z) = \frac{\sin z}{z^3}$ is holomorphic on $\mathbb{C}^*$. Let $\gamma$ be the unit circle $C(0,1)$. By Cauchy's differentiation formula,

$$\oint_\gamma f(z)dz = \pi i \cdot \frac{2!}{2\pi i}\oint_\gamma \frac{\sin z}{z^3}dz = \pi i \frac{d^2 \sin}{dz^2}(0) = -\pi i \sin 0 = 0.$$

## 4.2 Applications of Cauchy's Formulas

Cauchy's formulas have many implications and we shall state a number of them below. For each result presented, we shall see how holomorphic functions are much more rigid compared to real differentiable functions in general.

**Corollary 4.4.** *Every holomorphic function is infinitely complex differentiable and all of its derivatives $f', f'', \ldots$ are holomorphic.*

*Proof.* Let $f : U \to V$ be a holomorphic function. For every point $z \in U$, we can pick a small radius $\epsilon > 0$ such that the closed disk $\overline{\mathbb{D}(z, \epsilon)}$ lies in $U$. Take $\gamma$ to be the circle $C(z, \epsilon)$ and obtain $f^{(n)}(z)$ for every $n \in \mathbb{N}$ by Cauchy's differentiation formula. This proves that derivatives of all orders exist. The existence of $f^{(n+1)}$ automatically implies that $f^{(n)}$ is holomorphic on $U$. □

*Remark.* Recall that holomorphic functions are often called *analytic*. The term "*analytic*" refers to functions (real or complex) which are infinitely (real or complex) differentiable. There are plenty of examples of real differentiable functions which are not analytic (e.g. the indefinite integral of any continuous non-differentiable function).

Below is yet another important criterion of holomorphicity.

**Theorem 4.5** (Morera)**.** *Let $f$ be a continuous function on a domain $U$. If*

$$\oint_\gamma f(z)dz = 0$$

*for every closed contour $\gamma$ in $U$, then $f$ is holomorphic.*

*Proof.* By Theorem 3.6, the vanishing integral assumption guarantees the existence of a primitive $F$ of $f$ on $U$. By the previous corollary, the derivative of $F$, which is $f$, is holomorphic on $U$. □

**Corollary 4.6** (Cauchy's Inequality)**.** *Let $f$ be a holomorphic function on a domain $U$ and let $\overline{\mathbb{D}(z_0, r)}$ be a closed disk contained in $U$. Then*

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \max_{z \in C(z_0,r)} |f(z)|.$$

*Proof.* Apply the ML inequality to Cauchy's differentiation formula, taking $\gamma$ to be the circle $C(z_0, r)$. □

Cauchy's inequality itself has many important implications.

**Definition 12.** A function $f : U \to V$ is *bounded* if there is some $M > 0$ such that $|f(z)| \leq M$ for all $z \in U$.

**Theorem 4.7** (Liouville)**.** *Every bounded entire function is constant.*

*Proof.* Suppose $f$ is entire and bounded. There is some $M > 0$ where $|f(z)| \leq M$ for all $z \in \mathbb{C}$. The complex plane $\mathbb{C}$ contains closed disks $\mathbb{D}(z, r)$ centered at any point $z$ of arbitrarily large radius $r > 0$. By Cauchy's inequality,

$$|f'(z)| \leq \frac{M}{r} \to 0, \qquad \text{as } r \to \infty.$$

Since $|f'(z)|$ is independent of $r$, $f'$ is identically zero in $\mathbb{C}$. By Corollary 3.5, $f$ must be constant. □

**Example 27.** All polynomials of degree $\geq 1$, exp, sin, cos, sinh, and cosh are all unbounded.

*Remark.* There is no such analogue of Liouville's theorem for real functions. For example, the functions tanh and $\frac{1}{x^2+1}$ are bounded and infinitely differentiable (i.e. real analytic) in the whole $\mathbb{R}$.

One consequence of Liouville's theorem is the following.

**Corollary 4.8.** *The image of a non-constant entire function is dense in $\mathbb{C}$, i.e. it intersects every open disk in the plane.*

*Proof.* Suppose for a contradiction that the image avoids some open disk $\mathbb{D}(z_0, r)$, i.e. $|f(z) - z_0| \geq r$ for all $z \in \mathbb{C}$. Then, $\frac{1}{f(z)-z_0}$ is an entire function whose modulus is bounded above by $r$. By Liouville's theorem, this fraction must be a constant function, but then this raises a contradiction because $f$ is not constant. $\square$

Liouville's theorem also gives us a standard proof of the fundamental theorem of algebra.

**Theorem 4.9** (Fundamental Theorem of Algebra)**.** *Every complex polynomial $p(z)$ of degree $d \geq 1$ has exactly $d$ roots (counting multiplicity).*

*Proof.* Let $p(z) = \sum_{n=0}^{d} a_n z^n$ where $a_d \neq 0$ and suppose for a contradiction that $p$ has no roots. Then, $1/p(z)$ is an entire function. As $|z| \to \infty$,

$$\lim_{|z|\to\infty} \left|\frac{1}{p(z)}\right| = \lim_{|z|\to\infty} \frac{1}{|z^d|} \frac{1}{|a_d + \frac{a_{d-1}}{z} + \ldots + \frac{a_0}{z^d}|} = 0 \cdot \frac{1}{|a_d|} = 0.$$

Let's pick any small $\epsilon > 0$. By the definition of continuity, there is some $R > 0$ such that $\left|\frac{1}{p(z)}\right| \leq \epsilon$ whenever $|z| > R$. Since the closed disk $\overline{\mathbb{D}(0, R)}$ is a compact disk, the maximum value

$$M := \max_{|z|\leq R} \left|\frac{1}{p(z)}\right|$$

exists and is finite. Clearly, $1/p(z)$ is bounded because $\left|\frac{1}{p(z)}\right| \leq \max\{\epsilon, M\}$. By Liouville's theorem, $1/p$ is constant, but this contradicts the fact that $p(z)$ is a non-constant polynomial.

Therefore, $p$ has some root $z_1 \in \mathbb{C}$. This allows us to express $p$ as a product $p(z) = (z - z_1)q_1(z)$ for some polynomial $q_1(z)$ of degree $d - 1$. By the same reasoning, $q_1$ has a root $z_2 \in \mathbb{C}$ and $q_1(z) = (z - z_2)q_2(z)$ for some polynomial $q_2(z)$ of degree $d - 2$. Inductively, we can find $d$ roots of $p(z)$, namely $z_1$, $z_2$, $\ldots z_d$ (may be repeated). $\square$

Above is one of the many ways for us to prove the theorem. We will present a much shorter one in section 5.5.

## 4.3 Maximum Modulus Principle

**Lemma 4.10.** *Let $f$ be a holomorphic function on a domain $U$. If the modulus $|f(z)|$ is a constant function on $U$, then $f$ is constant too.*

*Proof.* Suppose $|f(z)| = c$ for all $z \in U$. If $c = 0$, then $f$ is identically zero. Let's assume $c > 0$. Writing $f$ as $f(x+iy) = u(x,y) + iv(x,y)$, observe that $u^2 + v^2 = c^2$. Taking the partial derivatives with respect to $x$ and $y$ respectively, we obtain:

$$uu_x + vv_x = 0, \qquad uu_y + vv_y = 0.$$

By Cauchy Riemann equations,

$$\begin{aligned} c^2(u_x^2 + u_y^2) = (u^2+v^2)(u_x^2+u_y^2) &= (uu_x - vu_y)^2 + (uu_y + vu_x)^2 \\ &= (uu_x + vv_x)^2 + (uu_y + vv_y)^2 = 0 \end{aligned}$$

As $c^2 \neq 0$, then $u_x = v_y = 0$ and $u_y = -v_x = 0$. Therefore, $u$ and $v$ are constant. □

**Lemma 4.11** (Maximum Modulus Principle - Local Version)**.** *Let $f$ be a holomorphic function on an open disk $\mathbb{D}(z_0, R)$. If $|f|$ attains a maximum at $z_0$, i.e. $|f(z)| \leq |f(z_0)|$ for all $z$, then $f$ is constant.*

*Proof.* Suppose for a contradiction that $|f(z)|$ is not constant. There must be some point $z$ in $\mathbb{D}(z_0, R)$ such that $|f(z)| < |f(z_0)|$. Let $re^{i\theta}$ be the polar form of $z - z_0$. By continuity of $|f(z)|$, there is a small $\delta > 0$ such that $|f(z_0 + re^{it})| < |f(z_0)|$ whenever $\theta - \delta < t < \theta + \delta$. Thus,

$$\int_{\theta-\delta}^{\theta+\delta} |f(z_0 + re^{it})| dt < \int_{\theta-\delta}^{\theta+\delta} |f(z_0)| dt = 2\delta |f(z_0)|. \tag{4.1}$$

The subset $I = [0, 2\pi] \backslash (\theta - \delta, \theta + \delta)$ has length $2\pi - 2\delta$. By ML inequality,

$$\int_I |f(z_0 + re^{it})| dt \leq (2\pi - 2\delta)|f(z_0)|.$$

Let $\gamma$ be the circle $C(z_0, r)$. By the mean value property,

$$\begin{aligned} |f(z_0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt \right| &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})| dt \\ &= \frac{1}{2\pi} \left[ \int_I |f(z_0 + re^{it})| dt + \int_{\theta-\delta}^{\theta+\delta} |f(z_0 + re^{it})| dt \right] \\ &< \frac{1}{2\pi} \big[ (2\pi - 2\delta)|f(z_0)| + 2\delta |f(z_0)| \big] = |f(z_0)|, \end{aligned}$$

where the strict inequality comes from (4.1). The above statement raises a contradiction. Thus, $|f|$ is constant. By the previous lemma, so is $f$. □

**Theorem 4.12** (Maximum Modulus Principle - Global Version)**.** *If $f$ be a non-constant holomorphic function on a domain $U$. Then,*

- *$|f|$ cannot attain a maximum on $U$.*
- *if additionally $U$ is bounded and $f$ is continuous along the boundary $\partial U$ of $U$, then $f$ attains a maximum at some point in $\partial U$.*

*Proof.* Suppose for a contradiction that $f$ attains a maximum at some point $z_0$ in $U$. Pick any point $w \in U$. Since $U$ is connected, we can always find some integer $N > 0$ and some finite sequence[1] of open disks $D_n := \mathbb{D}(z_n, r_n)$ in $U$ for $n = 0, \dots N$ such that $z_N = w$ and $z_{n+1} \in D_n$ for all $n < N$.

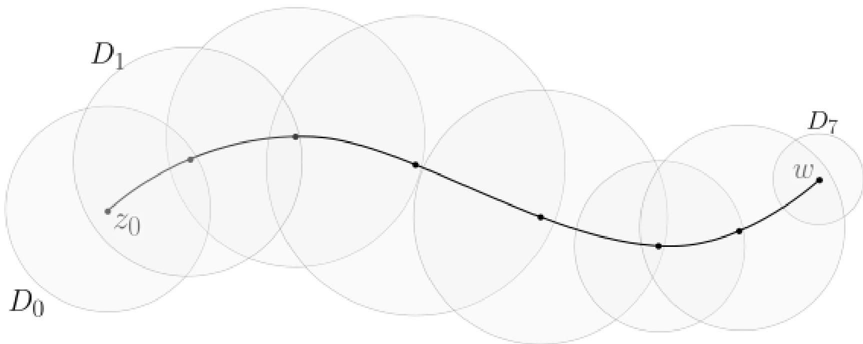



Figure 4.1: A chain of 8 disks.

By the local version of the maximum principle, $f \equiv f(z_0)$ in $D_0$. Since the center $z_1$ of $D_1$ lies in $D_0$, then the same lemma guarantees that $f \equiv f(z_0)$ in $D_1$. Inductively, if $f \equiv f(z_0)$ in $D_n$ for every $n$, and in particular, $f(w) = f(z_0)$. As the point $w$ picked is arbitrary, $f$ must be a constant function equal to $f(z_0)$ on $U$. This proves the first part of the theorem.

Suppose $f$ is non-constant and $U$ is a bounded domain. The union $U \cup \partial U$ of $U$ and its boundary $\partial U$ is a compact subset. By Theorem 2.6, $|f(z)|$ must attain a maximum at some point $z_0 \in U \cup \partial U$. We conclude that $z_0 \in \partial U$ because the first part of the theorem states that $z_0 \notin U$. □

There are plenty of examples of real differentiable functions which violate the maximum modulus principle. One of such is the function $\frac{1}{x^2+1}$ which has a global maximum at 0.

**Corollary 4.13** (Minimum Modulus Principle)**.** *Let $f$ be a non-constant holomorphic function on a domain $U$. Then,*

[1]This chain of disks is commonly known as *kreisketten* in German.

- *$f$ cannot attain a minimum on $U$ except at its zeros,*
- *if additionally $U$ is bounded, $f$ extends continuously to the boundary $\partial U$ and $f(z) \neq 0$ for all $z \in U$, then $f$ attains a minimum at some point in $\partial U$.*

*Proof.* If $f(z) \neq 0$ for all $z \in U$, then $1/f$ is a non-constant holomorphic function on $U$. Apply the maximum modulus principle to $1/f$. □

**Example 28.** Let's find the maximum and minimum of $\sin z$ on the closed square $S$ with vertices $\pi i$, $\pi + \pi i$, $\pi + 2\pi i$ and $2\pi i$. Recall from 2.3 that $|\sin z|^2 = \sin^2 x + \sinh^2 y$. Observe the following.

- If $0 \leq x \leq \pi$, $\sin^2 x$ achieves a maximum value of 1 at $x = \frac{\pi}{2}$ and a minimum value of 0 at $x = 0, \pi$.
- $\sinh^2 y$ is monotonically increasing on $\pi \leq y \leq 2\pi$.

Therefore, on $S$, $|\sin z|$ achieves a maximum value of $\sqrt{1 + \sinh^2 2\pi}$ at $z = \frac{\pi}{2} + 2\pi i$ and a minimum value of $\sinh \pi$ at $z = \pi i, \pi + \pi i$. These extremal values are achieved along the boundary of $S$.

**Example 29.** Let's find the maximum and minimum of $|f|$, where $f(z) = z^2 - 7z + 12$, on the closed unit disk $\bar{\mathbb{D}}$. Since $f$ only attains 0 outside of $\bar{\mathbb{D}}$ at 3 and 4, it is sufficient to check maxima and minima along the boundary, which is the unit circle. When $|z| = 1$,

$$\begin{aligned} |f(z)| &= |z-3||z-4| \leq (|z|+3)(|z|+4) = 20, \\ |f(z)| &= |z-3||z-4| \geq (3-|z|)(4-|z|) = 6. \end{aligned}$$

The triangle inequalities used above achieve equality at $z = -1$ and $z = 1$. Thus, the extremal values of $|f(z)|$ on $\bar{\mathbb{D}}$ are 20 and 6.

**Short Quiz 4**

1. What is the integral of $\frac{\sin z}{z-\pi}$ along the circle $C(0,5)$?
2. What is the integral of $\frac{2z^5}{(2z-1)^3}$ along the circle $C(0,5)$?
3. Which of these functions are bounded on $\mathbb{C}$?

$$|z|, \quad \frac{z}{|z|+1}, \quad \frac{1}{z-1}, \quad \frac{\cos(\pi/2 - z)}{\sin z}, \quad \sin(z^2).$$

4. How many roots does $(z^2+1)^2+1$ have?
5. Which of these functions are entire with bounded derivative?

$$iz, \quad iz^2, \quad e^{iz}, \quad \sin iz$$

6. Locate the extrema points of $|\cos z|$ on the square $\{x+iy|0 \leq x, y \leq \pi\}$.

Answers: 1. 0, 2. $5\pi i/8$, 3. The second and the fourth. 4. 4, 5. $iz$. 6. $\frac{\pi}{2}, \pi i, \pi + \pi i$.

# Chapter 5

# Series, Zeros, and Poles

## 5.1 Taylor Series

**Example 30.** The function $\frac{1}{z-1}$ is holomorphic on the unit disk $\mathbb{D}$. We can always this function in terms of a complex power series. Specifically, for $z \in \mathbb{D}$,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n.$$

Indeed, basic arithmetic results in sequences and series tell us that the partial sums can be expressed as

$$\sum_{n=0}^{N} z^n = \frac{1-z^{N+1}}{1-z}.$$

Since $|z| < 1$, $|z^{N+1}| \to 0$ and clearly $z^{N+1} \to 0$ too, as $N \to \infty$. Therefore, the partial sums converge to $\frac{1}{1-z}$ as $N \to \infty$.

**Example 31.** The principal branch of the logarithm $\mathrm{Log}(1-z)$ is holomorphic on $\mathbb{D}$ as well. As this is a primitive of $\frac{1}{z-1}$, we can obtain a power series for this function by integrating the power series for $\frac{1}{z-1}$ obtained. It is given by:

$$\mathrm{Log}(1-z) = -\sum_{n=0}^{\infty} \frac{z^n}{n}$$

Suppose a holomorphic function $f(z)$ on a domain $U$ coincides with some power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ about some point $z_0$ in $U$. Differentiating both functions at $z_0$ $n$ times, we find that the coefficients $a_n$ are unique as they must satisfy $f^{(n)}(z_0) = n!a_n$. Therefore, this power series must also be unique.

**Theorem 5.1** (Taylor's Theorem)**.** *If $f$ is a holomorphic function on an open disk $\mathbb{D}(z_0, r)$, then for all $z \in \mathbb{D}(z_0, r)$,*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad \textit{where } a_n = \frac{f^{(n)}(z_0)}{n!}.$$

*Proof.* We can always assume that $z_0 = 0$ without loss of generality because when $z_0 \neq 0$, we can replace the function $f$ with $f(z + z_0)$ where $z \in \mathbb{D}(0, r)$.

Pick any point $z \in \mathbb{D}(0, r)$. Let $\gamma$ be a circle $C(0, s)$ centered at 0 of some radius $s$ such that $|z| < s < r$. This curve $\gamma$ separates the point $z$ from the boundary of the disk $\mathbb{D}(0, r)$. For any point $w$ along this curve, since $|\frac{z}{w}| = \frac{|z|}{s} < 1$, we have the following identity:

$$\frac{1}{w - z} = \frac{1}{w} \cdot \frac{1}{1 - \frac{z}{w}} = \frac{1}{w} \sum_{n=0}^{\infty} \left(\frac{z}{w}\right)^n.$$

By Cauchy's formula and the above identity,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - z} dw \\ &= \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w} \sum_{n=0}^{\infty} \left(\frac{z}{w}\right)^n dw \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w^{n+1}} dw\right] z^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} a_n z^n. \end{aligned}$$ □

In particular, Taylor's theorem tells us that if we know the derivatives of $f$ of every order at a point $z_0$, then we know $f(z)$ for every $z$ $z_0$.

**Corollary 5.2.** *Let $f$ be an entire function and $z_0$ be any point in $\mathbb{C}$. Then, for any $z \in \mathbb{C}$,*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad \textit{where } a_n = \frac{f^{(n)}(z_0)}{n!}.$$

**Definition 13.** Given a holomorphic function $f$ on a domain $U$, The *Taylor series* for $f$ about a point $z_0 \in U$ is the infinite sum

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

When $z_0 = 0$, the expansion is often called the *MacLaurin series* for $f$. The *radius of convergence* of the Taylor series is the largest possible radius $R > 0$ such that the series converges for $|z - z_0| < r$. If there is no such maximum value (e.g. when $f$ is entire), we say that $R = \infty$.

**Example 32.** Examples 30 and 31 show the MacLaurin series of $\frac{1}{1-z}$ and $\operatorname{Log}(1-z)$. Both series cannot be extended beyond the unit disk since immediately at $z = 1$, both functions are not well-defined. Therefore, both have radius of convergence 1.

**Example 33.** The exponential function $e^{z-z_0}$ is an entire function. Since its $n^{\text{th}}$ derivative at $z_0$ is 1 for all $n$, it has a Taylor series about $z_0$ with infinite radius of convergence given below:

$$e^{z-z_0} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!}.$$

Using this series, we can also derive the corresponding series for the functions sin, cos, sinh and cosh.

**Example 34.** Let us compute the Taylor series for $f(z) = \frac{1}{2i-z}$ about 2.

$$\begin{aligned} \frac{1}{2i-z} &= \frac{1}{(2i-2)-(z-2)} = \frac{1}{2i-2} \cdot \frac{1}{1-\frac{z-2}{2i-2}} \\ &= \frac{1}{2i-2} \sum_{n=0}^{\infty} \left( \frac{z-2}{2i-2} \right)^n = \sum_{n=0}^{\infty} \frac{(z-2)^n}{(2i-2)^{n+1}} \\ &= \sum_{n=0}^{\infty} \left( -\frac{1+i}{4} \right)^{n+1} (z-2)^n. \end{aligned}$$

The radius of convergence is $2\sqrt{2}$ because the series converges when $|\frac{z-2}{2i-2}| < 1$, or equivalently, $|z-2| < 2\sqrt{2}$.

## 5.2 Zeros

**Definition 14.** Suppose $f$ is a holomorphic function on a domain $U$. We say that $f$ has a *zero* at a point $z_0 \in U$ order $k$ if $f^{(k)}(z_0) \neq 0$ and $f^{(n)}(z_0) = 0$ for all $n < k$. If $k = 1$, we say that the zero is *simple*.

**Proposition 5.3.** *Let $f$ be a holomorphic function on a domain $U$ and let $z_0 \in U$. The following are equivalent.*

*(a) $f$ has a zero at $z_0$ of order $k > 0$,*

*(b) the Taylor series of $f$ about $z_0$ is of the form $\sum_{n=k}^{\infty} a_n(z-z_0)^n$, $a_k \neq 0$,*

*(c) there is some holomorphic function $g$ on $U$ such that $g(z_0) \neq 0$ and for each $z \in U$, $f(z) = (z-z_0)^k g(z)$.*

*Proof.* $(a) \iff (b)$ follows immediately from Taylor's theorem, and $(c) \Rightarrow (b)$ follows from direct computation of derivatives using Leibniz's rule.

Assume $(b)$ holds. The function $g(z) = \frac{f(z)}{(z-z_0)^k}$ is holomorphic on $U \backslash \{z_0\}$. About $z_0$, the Taylor series of $f$ gives us a well-defined power series representation of $g$:

$$g(z) = \frac{\sum_{n=k}^{\infty} a_n(z-z_0)^n}{(z-z_0)^k} = \sum_{n=0}^{\infty} a_{n+k}(z-z_0)^n.$$

Hence, $g$ is holomorphic at $z_0$ and $g(0) = a_k \neq 0$. We then have $(b) \Rightarrow (c)$. □

**Lemma 5.4.** *Let $f$ be holomorphic on some disk $\mathbb{D}(z_0, r)$ and let $\{z_n\}_{n\in\mathbb{N}}$ be an infinite sequence of distinct zeros of $f$ such that $z_n \to z_0$. Then, $f \equiv 0$.*

*Proof.* By continuity of $f$, $f(z_0) = 0$. Let $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ be the Taylor series of $f$ about $z_0$. Assume for a contradiction that $f \not\equiv 0$, then there must be some non-zero coefficient within the Taylor series. Let $k > 1$ be the smallest number such that $a_k \neq 0$, then $f$ has a zero of order $k$ at $z_0$.

Let $g$ be a holomorphic function in Proposition 5.3. By continuity of $g$, since $g_(z_0) \neq 0$, there is a small $\delta > 0$ such that $g(z) \neq 0$ for $z \in \mathbb{D}(z_0, \delta)$. However, since $z_n \in \mathbb{D}(z_0, \delta)$ for sufficiently high $n$,

$$0 = f(z_n) = g(z_n)(z - z_n)^k \neq 0.$$

This is a contradiction. □

**Theorem 5.5** (Identity Theorem / Coincidence Principle)**.** *Let $f$ and $g$ be holomorphic functions on a domain $U$ and let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of distinct points in $U$ such that $z_n \to z_0 \in U$ and $f(z_n) = g(z_n)$ for all $N$. Then, $f \equiv g$.*

*Proof.* Define the holomorphic function $h = f - g$. To prove the theorem, it is sufficient to show that $h \equiv 0$ in $U$. Pick any point $w \in U$. Since $U$ is connected, there is some finite sequence of open disks $D_n := \mathbb{D}(w_n, r_n)$ in $U$ for $n = 0, \dots N$ such that $w_0 = z_0$, $w_N = w$ and $w_{n+1} \in D_n$. (See Figure 4.1.)

The function $h$ has a zero at every $z_n$ By the previous lemma, $h \equiv 0$ on $D_0$. Since $D_0 \cap D_1$ contains $w_1$ as well as a sequence of points converging to $w_1$, the same lemma tells us that $h \equiv 0$ on $D_1$. Inductively, we conclude that $h \equiv 0$ for all $z \in D_N$ and in particular, $h(w) = 0$. As $w$ is arbitrary, $h \equiv 0$. $\square$

The theorem essentially says that a holomorphic function on some domain is completely determined by its values on a countable subset of the domain. Such property again hints at how rigid holomorphic functions are compared to real differentiable functions.

**Example 35.** The only holomorphic function $f(z)$ which has zeros on the set of rational numbers $\mathbb{Q}$ is the zero function.

**Example 36.** The only holomorphic function $f(z)$ satisfying $f(\frac{1}{n}) = \frac{1}{n}$ for all $n \in \mathbb{N}$ is the identity function $f(z) = z$.

## 5.3 Laurent Series

It is often useful to include terms with negative powers in a power series. This allows the possibility of expressing a holomorphic function with singularity at a single point as a power series.

**Example 37.** The function $g(z) = \frac{2-z}{z^2-z^3}$ has a singularity at 0. The power series about 0 for $0 < |z| < 1$ is

$$\begin{aligned} g(z) &= \frac{2}{z^2} + \frac{1}{z(1-z)} = \frac{2}{z^2} + \frac{1}{z}\sum_{n=0}^{\infty} z^n \\ &= 2z^{-2} + z^{-1} + 1 + z + z^2 + \dots . \end{aligned}$$

We say that the series above is the Laurent series of $g$ at 0.

**Theorem 5.6** (Laurent's Theorem)**.** *Let $f$ be a holomorphic function on an annular domain $A = \{z \in \mathbb{C} : r < |z - z_0| < R\}$ of inner and outer radii $r \in [0, \infty)$ and $R \in (0, \infty]$ centered at $z_0 \in \mathbb{C}$. For all $z \in A$,*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \qquad \text{where } a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

*for any simple closed contour $\gamma$ in $A$ enclosing the disk $\mathbb{D}(z_0, r)$.*

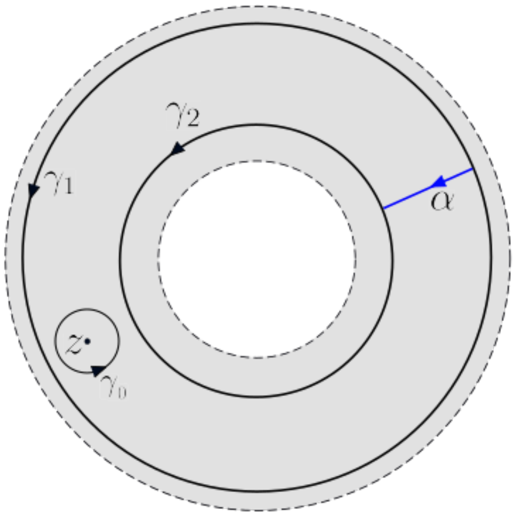



Figure 5.1: The curve $\alpha$ joining $\gamma_1$ and $\gamma_2$.

*Proof.* Let's assume again that $z_0 = 0$ because when $z_0 \neq 0$, we can replace the function $f$ with $f(z + z_0)$ where $r < |z| < R$. Pick any point $z \in A$ and positive numbers $s$ and $S$ such that $r < s < |z| < S < R$. Let $\gamma_1$ and $\gamma_2$ be simple closed curves parametrizing the circles $C(0, S)$ and $C(0, s)$ respectively.

If $w \in \gamma_1$, then $\left|\frac{z}{w}\right| < 1$ and thus,

$$\frac{1}{w-z} = \frac{1}{w} \cdot \frac{1}{1-\frac{z}{w}} = \frac{1}{w}\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n = \sum_{n=0}^{\infty}\frac{z^n}{w^{n+1}}. \tag{5.1}$$

However, if $w \in \gamma_2$, then $\left|\frac{w}{z}\right| < 1$ and thus,

$$\frac{1}{z-w} = \frac{1}{z} \cdot \frac{1}{1-\frac{w}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n = \sum_{m=-\infty}^{-1}\frac{z^m}{w^{m+1}}. \tag{5.2}$$

Pick an angle $\theta \in (-\pi, \pi]\backslash\{\mathrm{Arg}(z)\}$ and define a radial line segment $\alpha : [s, S] \to A$ where $\alpha(t) = te^{i\theta}$. (See Figure 5.1.) Let $\sigma$ be the closed curve obtained by concatenating $\gamma_1$, $\alpha^-$, $\gamma_2^-$ and $\alpha$ in order, and let $\gamma_0$ be any small circle centered at $z$ lying entirely between $\gamma_1$ and $\gamma_2$. Then, since $w \mapsto \frac{f(w)}{w-z}$ is holomorphic on the domain enclosed between $\sigma$ and $\gamma_0$, by Cauchy's formula and deformation theorem,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{\gamma_0} \frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\oint_{\sigma} \frac{f(w)}{w-z}dw \\ &= \frac{1}{2\pi i}\left(\oint_{\gamma_1} - \oint_{\alpha} - \oint_{\gamma_2} + \oint_{\alpha}\right)\frac{f(w)}{w-z}dw \\ &= \frac{1}{2\pi i}\oint_{\gamma_1} \frac{f(w)}{w-z}dw + \frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(w)}{z-w}dw. \end{aligned}$$

By (5.1) and (5.2), we can convert the two integrals into the desired series.

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{\gamma_1}\left(\sum_{n=0}^{\infty}\frac{z^n f(w)}{w^{n+1}}\right)dw + \frac{1}{2\pi i}\oint_{\gamma_2}\left(\sum_{m=-\infty}^{-1}\frac{z^m f(w)}{w^{m+1}}\right)dw \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{\gamma_1}\frac{f(w)}{w^{n+1}}dw\right)z^n + \sum_{m=-\infty}^{-1}\left(\frac{1}{2\pi i}\oint_{\gamma_2}\frac{f(w)}{w^{m+1}}dw\right)z^m \\ &= \sum_{n=0}^{\infty}a_n z^n + \sum_{m=-\infty}^{-1}a_m z^m = \sum_{n=-\infty}^{\infty}a_n z^n. \end{aligned}$$ □

**Definition 15.** The bi-infinite series in the theorem above is called the *Laurent series* of $f$ about the point $z_0$.

The Laurent series of a holomorphic function about a point is unique. This is because if it coincides with some series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$, then each $a_n$ necessarily satisfies the equation given in the theorem. Note that the $a_n$'s are independent of the choice of $\gamma$ by deformation theorem.

**Example 38.** The Laurent series of $\frac{1}{z-1}$ about 0 is

$$\frac{1}{z-1} = \frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=-\infty}^{-1} z^n.$$

This series converges when $|\frac{1}{z}| < 1$, or equivalently, when $1 < |z| < \infty$.

**Example 39.** Let's find a Laurent series for $f(z) = \frac{3z}{z^2+z-2}$ converging on the annulus $\{1 < |z+1| < 2\}$. By partial fractions, $f(z) = \frac{1}{z-1} + \frac{2}{z+2}$ has singularities at $-1$ and 2. The Taylor series of $\frac{1}{z-1}$ about $-1$ is

$$\frac{1}{z-1} = -\frac{1}{2}\cdot\frac{1}{1-\frac{z+1}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z+1}{2}\right)^n = -\sum_{n=0}^{\infty}2^{-n-1}(z+1)^n.$$

convergent when $|\frac{z+1}{2}| < 1$, i.e. in the domain $U_1 = \{z \,:\, |z+1| < 2\}$. The Laurent series of $\frac{2}{z+2}$ about $-1$ is

$$\frac{2}{z+2} = \frac{2}{z+1} \cdot \frac{1}{1+\frac{1}{z+1}} = \frac{2}{z+1} \sum_{n=0}^{\infty} \left(-\frac{1}{z+1}\right)^n = \sum_{n=-\infty}^{-1} 2(-1)^{n+1}(z+1)^n.$$

convergent when $|\frac{1}{z+1}| < 1$, i.e. in the domain $U_1 = \{z \mid 1 < |z+1|\}$. Combining the two series, we deduce that the Laurent series of $f$ about $-1$ is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z+1)^n, \text{ where } a_n = \begin{cases} -2^{-n-1}, & \text{if } n \geq 0, \\ 2(-1)^{n+1}, & \text{if } n < 0, \end{cases}$$

convergent in the annulus $U_1 \cap U_2 = \{z \mid 1 < |z+1| < 2\}$.

## 5.4 Singularities

We will use an asterisk on a disk to introduce a puncture at its centre:

$$\mathbb{D}(z_0, r)^* := \mathbb{D}(z_0, r) \backslash \{z_0\} = \{z \in \mathbb{C} \,:\, 0 < |z - z_0| < r\}.$$

**Definition 16.** A point $z_0$ is a *singularity* of a function $f$ if

- $f$ is not holomorphic at $z_0$, and
- for every $r > 0$, there exists a point in the punctured neighbourhood $\mathbb{D}(z_0, r)^*$ at which $f$ is holomorphic.

Additionally, if there is some $R > 0$ such that $f$ is holomorphic on some punctured neighbourhood $\mathbb{D}(z_0, r)^*$, then $z_0$ is an *isolated singularity* of $f$.

**Example 40.** The function $(z - z_0)^{-d}$ for any integer $d > 0$ has an isolated singularity at $z_0$. In fact, the singularities of every rational function $\frac{p(z)}{q(z)}$ are isolated since there are only finitely many of them.

**Example 41.** The function $\csc(2\pi/z)$ has singularities at 0 and $\frac{1}{k}$ for each $k \in \mathbb{Z}$. In particular, 0 is not isolated.

For the rest of the chapter, we will only focus on isolated singularities. When isolated, Laurent's theorem allows us to express the function in terms of a Laurent series on a punctured neighbourhood of the singularity.

**Definition 17.** Suppose the Laurent series of a holomorphic function $f$ about an isolated singularity $z_0$ is $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ valid in a for $0 < |z - z_0| < R$ for some radius $R > 0$. We say that $z_0$ is:

- an *essential singularity* if $a_n \neq 0$ for infinitely many integers $n < 0$,
- a *pole* of order $k$ if $a_{-k} \neq 0$ and $a_n = 0$ for all $n < -k$,
- a *removable singularity* if $a_n = 0$ for all $n < 0$.

We say that a pole is *simple* if it has order $k = 1$.

**Example 42.** $e^{1/z}$ has an essential singularity at 0 because for $z \in \mathbb{C}^*$,

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{6} \ldots$$

**Proposition 5.7.** *Let $f$ be a holomorphic function on a punctured domain $U \backslash \{z_0\}$. The following are equivalent:*

*(a) $f$ has a pole at $z_0$ of order $k$,*

*(b) there is a holomorphic function $g$ on $U$ such that $g(z_0) \neq 0$ and for all $z \neq z_0$,*

$$g(z) = (z - z_0)^k f(z).$$

*Proof.* Assume $(a)$ holds. Let the Laurent series of $f$ about $z_0$ be $\sum_{n=-k}^{\infty} a_n(z-z_0)^n$ where $a_{-k} \neq 0$. Define a holomorphic function $g$ on $U \backslash \{z_0\}$ by $g(z) = (z - z_0)^k f(z)$. Since the Laurent series of $g$ about $z_0$ is $\sum_{n=0}^{\infty} a_{n+k}(z - z_0)^n$, $g$ has a removable singularity at $z_0$. Setting $g(z_0) = a_{-k} \neq 0$, then $g$ is holomorphic at $z_0$ as well. Thus, $(a) \Rightarrow (b)$.

Assume $(b)$ holds. Let $\sum_{n=0}^{\infty} b_n(z - z_0)^n$ be the Taylor series of $g$ about $z_0$, then the Laurent series of $f$ about $z_0$ is

$$f(z) = \frac{g(z)}{(z - z_0)^k} = \sum_{n=-k}^{\infty} b_{n+k}(z - z_0)^n.$$

As $b_0 \neq 0$, $f$ has a pole of order $k$. This gives us $(b) \Rightarrow (a)$. □

**Definition 18.** A function $f$ is *meromorphic* on a domain $U$ if it is holomorphic except at some countably many number of poles.

When $z_0$ is a removable singularity of $f$, the Laurent series has no terms of negative powers and hence becomes a Taylor series. We can then remove the singularity and make $f$ holomorphic at $z_0$ by defining $f(z_0) = a_0$, where $a_0 = \lim_{z \to z_0} f(z)$ is the zeroth coefficient of the Taylor series.

**Example 43.** $\frac{\sin z}{z}$ has a removable singularity at 0 because for $z \in \mathbb{C}^*$,

$$\frac{\sin z}{z} = \frac{1}{z} \cdot \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

Since $\lim_{z \to 0} \frac{\sin z}{z} = 1$, we may set $\frac{\sin 0}{0} = 1$ so that it becomes an entire function.

**Theorem 5.8** (Riemann's Removable Singularity). *Suppose $U$ is a domain and $f$ is a holomorphic function on $U \backslash \{z_0\}$ with a singularity at $z_0$. The following are equivalent.*

*(a) $z_0$ is a removable singularity.*

*(b) $f$ is continuously extendable to $z_0$,*

*(c) $f$ is bounded on a small punctured disk $\mathbb{D}(z_0, r)^*$ centered at $z_0$.*

*Proof.* The implication $(a) \Rightarrow (b)$ is clear and $(b) \Rightarrow (c)$ follows from continuity at $z_0$. Suppose $(c)$ holds and assume without loss of generality that $z_0 = 0$. There is some upper bound $M > 0$ for $|f|$ on the punctured disk.

When $|z| \to 0$, $|zf(z)| \le |z|M \to 0$. As such, the function

$$g(z) = \begin{cases} z^2 f(z), & z \in \mathbb{D}(0, r)^*, \\ 0, & z = 0. \end{cases}$$

is continuous at 0. In fact, it is also holomorphic at 0 since

$$g'(0) = \lim_{z \to 0} \frac{z^2 f(z) - 0}{z} = \lim_{z \to 0} z f(z) = 0.$$

The Taylor series of $h$ will be of the form $b_2 z^2 + b_3 z^3 + \dots$. Since $f(z) = z^{-2} h(z)$ for any $z \in \mathbb{D}(0, r)^*$, the Laurent series for $f$ about 0 is $b_2 + b_3 z + \dots$ and it has no negative power terms. Therefore, 0 is a removable singularity. $\square$

## 5.5 Counting Zeros and Poles

Consider a closed curve $\gamma : [a, b] \to \mathbb{C}$ avoiding the origin 0. In polar coordinates, $\gamma(t) = r(t) e^{2\pi i \theta(t)}$ for some continuous functions $r(t)$ and $\theta(t)$ such that $r(a) = r(b)$ and $\theta(a) = \theta(b) \bmod 1$.

**Definition 19.** The *winding number* $W(\gamma)$ of $\gamma$ is the number of counterclockwise turns $\gamma$ makes around 0, i.e. $W(\gamma) = \theta(b) - \theta(a)$.

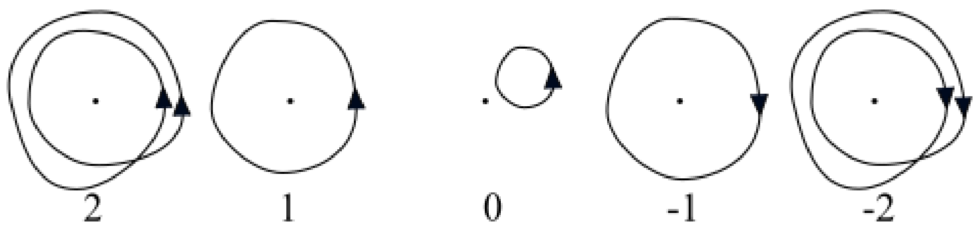



Figure 5.2: Winding number of various closed curves

**Lemma 5.9.** *Let $\gamma : [a, b] \to \mathbb{C}$ be a closed contour avoiding the origin* $0$. *Then,*

$$W(\gamma) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{z} dz.$$

*Proof.* Express $\gamma$ in terms of polar coordinates: $\gamma(t) = r(t)e^{2\pi i\theta(t)}$. Since $\gamma'(t) = r'(t)e^{2\pi i\theta(t)} + 2\pi i\theta(t)\gamma(t)$,

$$\begin{aligned}
\frac{1}{2\pi i} \oint_\gamma \frac{1}{z} dz &= \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)} dt \\
&= \frac{1}{2\pi i} \int_a^b \frac{r'(t)}{r(t)} + 2\pi i i\theta'(t) dt \\
&= \frac{1}{2\pi i} [\log r(b) - \log r(a)] + \theta(b) - \theta(a) \\
&= \theta(b) - \theta(a) = W(\theta).
\end{aligned}$$

□

**Example 44.** Let $\gamma(t) = e^{2\pi i t}$, $0 \leq t \leq 1$ parametrise the unit circle. The image of $\gamma$ under the power map $f(z) = z^n$, where $n \geq 1$, is $f \circ \gamma(t) = e^{2\pi i n t}$. The winding number of $f \circ \gamma$ is $n$, which coincides with the order of the zero of $f$ at $0$.

This observation is generalised by the argument principle.

**Theorem 5.10** (Argument Principle)**.** *Let $f$ be a meromorphic function on a simply connected domain $U$ and $\gamma$ be a simple closed contour in $U$ along which $f$ has no zeros or poles. Let $V$ be the domain enclosed by $\gamma$. Then,*

$$W(f \circ \gamma) = \frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} dz = Z - P,$$

*where $Z$ is the number of zeros of $f$ in $V$ counting multiplicities (i.e. each is counted as many times as its order), and $P$ is the number of poles of $f$ in $V$ counting multiplicities.*

*Proof.* Let the parametrization of $\gamma$ be $\gamma : [a, b] \to U$. From the previous lemma,

$$\begin{aligned} W(f \circ \gamma) &= \frac{1}{2\pi i} \oint_{f \circ \gamma} \frac{1}{z} dz = \frac{1}{2\pi i} \int_a^b \frac{(f \circ \gamma)'(t)}{f\big(\gamma(t)\big)} dt \\ &= \frac{1}{2\pi i} \int_a^b \frac{f'\big(\gamma(t)\big)}{f\big(\gamma(t)\big)} \gamma'(t) dt = \frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} dz \end{aligned}$$

Therefore, we have the first part of the equation. Suppose $\{z_1, \ldots z_m\}$ and $\{w_1, \ldots w_n\}$ are the sets of zeros and poles in $V$ respectively.

Pick any zero $z_j$ and let $k_j$ be its order. There is some meromorphic function $g_j$ such that $f(z) = (z - z_j)^{k_j} g_j(z)$ and $g_j$ is holomorphic at $z_j$ where $g_j(z_j) \neq 0$. Pick a small radius $\epsilon_j > 0$ so that inside the closed disk $\overline{\mathbb{D}(z_j, \epsilon_j)}$ $g_j$ have no poles nor zeros aside from $z_j$. Let $\gamma_j$ be the circle $C(z_1, \epsilon_j)$, then

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\gamma_j} \frac{f'(z)}{f(z)} dz &= \frac{1}{2\pi i} \oint_{\gamma_j} \frac{k_j (z - z_j)^{k_j - 1} g_j(z) + (z - z_j)^k g_j'(z)}{(z - z_j)^k g_j(z)} dz \\ &= \frac{k_j}{2\pi i} \oint_{\gamma_j} \frac{1}{z - z_j} dz + \frac{1}{2\pi i} \oint_{\gamma_j} \frac{g_j'(z)}{g_j(z)} dz = k_j, \end{aligned}$$

where the last equality follows from the fact that $g_j'(z)/g_j(z)$ is holomorphic on $\mathbb{D}(z_j, \epsilon_j)$ and Cauchy-Goursat.

Pick any pole $w_j$ and let $l_j$ be its order. There is some meromorphic function $h_j$ such that $f(z) = h_j(z)(z - w_j)^{-l_j}$ and $h_j$ is holomorphic at $w_j$ where $h_j(z_j) \neq 0$. Pick a small radius $\delta_j > 0$ so that inside the closed disk $\overline{\mathbb{D}(w_j, \delta_j)}$ $g$ have no poles nor zeros aside from $w_j$. Let $\sigma_j$ be the circle $C(w_j, \delta_j)$, then

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\sigma_j} \frac{f'(z)}{f(z)} dz &= \frac{1}{2\pi i} \oint_{\sigma_j} \frac{-l_j (z - z_j)^{l_j - 1} h_j(z) + (z - w_j)^{l_j} h_j'(z)}{(z - w_j)^{l_j} h_j(z)} dz \\ &= -\frac{l_j}{2\pi i} \oint_{\sigma_j} \frac{1}{z - w_j} dz + \frac{1}{2\pi i} \oint_{\sigma_j} \frac{h_j'(z)}{h_j(z)} dz = -l_j, \end{aligned}$$

where the last inequality follows from the fact that $h_j'(z)/h_j(z)$ is holomorphic on $\mathbb{D}(w_j, \delta_j)$.

The curve $\gamma$ can be split into some $m + n$ simple closed contours each of which encloses exactly one zero or pole. By deformation theorem, the integral along $\gamma$ is equal to the sum of the integrals along each of the contours $\gamma_j$ for $j = 1 \ldots m$ and $\sigma_j$ for $j = 1 \ldots n$. $\square$

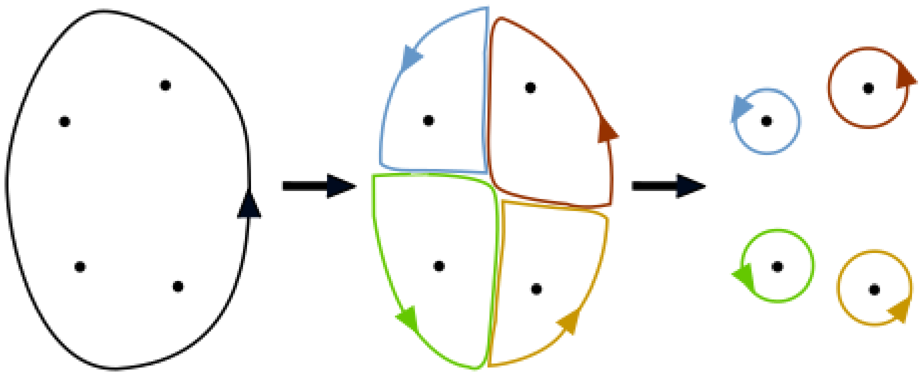


Figure 5.3: The contour $\gamma$ can be split into a collection of $m+n$ simple closed contours each of which encloses exactly one zero or pole and deformed into $\gamma_j$'s and $\sigma_j$'s.

The argument principle gives us two ways of computing the difference between the number of zeros and the number of poles inside some domain. One way is more geometric: by computing the winding number of the image of the boundary of the domain. The other is analytic: by computing a contour integral along the boundary. Both ways are equally useful.

**Example 45.** Let's compute the integral of $\sec z$ along a square $\gamma$ of side length 7 centered at 0 by using the fact that $\sec' z = \sec z \tan z$. There are four simple poles of sec enclosed by $\gamma$, namely $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$. sec has no zeros. By the argument principle,

$$\oint_\gamma \tan z \, dz = \oint_\gamma \frac{\sec z \tan z}{\sec z} dz = 2\pi i \cdot (-4) = -8\pi i.$$

**Theorem 5.11** (Rouché's Theorem)**.** *Let $f$ and $g$ be holomorphic functions on a domain $U$, $\gamma$ be a simple closed contour on $U$, and $V$ be the domain enclosed by $\gamma$ such that $V \subset U$. If $|g(z)| < |f(z)|$ for all $z$ along $\gamma$, then $f$ and $f+g$ have the same number of zeros, counting multiplicities, inside $V$.*

*Proof.* Define a meromorphic function $h(z) = \frac{g(z)}{f(z)} + 1$ on $U$. Along $\gamma$, $h$ is holomorphic and $|h(z) - 1| = \left|\frac{g(z)}{f(z)}\right| < 1$. Therefore, the contour $h \circ \gamma$ lies in the disk $\mathbb{D}(1,1)$ disjoint from 0 and consequently has zero winding number. Denote by $Z_{f+g}$ and $Z_f$ the respective numbers of zeros of $f+g$ and $f$ inside the domain enclosed by $\gamma$. For any $z$ along $\gamma$,

$$\begin{aligned}\frac{f'(z)+g'(z)}{f(z)+g(z)}-\frac{f'(z)}{f(z)}&=\frac{g'(z)f(z)-f'(z)g(z)}{f(z)\big(f(z)+g(z)\big)}\\&=\frac{g'(z)f(z)-f'(z)g(z)}{f(z)^2}\cdot\frac{1}{\frac{g(z)}{f(z)}+1}=\frac{h'(z)}{h(z)}.\end{aligned}$$

Therefore, by the argument principle,

$$\begin{aligned}Z_{f+g}-Z_f&=\frac{1}{2\pi i}\oint_\gamma\frac{f'(z)+g'(z)}{f(z)+g(z)}-\frac{f'(z)}{f(z)}dz\\&=\frac{1}{2\pi i}\oint_\gamma\frac{h'(z)}{h(z)}dz=W(h\circ\gamma)=0.\end{aligned}$$

□

**Example 46.** The polynomial $p(z)=z^5+3z^2+6z+1$ has exactly one zero inside the unit disk $\mathbb{D}$. Indeed, let's split $p$ into $f(z)=6z$ and $g(z)=z^5+3z^2+1$. The function $f$ has a simple zero at 0. When $|z|=1$,

$$|g(z)|=|z^5+3z^2+1|\le|z|^5+3|z|^2+1=5<6=|6z|=|f(z)|.$$

As such, inside $\mathbb{D}$, $p$ has the same number of zeros as $f$, which is 1.

Rouché's theorem also provides a much shorter proof of the fundamental theorem of algebra. In fact, it also gives us a rough estimate of where the zeros of a polynomial can be found.

*proof of the Fundamental Theorem of Algebra.* Let $p(z)=\sum_{n=0}^d a_nz^n$ be a degree $d$ polynomial. Split $p$ into $f(z)=a_dz^d$ and $g(z)=\sum_{n=0}^{d-1}a_nz^n$. Clearly, $f$ has a zero of order $d$ at 0. Pick any positive number $R$ such that

$$R>\max_{n=0,1,\dots d-1}\left|\frac{a_nd}{a_d}\right|^{\frac{1}{d-n}},$$

then $|a_n|R^n<\frac{|a_d|R^d}{d}$ for each $n=0,1,\dots d-1$. For $|z|=R$, by triangle inequality,

$$|g(z)|\le\sum_{n=0}^{d-1}|a_nz|^n=\sum_{n=0}^{d-1}|a_n|R^n<|a_d|R^d=|f(z)|.$$

By Rouché's theorem, $p=f+g$ has $d$ zeros, counting multiplicities, all of which lie inside the disk $\mathbb{D}(0,R)$. □

**Short Quiz 5**

1. Suppose the Taylor series of $\cos(iz+\pi)$ at $\pi$ is $a+b(z-\pi)+c(z-\pi)^2+\ldots$. What is the value of $c$?

2. Suppose f is an entire function. Which of the following criteria imply that f is the zero function?

   (a) $f(z) = 0$ for all natural numbers $z$

   (b) $f(z) = 0$ for all integers $z$

   (c) $f(z) = 0$ for all rational numbers $z$

   (d) $f(z) = 0$ for all real numbers $z$

$$\frac{\sin z}{z-\pi}, \qquad \frac{\cos z}{z-\pi}, \qquad e^{(z-\pi)^{-2}}, \qquad \tan((z-\pi)^{-1})$$

3. Which of the four functions above have a removable singularity at $\pi$?

4. Which of the four functions about have an essential singularity at $\pi$?

5. How many zeros of $z^4 + 2z^2 + 5z + 2$ lie inside the disk $\mathbb{D}(0,3)$ centered at 0 of radius 3?

Answers: 1. $\cos(i\pi + \pi)/2$, 2. $(c), (d)$, 3. $\frac{\sin z}{z-\pi}$, 4. $e^{(z-pi)^{-2}}$, 5. 2.

# Chapter 6

# Evaluation of Integrals

## 6.1 Residue Theory

Let $f$ be a meromorphic function on a domain $U$ and suppose its Laurent series representation at some point $z_0 \in U$ is $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$.

**Definition 20.** The *residue* $\text{Res}f(z_0)$ of $f$ at $z_0 \in U$ is defined to be the coefficient $a_{-1}$.

If $f$ is holomorphic at $z_0$, then clearly $\text{Res}f(z_0) = 0$. Otherwise, let's suppose $z_0$ is a pole of order $k$. If $g(z) = (z-z_0)^k f(z)$, then the Taylor series of $g$ about $z_0$ is

$$g(z) = a_{-k} + a_{-k+1}(z-z_0) + \ldots + a_{-1}(z-z_0)^{k-1} + \ldots$$

To extract out $a_{-1}$ from the series, we can take the $(k-1)^{\text{th}}$ derivative of $g$ evaluated at $z_0$:

$$g^{(k-1)}(z_0) = (k-1)!a_{-1}.$$

Therefore, when we are given the order $k$ of the pole, the residue can be computed as follows:

$$\text{Res}f(z_0) = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} \Big[(z-z_0)^k f(z)\Big].$$

**Theorem 6.1** (Residue Theorem)**.** *Let $f$ be a meromorphic function on a simply connected domain $U$ and $\gamma$ be a simple closed contour in $U$ along which $f$ has no poles. If $\{w_1, \ldots w_m\}$ is the set of poles of $f$ enclosed by $\gamma$, then*

$$\oint_\gamma f(z)dz = 2\pi i \sum_{j=1}^{m} \text{Res}f(w_j).$$

*Proof.* Pick a pole $w_j$ and let $k_j$ be its order. Let $\epsilon_j > 0$ be small enough such that the only pole of $f$ inside the closed disk $\overline{\mathbb{D}(w_j, \epsilon_j)}$ is $w_j$, and parametrize the boundary of this disk by

$$\gamma_j(t) = w_j + \epsilon_j e^{2\pi i t}, \qquad 0 \leq t \leq 1.$$

If the Laurent series of $f$ about $w_j$ is $\sum_{n=-k_j}^{\infty} a_n (z - w_j)^n$, then

$$\begin{aligned}
\oint_{\gamma_j} f(z)dz &= \sum_{n=-k_j}^{\infty} a_n \oint_{\gamma_j} (z - w_j)^n dz \\
&= \sum_{n=-k_j}^{\infty} a_n \int_0^1 \epsilon_j^n e^{2\pi i t n} \cdot \gamma_j'(t) dt \\
&= \sum_{n \geq -k_j, n \neq -1} a_n \int_0^1 2\pi i \epsilon_j^{n+1} e^{2\pi i t(n+1)} dt + a_{-1} \int_0^1 2\pi i dt \\
&= \sum_{n \geq -k_j, n \neq -1} a_n \epsilon_j^{n+1} \frac{e^{2\pi i t(n+1)}}{n+1} \bigg|_0^1 + 2\pi i a_{-1} \\
&= 2\pi i \operatorname{Res} f(w_j).
\end{aligned}$$

Similar to the proof of the argument principle, we finish this proof by splitting $\gamma$ into $k$ simple closed curves, each of which contains only one pole, and subsequently using deformation theorem to reduce the integral computation to a sum of the above integrals for each $j = 1 \ldots m$. □

In the proof, we do not actually use the properties of $w_j$'s being poles. As such, the residue theorem also applies to removable singularities (poles of zero order) and essential singularities (poles of infinite order).

**Example 47.** Let's evaluate $I = \oint_\gamma f(z)dz$ where $f(z) = \frac{1}{z^5 - z^3}$ and $\gamma$ is the circle $C(0, 1/2)$. The only pole of $f$ enclosed by $\gamma$ is 0 and its order is 3. By Residue Theorem,

$$\begin{aligned}
I = 2\pi i \operatorname{Res} f(0) &= \frac{2\pi i}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \left[ \frac{1}{z^2 - 1} \right] \\
&= \pi i \lim_{z \to 0} \frac{-6z^2 + 2}{(z^2 - 1)^3} = -2\pi i.
\end{aligned}$$

## 6.2 Jordan's Lemma

**Lemma 6.2** (Jordan's Lemma)**.** *Suppose $g$ is a holomorphic function on the domain $U = \{z \in \mathbb{C} : |z| > R_0, \operatorname{Im} z > 0\}$ for some radius $R_0 > 0$. Let*

*$\gamma_R = \{Re^{it} \mid 0 \le t \le \pi\}$ be a semicircular curve of radius $R > R_0$. Then, for any $\alpha > 0$,*

$$\Big|\int_{\gamma_R} g(z)e^{i\alpha z}dz\Big| \le \frac{\pi}{\alpha}\max_{z\in\gamma_R}|g(z)|.$$

*Proof.* Let $M_R = \max_{z\in\gamma_R}|g(z)|$. Then,

$$\begin{aligned}
\Big|\int_{\gamma_R} g(z)e^{i\alpha z}dz\Big| &= \Big|\int_0^{\pi} g(Re^{it})e^{\alpha R(i\cos t-\sin t)}\cdot iRe^{it}dt\Big| \\
&\le \int_0^{\pi} \big|g(Re^{it})e^{\alpha R\sin t}\cdot iRe^{i(t+\alpha R\cos t)}\big|dt \\
&= R\int_0^{\pi} |g(Re^{it})|e^{-\alpha R\sin t}dt \\
&\le RM_R\int_0^{\pi} e^{-\alpha R\sin t}dt.
\end{aligned}$$

To prove the theorem, it is sufficient to prove the following inequality:

$$\int_0^{\pi} e^{-\alpha R\sin t}dt \le \frac{\pi}{\alpha R}. \tag{6.1}$$

Since $\sin t$ is convex on $[0, \frac{\pi}{2}]$, $\sin t \ge \frac{2t}{\pi}$ on this interval. Since $\alpha R > 0$, $e^{-\alpha R\sin t} \le e^{-2\alpha Rt/\pi}$ for $0 \le t \le \frac{\pi}{2}$. Then, by symmetry of $\sin t$ about $t = \frac{\pi}{2}$,

$$\int_0^{\pi} e^{-\alpha R\sin t}dt = 2\int_0^{\pi/2} e^{-\alpha R\sin t}dt \le 2\int_0^{\pi/2} e^{-2\alpha Rt/\pi}dt = \frac{\pi(1-e^{-\alpha R})}{\alpha R} < \frac{\pi}{\alpha R}.$$

Thus, (6.1) holds and subsequently, the proof is done. □

The lemma is particularly useful when $g(z) \to 0$ as $|z| \to \infty$. This additional assumption is equivalent to saying that $M_R \to 0$ as $R \to \infty$, and by Jordan's lemma,

$$\Big|\int_{\gamma_R} g(z)e^{i\alpha z}dz\Big| \to 0.$$

When $\alpha = 0$, the upper bound in Jordan's lemma is undefined, but we already have ML inequality in hand.

## 6.3 Definite Integrals

Residue theory, ML inequality and Jordan's lemma are common tools used to evaluate real integrals. We will start with a simple integral $I$ of the form

$$I = \int_0^{2\pi} F(\cos(x), \sin(x))dx,$$

for some generic rational function $F$. With the substitution $z = e^{ix}$ where $0 \leq x \leq 2\pi$, we can transform $I$ into a contour integral

$$I = \oint_{C(0,1)} F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz},$$

which can be solved by residue theorem.

**Example 48.** Let's evaluate

$$I = \int_0^{2\pi} \frac{1}{2-\cos x}dx.$$

By the above substitution,

$$I = \oint_{C(0,1)} \frac{1}{2-\frac{z+z^{-1}}{2}}\frac{dz}{iz} = 2i\oint_{C(0,1)} \frac{1}{z^2-4z+1}dz.$$

Denote the new integrand by $f$. Observe that $f$ has simple poles at $2 \pm \sqrt{3}$, but only $2-\sqrt{3}$ is inside the unit disk. By residue theorem,

$$I = 2i \cdot 2\pi i \mathrm{Res} f(2-\sqrt{3}) = -4\pi \lim_{z \to 2-\sqrt{3}} \frac{1}{z-2-\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

Our aim is to generalise to an improper integral $I = \int_J f(x)dx$ where $J$ is either $(0,\infty)$ or $(-\infty,\infty)$. The general method is as follows:

1. Pick any arbitrarily large $R > 0$.
2. Cook up a simple closed contour $\gamma$ in $\mathbb{C}$ consisting of $\gamma_1 = (0,R)$ or $(-R,R)$ and a few other smooth pieces $\gamma_2, \ldots \gamma_m$, such that as $R \to \infty$, the number of poles enclosed by $\gamma$ remains constant.
3. Evaluate $I_0 = \oint_\gamma f(z)dz$ using residue theorem.
4. Reduce the limit as $R \to \infty$ of each $I_k = \int_{\gamma_k} f(z)dz$, $k \neq 1$, to either a multiple of $I$ or a constant value (typically 0) by Jordan's lemma or ML inequality.
5. Evaluate $I$ as the limit as $R \to \infty$ of $I_1 = \int_{\gamma_1} f(z)dz$.

We will explore in detail different types of integrals you can solve according to the type of integrand $f$ and the shape of the contour $\gamma$ used.

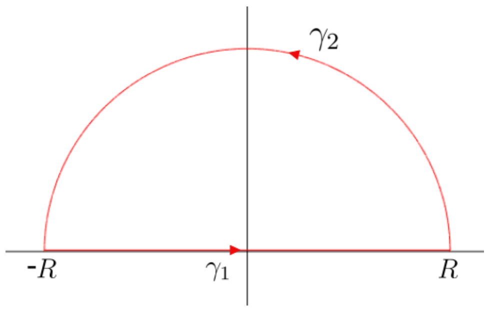
## Semicircular Contour

Consider integrals of the type

$$I = \int_{-\infty}^{\infty} f(x)dx,$$

where the function $f$ has no singularities in $\mathbb{R}$ and has a finite number of poles throughout $\mathbb{C}$. To evaluate $I$, we need to use a semicircular contour $\gamma$ consisting of

$$\gamma_1 = [-R, R] \quad \text{and} \quad \gamma_2 = \{Re^{it} \mid 0 \le t \le \pi\},$$




The most crucial part is to show that $I_2 \to 0$ as $R \to \infty$. Not all functions will satisfy this, but when it does, the value of $I$ follows immediately:

$$I = \lim_{R\to\infty} I_1 = \lim_{R\to\infty} I_0 - I_2 = I_0.$$

**Example 49.** Suppose the integrand is $f(x) = (x^6+1)^{-1}$. This function can be extended to a meromorphic function on $\mathbb{C}$ and it has 6 simple poles forming the set $\{z \mid z^6 = -1\} = \{\pm i, e^{\pm\pi i/6}, e^{\pm 5\pi i/6}\}$. The ones on the upper half plane are $i, e^{\pi i/6}, e^{5\pi i/6}$.

By L'Hopital's rule, the residue of $f$ at any pole $c$ is

$$\text{Res} f(c) = \lim_{z\to c} \frac{z-c}{z^6+1} = \lim_{z\to c} \frac{1}{6z^5} = \frac{c}{6c^5} = -\frac{c}{6}.$$

Let's pick a large $R > 0$ and define $\gamma$ as outlined before. By residue theorem,

$$\begin{aligned} I_0 = \oint_\gamma f(z)dz &= 2\pi i\big(\text{Res} f(i) + \text{Res} f(e^{\pi i/6}) + \text{Res} f(e^{5\pi i/6})\big) \\ &= 2\pi i\left(-\frac{i}{6} - \frac{e^{\pi i/6}}{6} - \frac{e^{5\pi/6}}{6}\right) = -\frac{\pi i}{3}\big(i + e^{\pi i/6} + e^{5\pi i/6}\big) = \frac{2\pi}{3}. \end{aligned}$$

Next, we will show that $I_2 \to 0$. By ML and triangle inequalities, as $R \to \infty$,

$$\left| \int_{\gamma_2} f(z)dz \right| \leq L(\gamma_2) \max_{z \in \gamma_2} |f(z)| = \frac{\pi R}{\min_{|z|=R, \mathrm{Im} z \geq 0} |z^6 + 1|} \leq \frac{\pi R}{R^6 - 1} \to 0.$$

Therefore, $I = I_0 = \frac{2\pi}{3}$.

**Example 50.** Calculating the integrals of functions such as

$$\frac{\cos x}{x^2+1} \quad \text{and} \quad \frac{\sin x}{x^2+1}$$

over $\mathbb{R}$ is common in Fourier analysis. To do so, it is easier to firstly integrate

$$f(z) = \frac{e^{iz}}{z^2+1}$$

and then take its real and imaginary parts (since $\cos z = \mathrm{Re}\, e^{iz}$ and $\sin z = \mathrm{Im}\, e^{iz}$). The function $f$ has simple poles at $\pm i$. With the same setup as before,

$$I_0 = \oint_\gamma f(z)dz = 2\pi i \,\mathrm{Res} f(i) = 2\pi i \frac{e^{-1}}{i+i} = \pi e^{-1}.$$

Since

$$\lim_{R\to\infty} \left| \frac{1}{z^2+1} \right| \leq \lim_{R\to\infty} \frac{1}{R^2-1} = 0,$$

we can immediately conclude by Jordan's lemma that $I_2 \to 0$ as $R \to \infty$. Therefore, $I = \pi e^{-1}$. Taking the real and imaginary parts,

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1} dx = \pi e^{-1}, \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin x}{x^2+1} dx = 0.$$

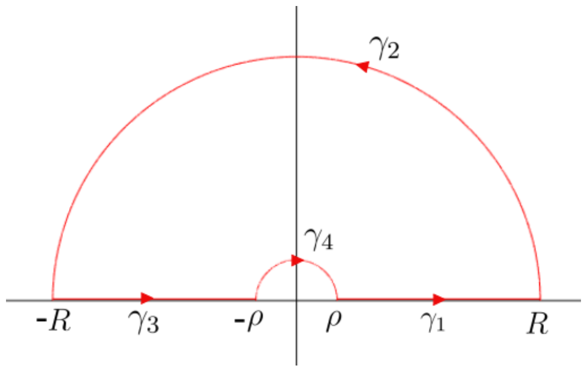
## Indented Semicircular Contour

Consider integrals of the type

$$I = \int_{-\infty}^{\infty} f(x)dx,$$

where the function $f$ has a finite number of poles throughout $\mathbb{C}$, with 0 being the only real pole. Due to the presence of singularity at 0, we need to add a little dent to our semicircular contour. We pick an arbitrarily small $\rho > 0$ and construct $\gamma$ by gluing together the following curves:

$$\gamma_1 = [\rho, R], \quad \gamma_2 = \{Re^{it} | 0 \leq t \leq \pi\}, \quad \gamma_3 = [-R, -\rho], \quad \gamma_4 = \{\rho e^{it} | 0 \leq t \leq \pi\}.$$




Once we show that $I_2$ as $R \to \infty$, the value of $I$ can be computed by taking the following limit:

$$I = \lim_{R\to\infty,\rho\to 0} I_1 + I_3 = I_0 - \lim_{\rho\to 0} I_4.$$

*Remark.* When $f$ has a singularity at a non-zero point in $\mathbb{R}$, we may introduce a dent around that point similar to our construction above.

**Example 51.** Let's compute the integral $I$ of $f(x) = e^{ix}/x$ over $\mathbb{R}$. $f$ has a single pole at 0. Introduce the indented semicircular contour $\gamma$ as outlined above. Since $f$ is holomorphic along $\gamma$ and on the region enclosed by $\gamma$, $I_0 = 0$ by Cauchy-Goursat.

We can show by Jordan's lemma that $I_2 \to 0$ as $R \to \infty$. When $\rho \to 0$,

$$\lim_{\rho\to 0} I_4 = \lim_{\rho\to 0} \int_{\gamma_4} \frac{e^{iz}}{z} dz = -\lim_{\rho\to 0} \int_0^\pi \frac{e^{i\rho e^{it}}}{\rho e^{it}} i\rho e^{it} dt = -i \int_0^\pi \left(\lim_{\rho\to 0} e^{i\rho e^{it}}\right) dt = -\pi i.$$

Therefore, $I = \pi i$. Taking the real and imaginary parts, we obtain the following two integrals as well:

$$\int_{-\infty}^{\infty} \frac{\cos x}{x} dx = 0, \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$$

## Sector Contour

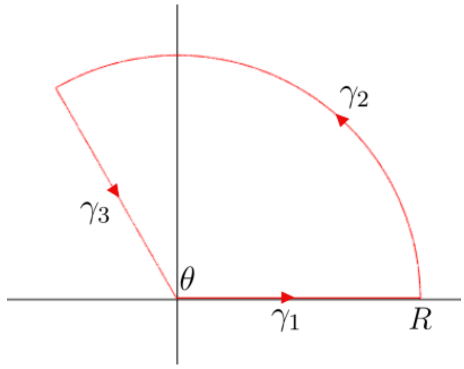
Consider integrals of the type

$$I = \int_0^\infty f(x)dx,$$

where $f$ has a finite number of poles throughout $\mathbb{C}$, none is located along $[0, \infty)$, and $f$ exhibits some kind of rotational symmetry. More precisely, there is some angle $\theta$ and constant $c$ such that $f(e^{i\theta} z) = c f(z)$.

Let $\gamma$ be a sector contour consisting of

$$\gamma_1 = [0, R], \qquad \gamma_2 = \{Re^{it} \mid 0 \le t \le \theta\}, \qquad \text{and} \qquad \gamma_3 = \{re^{i\theta} \mid 0 \le r \le R\}.$$




Rotational symmetry will imply that there is some constant $C$ such that

$$\int_{\gamma_3} f(z)dz = C\int_{\gamma_1} f(z)dz.$$

After showing by ML inequality that $I_2 \to 0$ as $R \to \infty$,

$$I = \lim_{R\to\infty} I_1 = \lim_{R\to\infty} I_0 - I_2 - I_3 = I_0 - CI$$

Therefore, $I = (1+C)^{-1}I_0$.

**Example 52.** Let's evaluate $I$ when $f(x) = (1+x^3)^{-1}$. This integrand has rotational symmetry of angle $\theta = \frac{2\pi}{3}$ since

$$f(e^{2\pi i/3}z) = \frac{1}{1+e^{2\pi i}z^3} = \frac{1}{1+z^3} = f(z).$$

The poles of $f$ are $-1$ and $e^{\pm i\pi/3}$, each of which is simple. Pick a large $R$ and define a sector contour $\gamma$ of angle $\frac{2\pi}{3}$ as outlined above. Since $e^{i\pi/3}$ is the only pole enclosed by $\gamma$,

$$I_0 = 2\pi i\,\mathrm{Res} f(e^{i\pi/3}) = 2\pi i \lim_{z\to e^{i\pi/3}} \frac{z-e^{i\pi/3}}{z^3+1} = 2\pi i \lim_{z\to e^{i\pi/3}} \frac{1}{3z^2} = \frac{2\pi e^{\pi i/3}}{3}.$$

Let's parametrize $\gamma_3$ by $\gamma_3(r) = re^{2\pi i/3}$ as $r$ varies from $R$ to 0. Then,

$$I_3 = \int_R^0 \frac{e^{2\pi i/3}}{1+r^3}dr = -e^{2\pi i/3}\int_0^R \frac{1}{1+r^3}dr = -e^{2\pi i/3}I_1.$$

By ML inequality, we have that $I_2 \to 0$ because

$$\left|\int_{\gamma_2} f(z)dz\right| \le L(\gamma_2)\max_{z\in\gamma_2}|f(z)| = \frac{2\pi R/3}{\min_{|z|=R,\mathrm{Im}z\ge 0}|z^3+1|} \le \frac{2\pi R}{3(R^3-1)} \to 0.$$

Thus,

$$I = (1-e^{2\pi i/3})^{-1}I_0 = \frac{2\pi}{3\sqrt{3}}.$$

## Keyhole Contour

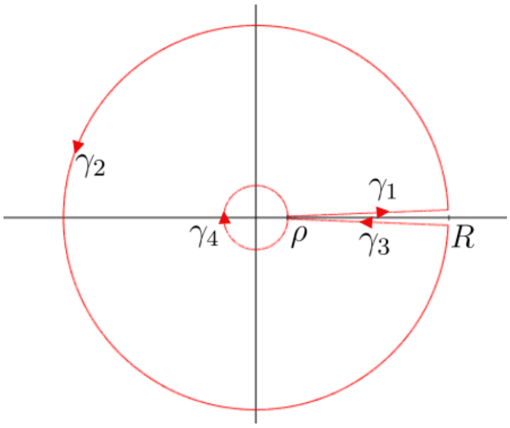
Consider integrals of the type

$$I = \int_0^\infty f(x)dx.$$

where $f$ contains some term $x^a$ where $0 < a < 1$ is a rational number. We have to pick a branch cut and make sure that our contour avoids it. The best choice of branch cut is usually the positive real axis $[0, \infty)$. There are a few variants of the contour $\gamma$, but it is typically split into four parts:

$$\gamma_1 = \{re^{i\epsilon} \mid \rho \le r \le R\}, \qquad \gamma_2 = \{Re^{it} \mid \epsilon \le t \le 2\pi - \epsilon\},$$
$$\gamma_3 = \{re^{i(2\pi-\epsilon)} \mid \rho \le r \le R\}, \qquad \gamma_4 = \{\rho e^{it} \mid \epsilon \le t \le 2\pi - \epsilon\}.$$

Additional parameters $\rho$ and $\epsilon$ are needed and they are taken to be arbitrarily small.




**Example 53.** Let's evaluate the integral $I$ of

$$f(x) = \frac{\sqrt{x}}{(x+1)^2}$$

over $(0, \infty)$. Taking the branch cut of the square root to be $\arg z = 0$, we complexify $f$ to a meromorphic function on $\mathbb{C}\backslash[0, \infty)$ with a pole at $-1$ of order 2. By residue theorem,

$$I_0 = 2\pi i \operatorname{Res} f(-1) = 2\pi i \lim_{z \to -1} \frac{d}{dz}\sqrt{z} = \pi.$$

The limits of $I_1$ and $I_3$ are:

$$\begin{aligned}
\lim_{R\to\infty,\rho,\epsilon\to 0} I_1 &= \lim_{R\to\infty,\rho,\epsilon\to 0} \int_{\gamma_1} \frac{\sqrt{z}}{(z+1)^2} dz = \lim_{R\to\infty,\rho,\epsilon\to 0} \int_{\rho}^{R} \frac{\sqrt{r} e^{i\epsilon/2}}{(re^{i\epsilon}+1)^2} e^{i\epsilon} dr \\
&= \int_0^\infty \frac{\sqrt{r}}{(r+1)^2} dr = I. \\
\lim_{R\to\infty,\rho,\epsilon\to 0} I_3 &= \lim_{R\to\infty,\rho,\epsilon\to 0} \int_{\gamma_3} \frac{\sqrt{z}}{(z+1)^2} dz = \lim_{R\to\infty,\rho,\epsilon\to 0} \int_{R}^{\rho} \frac{\sqrt{r} e^{i(\pi-\epsilon/2)}}{(re^{-i\epsilon}+1)^2} e^{i(2\pi-\epsilon)} dr \\
&= -e^{i\pi} \int_0^\infty \frac{\sqrt{r}}{(r+1)^2} dr = I.
\end{aligned}$$

By ML inequality and triangle inequality, as $R \to \infty$ and $\rho, \epsilon \to 0$,

$$\begin{aligned}
|I_2| &\leq (2\pi - 2\epsilon) R \max_{z\in\gamma_2} \left| \frac{\sqrt{z}}{(z+1)^2} \right| \leq (2\pi - 2\epsilon) \frac{R^{3/2}}{(R-1)^2} \to 0, \\
|I_4| &\leq (2\pi - 2\epsilon) \rho \max_{z\in\gamma_4} \left| \frac{\sqrt{z}}{(z+1)^2} \right| \leq (2\pi - 2\epsilon) \frac{\rho^{3/2}}{(1-\rho)^2} \to 0.
\end{aligned}$$

Collecting all the integrals together,

$$\begin{aligned}
I_1 + I_2 + I_3 + I_4 &= I_0 \\
I + 0 + I + 0 &= \pi \\
\therefore I &= \frac{\pi}{2}.
\end{aligned}$$

# Chapter 7

# Harmonic Functions

## 7.1 Harmonicity

**Definition 21.** Let $U \subset \mathbb{R}^2$ be a non-empty open subset. A real-valued function $u : U \to \mathbb{R}$ is *harmonic* if it is continuously twice differentiable and it satisfies the Laplace's equation:

$$\Delta u := u_{xx} + u_{yy} = 0.$$

The operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is known as the *Laplacian / Laplace operator*. There are many instances in which polar coordinates are used. With the usual change of variables $(x, y) = (r\cos\theta, r\sin\theta)$, this operator becomes

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

in polar coordinates.

**Example 54.** Affine functions $u(x, y) = ax + by + c$ for some real constants $a, b, c$ are harmonic on $\mathbb{R}^2$.

**Example 55.** The function $u(x, y) = \sin x \cosh y$ is harmonic on $\mathbb{R}^2$.

**Example 56.** Denote the real and imaginary parts of the principal logarithm $\mathrm{Log} : \mathbb{C}\backslash(-\infty, 0]$ as $u$ and $v$ accordingly. In Cartesian coordinates,

$$u(x, y) = \ln\sqrt{x^2 + y^2}, \qquad v(x, y) = \tan^{-1}\left(\frac{y}{x}\right).$$

It may not seem clear at first that both $u$ and $v$ are harmonic, but in polar coordinates, $u(r, \theta) = \ln r$ and $v(r, \theta) = \theta$ and it is much easier to show that these harmonic whenever $r > 0$ and $-\pi < \theta < \pi$.

Harmonic functions naturally appear in many fields of applied mathematics and physics. In probability theory, the probability of that a Brownian motion inside a domain hits part of the boundary is governed by harmonic functions. In fluid dynamics, the velocity potential in an incompressible irrotational fluid satisfies the Laplace's equation. In physics, electrostatic and gravitational potential satisfies the Poisson equation, a generalisation of Laplace's equation. One of the main reasons complex analysis is often an essential tool for many physicists is that holomorphic functions and harmonic functions are very much interrelated.

**Proposition 7.1.** *Let $f = u + iv$ be a holomorphic function on a domain $U \subset \mathbb{C}$, then both $u$ and $v$ are harmonic.*

*Proof.* By Cauchy-Riemann equations,

$$\Delta u = u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0, \qquad \Delta v = v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0.$$

Hence, $u$ and $v$ are harmonic. □

**Proposition 7.2.** *Let $u$ be a harmonic function on a simply connected domain $U$, then there is a harmonic function $v$ on $U$ such that $f = u + iv$ is holomorphic on $U$, and it is unique up to an additive constant.*

*Proof.* It is sufficient to find a solution $v$ to the first order partial differential equations $v_x = -u_y$ and $v_y = u_x$. By (2.1) and (2.2), this is equivalent to finding a primitive $f = u + iv$ of the holomorphic function $f' = u_x - iu_y$. By Corollary 3.8, such a primitive exists since $U$ is simply connected, and it is unique up to an additive constant $a + ib$. Since we fix the real part of $f$ to be $u$, then $a = 0$ and all harmonic conjugates of $u$ are $v(x, y) + b$ for some real constant $b$. □

In the theorem above, we say that $v$ is a *harmonic conjugate* to $u$.

**Example 57.** The function $u(x, y) = x^2 - y^2$ is a harmonic function on $\mathbb{R}^2$. Suppose $v$ is a harmonic conjugate of $u$. The partial derivatives of $u$ are $u_x = 2x$ and $u_y = -2y$. By Cauchy-Riemann,

$$v_x = -u_y = 2y, \qquad v_y = 2x.$$

By integrating $v_x$ with respect to $x$, $v(x, y) = 2xy + c(y)$ for some real differentiable function $c(y)$. Differentiating $v$ with respect to $y$, we obtain that $v_y = 2x + c'(y) = 2x$, which implies that $c'(y) = 0$. Therefore, any harmonic conjugate of $u$ is of the form

$$v(x, y) = 2xy + c,$$

for any real constant $c$. Indeed, the corresponding holomorphic function is a quadratic $f(z) = z^2 + ic$.

**Example 58.** The function $u(r,\theta) = \ln r$, written in polar coordinates, is harmonic in $\mathbb{R}^2\backslash\{0\}$. The proposition above fails because the punctured plane is multiply connected. Indeed, we see that $v(r,\theta) = \theta$ is the harmonic conjugate of $u$ but only after we introduce a branch cut; $v$ cannot be extended to a harmonic conjugate on the whole $\mathbb{R}^2\backslash\{0\}$.

## 7.2 Key Properties

We shall transfer a number of properties on holomorphic functions we have already known to harmonic functions. From now on, harmonic functions $u(x,y)$ will sometimes be expressed as a function $u(x+iy)$ of one complex variable without any ambiguity.

**Proposition 7.3** (Mean Value Property)**.** *Let $u$ be a harmonic function on a domain $U$. For any closed disk $\overline{\mathbb{D}(z_0,\epsilon)}$ in $U$,*

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+\epsilon e^{it})dt.$$

*Proof.* Let $f$ be a holomorphic function on $U$ such that $u(z) = \mathrm{Re} f(z)$. By Corollary 4.2,

$$\begin{aligned} u(z_0) = \mathrm{Re} f(z_0) &= \mathrm{Re}\left[\frac{1}{2\pi}\int_0^{2\pi} f(z_0+\epsilon e^{it})dt\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(z_0+\epsilon e^{it})dt. \end{aligned}$$

□

We may also take the mean value of $u$ over the whole disk enclosed by the circle and obtain the same result.

**Corollary 7.4** (Volume Mean Value Property)**.** *Let $u$ be a harmonic function on a simply connected domain $U \subset \mathbb{C}$. The average value of $u$ inside any closed disk $\overline{\mathbb{D}(z_0,r)}$ lying inside $U$ is equal to $u(z_0)$, i.e.*

$$u(z_0) = \frac{1}{\pi\epsilon^2}\int_{\mathbb{D}(z_0,\epsilon)} u(z)dA.$$

*Proof.* By expressing the area element $dA$ in polar coordinates,

$$\begin{aligned} \frac{1}{\pi\epsilon^2}\int_{\mathbb{D}(z_0,\epsilon)} u(z)dA &= \frac{1}{\pi\epsilon^2}\int_0^{\epsilon}\int_0^{2\pi} u(z_0+re^{it})rdtdr \\ &= \frac{2}{\epsilon^2}\int_0^{\epsilon}\left(\frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{it})dt\right)rdr \\ &= \frac{2}{\epsilon^2}\int_0^{\epsilon} u(z_0)\cdot rdr = u(z_0). \end{aligned}$$

□

It is worth noting that the only ingredients of the proof of the maximum modulus principle of holomorphic functions in Lemma 4.11 and Theorem 4.12 are continuity, mean value property and connectivity of the domain. Since these three properties are satisfied by every harmonic function on a domain, harmonic functions automatically satisfy the maximum modulus principle. Nonetheless, we shall give a slicker proof using harmonic conjugates.

**Theorem 7.5.** *Harmonic functions satisfy the maximum modulus principle.*

*Proof.* Let $u$ be a harmonic function on a domain $U \subset \mathbb{C}$. Pick a harmonic conjugate $v$ of $u$ on $U$ and define $f(x+iy) = u(x,y) + iv(x,y)$ on $U$. The function is a non-constant holomorphic function $e^{f(z)}$ with modulus is $|e^{f(z)}| = e^u(x,y)$. Applying the maximum modulus principle to $e^{f(z)}$, then $e^u$ and therefore $u(x,y)$ do not attain maximum on $U$. We can apply the same argument on $-u$ to conclude that $u$ does not attain minimum on $U$. Therefore, $|u|$ does not attain maximum on $U$, and if $U$ is bounded, maxima are achieved only on the boundary. $\square$

## 7.3 Poisson Integral Formula

Mean value property is an analogue of the Cauchy integral formula, but it only tells us the value of the function at the center of the circle. In this section, we wish to obtain a better analogue of the Cauchy integral formula to obtain the value of the function at any point enclosed by the circle.

**Definition 22.** The *Poisson kernel* $P(r,\theta)$ for the unit disk $\mathbb{D}$ is given by

$$P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2},$$

where $re^{i\theta} \in \mathbb{D}$.

By the cosine rule, the denominator is equal to the square of the length of a side of a triangle whose other sides have length 1 and $r$ and make an interior angle $\theta$ (mod $2\pi$). Let $z = re^{i\theta} \in \mathbb{D}$, then this triangle can be chosen to have vertices 0, 1 and $z$, and the denominator is equal to $|1-z|^2$. In particular, $P(r,\theta) > 0$.

The Poisson kernel can also be rewritten as the real part of a rational function of $z$:

$$P(r,\theta) = \frac{1-|z|^2}{|1-z|^2} = \operatorname{Re}\left[\frac{1+z}{1-z}\right].$$

In particular, the Poisson kernel is harmonic on $\mathbb{D}$ because this rational function is holomorphic on $\mathbb{D}$.

**Theorem 7.6** (Poisson Integral Formula)**.** *Let $u(r,\theta)$ be a harmonic function on a neighbourhood of $\overline{\mathbb{D}}$, written as a function in polar coordinates. For any point $z = re^{i\theta} \in \mathbb{D}$,*

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} P(r, t-\theta)\, u(1,t)dt.$$

*Proof.* Let $z_1 = 1/\bar{z} = r^{-1}e^{i\theta}$ be the reflection of $z$ in the unit circle. Since $z_1$ is outside the closed unit disk, if we denote by $f$ a holomorphic function whose real part is $u$, then by Cauchy-Goursat,

$$\oint_{\partial\mathbb{D}} \frac{f(w)}{w - z_1} dw = 0.$$

By Cauchy integral formula,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \frac{f(w)}{w-z} dw \\ &= \frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \frac{f(w)}{w-z_1} dw \\ &= \frac{1}{2\pi i}\oint_{\partial\mathbb{D}} f(w)\left(\frac{1}{w-z} - \frac{1}{w-z_1}\right) dw. \end{aligned} \tag{7.1}$$

For any $w = e^{it}$ on the unit circle $\partial\mathbb{D}$,

$$\begin{aligned} w\left(\frac{1}{w-z} - \frac{1}{w-z_1}\right) &= w\left(\frac{1}{w-z} - \frac{1}{w - w\bar{w}/\bar{z}}\right) \\ &= \frac{w}{w-z} + \frac{\bar{z}}{\bar{w}-\bar{z}} \\ &= \frac{w\bar{w} - z\bar{z}}{w\bar{w} - w\bar{z} - \bar{w}z + z\bar{z}} \\ &= \frac{1-r^2}{1 - 2r\cos(t-\theta) + r^2} = P(r, t-\theta). \end{aligned} \tag{7.2}$$

Using parametrization $w = e^{it}$, $0 \le t \le 2\pi$, of the unit circle $\partial\mathbb{D}$, $dw = iw\,dt$. Then, combining (7.1) and (7.2),

$$\begin{aligned} f(z) &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\left(\frac{w}{w-z} - \frac{w}{w-z_1}\right) dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})P(r, t-\theta)dt. \end{aligned}$$

Taking the real part of both sides of the equation, we immediately obtain the Poisson integral formula. $\square$

## 7.4 Dirichlet Problem for $\mathbb{D}$

Solving Laplace's equation with given boundary conditions goes under the name of Dirichlet's problem, and is one of the fundamental problems of partial differential equations (PDE). Specifically, we wish to find a solution $u$ of the equation $\Delta u = 0$ on some domain such that along the boundary of the domain, $u$ takes certain prescribed values.

**Theorem 7.7** (Dirichlet Problem on $\mathbb{D}$). *For any real-valued function $F : [0, 2\pi] \to \mathbb{R}$ which is integrable, i.e. $\int_0^{2\pi} |F(t)|dt < \infty$, there is a unique solution $u : \mathbb{D} \to \mathbb{R}$ to the following PDE problem:*

$$\begin{aligned} \Delta u(r,\theta) &= 0, && \text{if } 0 \le r < 1, 0 \le \theta < 2\pi, \\ \lim_{r\to 1} u(r,\theta) &= F(\theta), && \text{for all } 0 \le \theta < 2\pi \text{ at which } F \text{ is continuous.} \end{aligned}$$

*Proof.* Assume $u$ and $v$ are two such solutions, then $u - v$ is a harmonic function on $\mathbb{D}$ which is identically zero along $\partial\mathbb{D}$. By the maximum modulus principle, $u - v \equiv 0$. Therefore, $u \equiv v$. We have then proven uniqueness.

What remains is to prove existence of the solution $u$. If it does exist, then it will satisfy the Poisson kernel formula we previously obtained. Let's define $u$ using the Poisson kernel:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} P(r, t-\theta)\, F(t)dt.$$

On $\mathbb{D}$, $u$ is continuously twice differentiable because $P$ is continuously twice differentiable. By chain rule, for any fixed value of $t$,

$$\frac{\partial}{\partial r}P(r,t-\theta) = \frac{\partial P}{\partial r}(r,t-\theta), \qquad \frac{\partial}{\partial \theta}P(r,t-\theta) = -\frac{\partial P}{\partial \theta}(r,t-\theta).$$

Therefore, since $P$ is harmonic,

$$\begin{aligned} \Delta\big(P(r,t-\theta)\big) &= \frac{\partial^2 P}{\partial r^2}(r,t-\theta) + \frac{1}{r}\frac{\partial P}{\partial r}(r,t-\theta) + \frac{1}{r^2}(-1)^2\frac{\partial^2}{\partial \theta^2}(r,t-\theta) \\ &= (\Delta P)(r,t-\theta) = 0. \end{aligned}$$

This immediately implies that $u$ is harmonic on $\mathbb{D}$:

$$\Delta u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \Delta P(r,t-\theta)\, F(t)dt = 0.$$

What remains is to check that $u$ satisfies the boundary condition. By the mean value property,

$$\frac{1}{2\pi}\int_0^{2\pi} P(r,t-\theta)dt = P(0,t-\theta) = 1. \tag{7.3}$$

Consequently,

$$u(r,\theta) - F(\theta) = \frac{1}{2\pi}\int_0^{2\pi} P(r,t-\theta)\,(F(t) - F(\theta))dt.$$

Since $P$ is positive on $\mathbb{D}$,

$$|u(r,\theta) - F(\theta)| \le \frac{1}{2\pi}\int_0^{2\pi} P(r,t-\theta)\,|F(t) - F(\theta)|dt. \tag{7.4}$$

To evaluate the last integral, we will split the interval into $I = (\theta-\delta,\theta+\delta)$ (mod $2\pi$) and $J = [0,2\pi]\backslash I$. On $I$,

$$\begin{aligned}\frac{1}{2\pi}\int_I P(r,t-\theta)\,|F(t) - F(\theta)|dt &\le \frac{1}{2\pi}\left(\int_I P(r,t-\theta)dt\right)\max_{\theta-\delta\le t\theta+\delta}|F(t)-F(\theta)| \\ &\le \frac{1}{2\pi}\max_{\theta-\delta\le t\theta+\delta}|F(t)-F(\theta)| \qquad (7.5)\end{aligned}$$

The last inequality follows from (7.3). By continuity of $F$, $\max_{\theta-\delta\le t\theta+\delta}|F(t)-F(\theta)| \to 0$ as $\delta \to 0$.

When $t \in J$, $P(r,t-\theta)$ is well defined because it avoids the singularity at $t=\theta$. The distance between $e^{i\theta}$ and the set $\{z \in \mathbb{D} \mid \arg z \in J\}$ is $\sin\delta$, so if $t \in J$, $|1 - re^{i(t-\theta)}| = |e^{i\theta} - re^{it}| \ge \sin\delta$. Therefore,

$$\begin{aligned}\frac{1}{2\pi}\int_J P(r,t-\theta)\,|F(t) - F(\theta)|dt &= \frac{1}{2\pi}\int_J \frac{1-r^2}{|1-re^{i(t-\theta)}|^2}\,|F(t)-F(\theta)|dt \\ &\le \frac{1-r^2}{2\pi\sin^2\delta}\int_J |F(t)-F(\theta)|dt \\ &\le \frac{D(1-r^2)}{2\pi\sin^2\delta}.\end{aligned}$$

where $D = \int_0^{2\pi}|F(t)|dt$. This upper bound converges to 0 as $r \to 1$. Combining this with (7.5), we have shown that the upper bound in (7.4) converges to 0 as $r \to 1$ and subsequently $\delta \to 0$. In other words, $u(r,\theta) \to F(\theta)$ as $r \to 1$. Hence, $u$ extends continuously to the boundary condition $F$. □

*Remark.* By rescaling and translation, it is straightforward that the uniqueness and existence of solution of the Dirichlet problem for any arbitrary disk $\mathbb{D}(z_0, r_0)$ also hold. In general, the theorem also holds for arbitrary bounded simply connected domain $U$ if its boundary $\partial U$ is a smooth curve. Proving such generalisation requires more involved argument and we shall not attempt to do it here.

**Example 59.** For any real constant $c \in \mathbb{R}$, every harmonic function $u : \mathbb{D} \to \mathbb{R}$ that is identically $c$ along the boundary $\partial\mathbb{D}$ must be constant, i.e. $u \equiv c$ on $\mathbb{D}$.

**Example 60.** Let $F(\theta) = 1$ when $0 \leq \theta < \pi$ and $F(\theta) = 0$ when $\pi \leq \theta < 2\pi$. The unique harmonic function on the unit disk with boundary value $F(\theta)$ is

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_0^\pi P(r,t-\theta)dt = \frac{1}{\pi}\tan^{-1}\left(\frac{1+r}{1-r}\tan\frac{t-\theta}{2}\right)\Bigg|_0^\pi \\ &= \frac{1}{\pi}\tan^{-1}\left(\frac{1+r}{1-r}\tan\frac{\pi-\theta}{2}\right) + \frac{1}{\pi}\tan^{-1}\left(\frac{1+r}{1-r}\tan\frac{\theta}{2}\right) \\ &= \frac{1}{\pi}\tan^{-1}\left(\frac{r^2-1}{2r\sin\theta}\right). \end{aligned}$$

One implication of the principle is the fact that continuous functions which satisfy the mean value property are automatically harmonic. This holds even without the assumption that the function is differentiable.

**Corollary 7.8.** *Let $u : U \to \mathbb{R}$ be a continuous function on a domain $U$. Then, $u$ is harmonic if and only if it satisfies the mean value property (MVP).*

*Proof.* It is clear from Proposition 7.3 that harmonicity implies mean value property. Assume now that $u$ satisfies the MVP. Let's pick a point $z_0 \in U$ in complex coordinates. By openness, there is some $r > 0$ such that $U$ contains the closed disk $\overline{\mathbb{D}(z_0, r)}$. By the Dirichlet principle, there is a unique continuous function $v$ on $\overline{\mathbb{D}(z_0, r)}$ that coincides with $u$ on the circle $C(z_0, r)$ and is harmonic on the open disk $\mathbb{D}(z_0, r)$.

The function $u - v$ on $\overline{\mathbb{D}(z_0, r)}$ satisfies the MVP because both $u$ and $v$ satisfy the MVP. Therefore, $u - v$ satisfies the maximum modulus principle too. Since $u - v \equiv 0$ on the boundary, $u - v \equiv 0$ on $\mathbb{D}(z_0, r)$. In particular, $u$ is harmonic on the open disk $\mathbb{D}(z_0, r)$. Since $z_0$ is an arbitrary point in $U$, $u$ must be harmonic everywhere on $U$. $\square$

## 7.5 Applications in Fluid Dynamics

We shall discuss one immediate application of the study of harmonic and holomorphic functions in fluid mechanics. Consider the case of a fluid flowing on a planar domain $\Omega \subset \mathbb{R}^2$. The (instantaneous) velocity of the flow of the fluid at each point $(x, y) \in \Omega$ is assumed to be determined by a smooth vector field $V(x, y) = (p(x, y), q(x, y))$.

**Definition 23.** The flow is *incompressible* if and only if the vector field $V$ has zero divergence:

$$\text{div } V = \nabla \cdot V = p_x + q_y \equiv 0.$$

The flow is *irrotational* if and only if $V$ has zero curl:

$$\text{curl } V = \nabla \times V = q_x - p_y \equiv 0.$$

Incompressibility essentially means that the fluid has constant uniform density throughout. The reason behind the vanishing divergence relies on what is known in physics as the "continuity equation". Irrotationality means that the flow has zero local circulation (often known as vorticity). This can be described more mathematically in the following way.

Let $\gamma$ be a simple closed contour on $\Omega$ and suppose the region $R$ enclosed by $\gamma$ is contained in $\Omega$. The circulation of the fluid along $\gamma$ is defined to be the line integral of $V$ along $\gamma$, i.e.

$$\oint_\gamma p(x,y)dx + q(x,y)dy.$$

By Green's theorem, this integral can be rewritten as

$$\int\int_R (q_x - p_y)dxdy$$

Notice that the integrand coincides with the curl of $V$. Irrotationality essentially ensures that the circulation along $\gamma$ is zero because this integral always vanishes. (Alternatively, you may say that the vector field is conservative.)

When both assumptions are met, we say that the fluid has an *ideal* fluid flow. Observe that they very much resemble the Cauchy-Riemann equations.

**Proposition 7.9.** *The vector field $V = (p,q)$ on the domain $\Omega$ induces an ideal fluid flow if and only if the complex function*

$$f(x+iy) = p(x,y) - iq(x,y)$$

*is holomorphic on $\Omega \subset \mathbb{C}$.*

For an ideal fluid flow, the function $f$ above is often called the *complex velocity* of the fluid flow.

One implication of irrotationality is that $f$ admits a primitive $F(x+iy) = \phi(x,y) + i\psi(x,y)$ on $\Omega$. By Cauchy-Riemann equations and equation 2.1,

$$p - iq = f = \frac{dF}{dz} = \phi_x + i\psi_x = \phi_x - i\phi_y.$$

In particular, $\phi$ satisfies $\nabla\phi = (\phi_x, \phi_y) = (p, q) = V$. Moreover, both $\phi$ and $\psi$ are necessarily harmonic because they are the real and imaginary parts of the holomorphic function $F$.

**Definition 24.** A *complex potential* of the fluid flow is a primitive $F$ of $f$ on $\Omega$. The real part $\phi$ of $F$ is called the *velocity potential* and the imaginary part $\psi$ of $F$ is called the *stream function.*

In summary, if we complexify $V$ and use the notation $V = p + iq$ instead, we have the equation $V = \bar{f} = \overline{F'}$. By decomposing $F$, we obtain two harmonic functions $\phi$ and $\psi$ which are harmonic conjugates of each other and they satisfy the equations $\phi_x = \psi_y = p$ and $\phi_y = -\psi_x = q$.

**Definition 25.** The level sets of the velocity potential $\{(x, y) \in \Omega \,|\, \phi(x, y) = c\}$ for some real constant $c$ are called the *equipotentials* of the flow. The level sets of the stream function $\{(x, y) \in \Omega \mid \psi(x, y) = d\}$ for some real constant $d$ are called the *streamlines* of the flow.

**Proposition 7.10.** *Equipotentials and streamlines are always perpendicular to each other at their points of intersection.*

*Proof.* The gradient $\nabla\phi$ is always normal to any equipotential $\{\phi = c\}$ and The gradient $\nabla\psi$ is always normal to any streamline $\{\psi = d\}$. By Cauchy-Riemann equations,

$$\nabla\phi \cdot \nabla\psi = \phi_x\psi_x + \phi_y\psi_y = \psi_y\psi_x + (-\psi_x)\psi_y = 0.$$

As the dot product always vanishes, perpendicularity is shown. $\square$

The proof above also shows that streamlines have tangent vectors $\nabla\phi$. Given any fluid particle $(x_0, y_0)$ in $\Omega$, since the vector field $V$ coincides with $\nabla\phi$ and determines its flow trajectory, the particle always flows along a unique streamline $\{(x, y) \mid \psi(x, y) = \psi(x_0, y_0)\}$.

**Example 61.** A constant vector field throughout the plane $V = (a, b)$ is trivially an ideal flow. An easy choice of complex potential would be $F(z) = (a - ib)z$ on $\mathbb{C}$. The corresponding complex potential and stream function are $\phi(x, y) = ax + by$ and $\psi(x, y) = bx - ay$. As such, all equipotentials are straight lines of slope $-a/b$ and all streamlines are straight lines of slope $b/a$.

**Example 62.** The vector field $V(x, y) = (x, -y)$ induces an ideal fluid flow on the plane. The complex potential can be chosen to be $F(z) = \frac{z^2}{2} = \frac{x^2-y^2}{2} + ixy$. $V$ induces hyperbolic streamlines $\{xy = d\}$. The equipotentials of $V$ are hyperbolas of the form $\{x^2 - y^2 = 2c\}$. At the origin, the flow has a unique fixed point, which is a saddle point of the flow. Perpendicularity of the equipotentials and the streamlines fails here because the vector field vanishes.

## Short Quiz 7

1. Which of the following functions are harmonic on $\mathbb{R}^2$?

$$x^3 - y^3, \qquad (x-y)^{2020}, \qquad \cos x \cosh y, \qquad \cosh x \cosh y$$

Answers: 1. $\cos x \cosh y$.