Problem Set 4

Assessed problems (and sub-problems) are marked by the asterisk *. All closed curves are assumed to be positively oriented, unless stated otherwise.

1. Find the Taylor series expansion of:

   (a)* $f(z) = e^{2\pi z}$ about 2π,

   (a) $f(z) = \frac{1}{z^2+1}$ about 0,

   (b) $f(z) = \sin z$ about $\frac{\pi}{2}$.

2. Suppose f is holomorphic function on a domain U. Show that if f is constant on some non-empty open subset $V \subset U$, then f is constant on U.

3. * Let f be an entire function which restricts to a real function $f : \mathbb{R} \to \mathbb{R}$ on the real axis. Show that for all $z \in \mathbb{C}$, $\overline{f(\bar{z})} = f(z)$. (Hint: refer to Problem set 2 Qn 1.)

4. Does $g(z) = \sum_{n=0}^{\infty} \frac{1}{z^n}$ define a holomorphic function on the domain $\{|z| > 1\}$? Explain why.

5. Find the Laurent series expansion of:

   (a) $f(z) = \frac{z}{z^2+1}$ about i,

   (b) $f(z) = \frac{2}{z-2} + \frac{1}{4-z}$, convergent on $\{2 < |z| < 4\}$,

   (c)* $f(z) = \frac{3-3z}{2z^2-5z+2}$, convergent on $\{\frac{1}{2} < |z-1| < 1\}$.

6. Find and classify all singularities of the following function.

$$\text{(a) } \cot z, \qquad \text{(b)} \frac{1}{\sin z - \sin(2z)}, \qquad \text{(c)} \frac{z^2 + \cos(\pi z)}{z^2(z^2-1)}.$$

7. * Suppose f and g are entire functions such that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that all singularities of f/g are removable and that there is a constant $a \in \mathbb{C}$ such that $f(z) = ag(z)$ for all $z \in \mathbb{C}$.

8. (Schwarz Lemma) Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic self map of the unit disk such that $f(0) = 0$.

   (a) Explain why $g(z) = f(z)/z$ is a well-defined holomorphic function on $\mathbb{D}$,

(b) Show by maximum modulus principle that $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$,

(c) Show that if $|f'(0)| = 1$ or $|f(w)| = |w|$ for some point $w \in \mathbb{D}^*$, then f is a rotation[1].

9. * Let γ be the curve parametrising the square with vertices $\pm\frac{\pi}{4} - \frac{\pi i}{4}$ and $\pm\frac{\pi}{4} + \frac{\pi i}{4}$. Sketch the image of γ under the function $\cos 2z - 1$ and use your sketch to find the number of zeros of $\cos 2z - 1$ enclosed by γ.

10. Show that for each $n \in \mathbb{N}$, the number of zeros of $e^{z-1} + 2z^n$ inside the unit disk $\mathbb{D}$ is n.

11. * How many zeros does the polynomial $z^5 + 5z + 1$ have in the annular domain $\{1 \leq |z| < 2\}$?

[1]In particular, this lemma says that 0 is either an attracting fixed point ($|f'(0)| < 1$) or f is a rotation ($|f'(0)| = 1$).