Practice Questions

1. Find the Laurent series of:
   - $z\cos\left(\frac{1}{z}\right)$ about 0,
   - $\dfrac{3}{(z^2-1)(z^2-4)}$ on $\{|z|<1\}$,
   - $\dfrac{3}{(z^2-1)(z^2-4)}$ on $\{1<|z|<2\}$.

2. Classify all the singularities of:
   - $\dfrac{z^2+\cos(\pi z)}{z^2(z^2-1)}$
   - $\dfrac{\sin z}{\sin 3z}$
   - $\dfrac{1}{z^4 \sin z}$

3. Compute the following residues
   - $\operatorname*{Res}_{z=i} \dfrac{z^{\frac{1}{2}}}{(z^2+1)^2}$
   - $\operatorname*{Res}_{z=0} \dfrac{z}{\sinh z - z}$

4. Let $f:\mathbb{D}\to\mathbb{D}$ be a holomorphic fn. with Taylor series $\sum_{n\geq 0} a_n z^n$. Show that $|a_n|\leq 1$ for all n.

5. Let $f:\mathbb{D}\to\mathbb{C}$ be a holomorphic fn with Taylor series $\sum_{n\geq 0} a_n z^n$ & $|f(z)| < \frac{1}{1-|z|}$ for all $z\in\mathbb{D}$.

Show that $|a_n| \leq \frac{(n+1)^{n+1}}{n^n}$.

(6) Show that the eqn $e^z = 3z^n$ has n roots inside $\mathbb{D}$.

(7) Prove the following integral identities below:

- $\int_0^\infty \frac{1}{1+x^n}\,dx = \frac{\pi/n}{\sin(\pi/n)}$.

- $\int_0^\pi \sin^{2n}\theta \, d\theta = \frac{(2n)!\,\pi}{2^{2n}(n!)^2}$, $n \geq 1$.

(8) An ideal fluid flow on $\mathbb{R}^2 \setminus \{0,0\}$ is governed by the complex potential $F(z) = \frac{1}{z}$. Sketch the streamlines & equipotentials and describe the flow.

(9) Is $u = x^3 - 3xy^2$ harmonic on $\mathbb{R}^2$? Find all harmonic conjugates v & find all holomorphic functions f such that:

$$\operatorname{Re} f = x^3 - 3xy^2 \ \& \ f(i) = i.$$

(10) If $f = u + iv$ is entire & $u+v$ is is bounded, is f constant?

(11) Let f be an entire function satisfying

$$f(z) = f(z+1) = f(z+i)$$

for all $z \in \mathbb{C}$. Show that f is constant.