Problem Set 3

Assessed problems (and sub-problems) are marked by the asterisk *. All closed curves are assumed to be positively oriented, unless stated otherwise.

1. Evaluate the integral
$$\oint_\gamma \frac{1}{4z^2+9}dz.$$
for each of the following cases:

   (a) γ is the rectangle with vertices ± 2 and $\pm 2 - i$,

   (b) γ is the circle $C(2+2i, 3)$,

   (c)* $\gamma(t) = \pi e^{-\pi i t}$ where $0 \le t \le 2$.

2. * Use Cauchy's formulas to compute and simplify the integral of f along the circle $C(0,2)$ for each of the following functions.
$$\text{(a) } f(z) = \frac{z+2}{(z-1)(z-3)}, \quad \text{(b)}^* \ f(z) = \frac{e^{e^z}}{z - \frac{i\pi}{2}}, \quad \text{(c)}^* \ f(z) = \frac{\sinh(\pi z)}{z^4}.$$

3. * Suppose an entire function f satisfies $|f(z)| \le \pi|z|$ for all $z \in \mathbb{C}$.

   (a) Evaluate $f''(z)$ for each $z \in \mathbb{C}$ using Cauchy's inequality.

   (b) Show that f must be a linear function az for some $a \in \overline{\mathbb{D}(0,\pi)}$.

4. Let f be an entire function such that $0 < |f^{(6)}(z)| \le 2020$ for all $z \in \mathbb{C}$. Explain why f must be a polynomial and state its degree.

5. * Let f be an entire function such that $|f(z)| \ge 1$ for all $z \in \mathbb{C}$. Show that f is constant.

6. Let's prove Liouville's theorem in a different way. Suppose f is a bounded entire function. Pick any two distinct points $z_1, z_2 \in \mathbb{C}$ and pick a large positive number R such that $|z_1|, |z_2| < R$.

   (a) Show that there is some constant $k > 0$ such that
$$\left| \oint_{C(0,R)} \frac{f(z)}{(z-z_0)(z-z_1)} dz \right| \le \frac{kR}{(R-|z_0|)(R-|z_1|)}.$$

   (b) Apply Cauchy's integral formula to the inequality above to show that $f(z_1) = f(z_2)$.

7. Let
$$f(z) = \begin{cases} z^2, & \text{if } z \in \mathbb{D}, \\ 2, & \text{if } 2 < |z| < 3, \end{cases}$$
be a function on the open set $U = \mathbb{D} \cup \{2 < |z| < 3\}$. Show that f is a non-constant holomorphic function on U which attains a maximum. Does this contradict the maximum modulus principle?

8. * Find all points on which the modulus of the function $f(z) = z^3 + 1$ on the closed disk $\{|z| \leq 2\}$ attains its maximum value.

9. Find the smallest radius $r > 0$ of the disk $\mathbb{D}(0, r)$ containing the image of the function $e^{(1+i)z}$ on the open square $\{x + iy \mid 1 < x, y < \pi\}$.

10. * Argand was the first to rigorously prove the fundamental theorem of algebra. We shall follow along his thought process.

    (a) Prove D'Alembert's lemma: for every polynomial f of degree $d \geq 1$, every point z_0 such that $f(z_0) \neq 0$ and every $\epsilon > 0$, we can always find a point z such that $|z - z_0| < \epsilon$ and $|f(z)| < |f(z_0)|$.

    (b) Let $f(z) = \sum_{n=0}^{d} a_n z^n$ be some polynomial where $a_d \neq 0$. Show that if $R \geq 1$ is a real number satisfying
    $$R \geq \frac{1 + \sum_{n=0}^{d-1} |a_n|}{|a_d|},$$
    then $|f(z)| \geq R^{d-1}$ whenever $|z| \geq R$.

    (c) Use the two results above to prove the fundamental theorem of algebra.