Problem Set 5

Assessed problems (and sub-problems) are marked by the asterisk *. All closed curves are assumed to be positively oriented, unless stated otherwise.

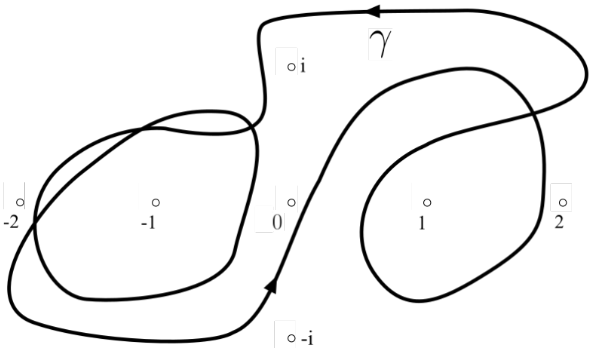
1. * Compute the residues of the following functions.

   (a) $\cot z$ at 0,

   (b)* $\frac{1}{\cos z+1}$ at π.

2. Evaluate along the smooth curve γ given below the contour integral of $f(z)$ for the following functions.

   (a) $f(z) = \dfrac{3z+1}{z^2+z-2}$, (b) $f(z) = e^{1/z}$, (c) $f(z) = \csc(\pi z)$.



3. * Use the method of residues to compute the integral

$$\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2},$$

   where $-1 < a < 1$.

4. Use the method of residues to compute the following improper integrals.

   (a) $\displaystyle\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx$, (b)* $\displaystyle\int_{-\infty}^\infty \frac{x\sin x}{(x^2+1)(x^2+4)}dx$.

   (c) $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{x-\pi}dx$, (d)* $\displaystyle\int_0^\infty \frac{1}{1+x^4}dx$,

   (e) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}(x^2+9)}dx$, (f)* $\displaystyle\int_0^\infty \frac{\sqrt[3]{x}}{(x+1)^3}dx$.

5. Locate the principal branch cut for $\log(z^2+1)$. Use the method of residues to derive the following integration formula.

$$\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}dx = \pi \ln 2.$$

6. * Show that if u is harmonic on a domain $W \subset \mathbb{R}^2$, the complex derivative $\frac{\partial U}{\partial z}$ of the complex function $U(x+iy) = u(x,y)$ on $W \subset \mathbb{C}$ is holomorphic on W.

7. * Show that the function

$$u(x,y) = \frac{y}{x^2+y^2}$$

is harmonic everywhere except at $(0,0)$.

8. Prove Liouville's theorem for harmonic functions: every bounded harmonic function on $\mathbb{R}^2$ is constant.

9. * Prove a variant of the coincidence principle for harmonic functions: whenever two harmonic functions u_1 and u_2 on a domain $U \subset \mathbb{R}^2$ satisfy $u_1 \equiv u_2$ on some non-empty open subset $V \subset U$, then $u_1 \equiv u_2$ on U. (Hint: refer to Qn. 6 above and/or Problem set 4 Qn. 2.)

10. Let f be a holomorphic function on a domain U and let g be a harmonic function where $g(x,y) = |f(x+iy)|^2$ whenever $x+iy \in U$. Show that f is a constant function.

11. Let $0 < r < 1$ and $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

    (a) Show that the Laurent series expansion for the function $w \mapsto r/(w-r)$ on the domain $\{|w| > |r|\}$ is $\sum_{n=1}^{\infty} r^n w^{-n}$,

    (b) By considering the real and imaginary parts of the Laurent expansion above, prove the following identities:

$$\frac{r(\cos\theta - r)}{1-2r\cos\theta + r^2} = \sum_{n=1}^{\infty} r^n \cos n\theta, \qquad \frac{r\sin\theta}{1-2r\cos\theta+r^2} = \sum_{n=1}^{\infty} r^n \sin n\theta.$$

    (c) Hence, show that the Poisson kernel has the following cosine series expansion:

$$P(r,\theta) = 1 + 2\sum_{n=1}^{\infty} r^n \cos n\theta.$$

12. Find the unique harmonic function $u(r, \theta)$ on the unit disk satisfying the following boundary conditions[1].

(a) $\lim_{r \to 1} u(r, \theta) = \begin{cases} 1, & \text{if } 0 \leq \theta < \frac{\pi}{2}, \\ 0, & \text{if otherwise.} \end{cases}$

(b) $\lim_{r \to 1} u(r, \theta) = \cos \theta.$

[1] Perhaps it may be easier to solve parts (c) and (d) using the cosine series expansion of $P(r, \theta)$ in Exercise 7.