From Herman Rings to Herman Curves

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Abstract of the Dissertation

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Given a holomorphic map on a Riemann surface, an invariant set on which the map is conjugate to irrational rotation is topologically equivalent to either a disk (*Siegel disk*), an annulus (*Herman ring*), or a Jordan curve (*Herman curve*). The last one is the least understood. The goal of this dissertation is threefold:

1. We obtain *a priori bounds* for a family of rational maps with Herman rings that are independent of the conformal moduli. This is done via careful analysis of near-degenerate surfaces in the spirit of Kahn, Lyubich, and D. Dudko. As a major application, we study the limits of degenerating Herman rings and obtain the first examples of Herman curves with bounded type rotation number which are not equivalent to round circles.

2. We study the rigidity properties of rational maps admitting bounded type Herman curves that we constructed. We also prove a rigidity theorem for critical quasicircle maps, i.e. analytic self-homeomorphisms of a quasicircle with a single critical point. This implies dynamical universality and exponential convergence of renormalizations towards a horseshoe attractor.

3. We prove the hyperbolicity of renormalization periodic points of critical quasicircle maps by developing an operator called *Corona Renormalization*, a doubly connected version of *Pacman Renormalization* for Siegel disks. The proof is inspired by rigidity results on the escaping dynamics of transcendental entire functions.

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Chapter 1

Introduction

The Fatou set $F(f)$ of a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree at least two is defined as the set of points in the Riemann sphere $\hat{\mathbb{C}}$ around which the set of iterates of f is equicontinuous, whereas the Julia set $J(f)$ is the complement $\hat{\mathbb{C}} \backslash F(f)$. Fatou's classification states that every periodic component of the Fatou set of a rational map f must be either a basin of attraction of a periodic point, or a *rotation domain*, i.e. a domain in which the first return map is conjugate to an irrational rotation.

A rational map f is called *hyperbolic* if every critical point of f is attracted towards an attracting periodic cycle. Hyperbolic maps are very well understood. For example, it is well known that the Julia set of a hyperbolic rational map has zero Lebesgue measure; in layman's terms, there is 0% probability of picking a random point in $\hat{\mathbb{C}}$ at which f is locally chaotic. The following is a central conjecture in rational dynamics, dating back to Fatou.

Density of Hyperbolicity Conjecture. *For any integer $d \geq 2$, hyperbolic maps are dense in the space* Rat_d *of all degree d rational maps.*

The existence of rotation domains is a clear obstruction to hyperbolicity, which makes it worth studying. Every rotation domain is either simply connected, in which case it is called a *Siegel disk*, or doubly connected, in which case it is called a *Herman ring*. Rational maps admitting Siegel disks can be found on the boundary of hyperbolic components of Rat_d. In contrast, the location of rational maps admitting Herman rings is more mysterious.

Siegel disks have been actively studied in the last few decades. In the second half of the last century, the study of local dynamics near a neutral fixed point has essentially received a complete treatment by the works of Brjuno, Herman, Yoccoz, and Perez-Marco. At the same time, the semi-local theory for Siegel disks of quadratic maps $e^{2\pi i\theta}z + z^2$ started with the introduction of Douady-Ghys surgery [Dou87; Ghy84], which was based on the work of Herman and Świątek [Her86; Świ88]. In contrast, the absence of periodic points associated to

Herman rings makes them more difficult to study than Siegel disks. The construction of the first examples of Herman rings was based on the study of linearizability of analytic circle diffeomorphisms by Arnol'd and Herman. A more general construction was later established by Shishikura. In [Shi87], Shishikura developed surgery procedures to construct Herman rings out of two Siegel disks, and to convert Herman rings into Siegel disks.

Given an irrational number $\theta \in (0,1)$ with continued fraction expansion

$$\theta = [0; a_1, a_2, a_3, \dots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}},$$

we say that θ is *of bounded type* if a_n's are uniformly bounded above, *pre-periodic* if there are positive integers m and p such that $a_n = a_{n+p}$ for all $n \geq m$, and *periodic* if additionally $m = 1$. We will denote corresponding spaces by Θ_{bdd}, Θ_{pre} and Θ_{per} respectively.

Unlike Siegel disks, every rational map admitting a Herman ring H admits a non-trivial moduli space arising from the deformation of the complex structure of H. In the first half of this dissertation, we develop a machinery to obtain geometric bounds on such a moduli space and control the degeneration near the boundary. We will later discuss how in some cases (and we expect this to be true in general), the limit of degenerating Herman rings with bounded type rotation number must be a *Herman curve.*

Definition 1.0.1. A periodic Jordan curve $\mathbf{H}$ of a rational map f is called a *Herman curve* if the first return map of f on $\mathbf{H}$ is conjugate to irrational rotation, and $\mathbf{H}$ is not contained in the closure of a rotation domain.

A Jordan curve $\mathbf{H} \subset \hat{\mathbb{C}}$ is called a $(K$-$)$*quasicircle* if it is the image of the unit circle under a $(K$-$)$quasiconformal homeomorphism of $\hat{\mathbb{C}}$. Under the bounded type assumption, the boundary components of the Herman rings as well as the limiting Herman curves in consideration will all be K-quasicircles, and the control of the dilatation K is precisely our *a priori bounds.* As a major application, we obtain examples of *critical quasicircle maps* which are not equivalent to round circles.

Definition 1.0.2. A $($*uni-*$)$*critical quasicircle map* is an orientation-preserving self homeomorphism $f : \mathbf{H} \to \mathbf{H}$ of a quasicircle which extends to a holomorphic map on a neighborhood of $\mathbf{H}$ and has exactly one critical point on $\mathbf{H}$.

For trivial reasons, we are exclusively concerned with the case when the rotation number θ of f is irrational. There are two special well-studied special cases: the first is critical circle maps when $\mathbf{H} = S^1$), and the second is Siegel maps when $\mathbf{H}$ is the boundary of a Siegel disk. Our method of degeneration of Herman rings yields the existence of many more critical quasicircle maps beyond these two special cases. The first work on critical quasicircle maps in full generality was done by Petersen [Pet96] who proved Denjoy distortion estimates and showed that f is quasisymmetrically conjugate to irrational rotation if and only if θ is of bounded type. Though we will primarily be working in the bounded type regime, let us note that such estimates imply that, for any irrational θ, f is topologically conjugate to irrational rotation.

Various classes of one-dimensional dynamical systems exhibit remarkable universal properties. The two main examples of universality include the golden mean universality phenomena empirically observed in smooth families of critical circle maps by Feigenbaum et al. [FKS82] and Östlund et al. [Öst+83], as well as the Feigenbaum-Coullet-Tresser universality observed in unimodal maps [Fei78; Fei79; TC78; CT79]. Both cases have been successfully justified via renormalization theory. In the second half of this dissertation, we show that our class of dynamical systems also exhibit universality. We initiate the study of renormalization theory of critical quasicircle maps, extending the classical renormalization theory of critical circle maps, via various old and new techniques in complex dynamics.

1.1 Main results

The main results of this dissertation are captured in three separate articles [Lim23a; Lim23b; Lim24]. Let us summarize them in greater detail. Throughout, we will fix a pair of integers $d_0, d_\infty \geq 2$ and set $d := d_0 + d_\infty - 1$. Any irrational number $\theta \in (0,1)$ in consideration will always be assumed to be of bounded type.

1.1.1 A priori bounds of Herman rings

Douady-Ghys surgery procedure can be applied to prove that bounded type Siegel disks of quadratic maps are quasidisks containing a critical point on the boundary. Such a result was generalized by Zakeri [Zak99] for cubic polynomials, by Shishikura for polynomials of arbitrary degree, and ultimately by Zhang [Zha11] for rational maps. Moreover, Zhang proved *a priori bounds* for Siegel disks: the dilatation of the boundary of every invariant Siegel disk with rotation number $\theta \in \Theta_{bdd}$ of a rational map f depends only on the degree of f and the bound

$$\beta(\theta) := \max_i a_i < \infty$$

of the continued fraction expansion $[0; a_1, a_2, \dots]$ of θ.

By Shishikura's surgery, Zhang's results also translate to Herman rings as follows. Every boundary component of a bounded type invariant Herman ring $\mathbb{H}$ of a rational map f is a

K-quasicircle containing a critical point, where the dilatation K depends only on the degree of f, the bound $\beta(\theta)$ associated to its rotation number θ, and the modulus of $\mathbb{H}$. We will develop the machinery to remove the dependence of the modulus and obtain *a priori bounds*.

Definition 1.1.1. We define $\mathcal{H} = \mathcal{H}_{d_0,d_\infty,\theta}$ to be the space of all degree $d_0 + d_\infty - 1$ rational maps f such that

(I) 0 and ∞ are superattracting fixed points of f with local degrees $d_0 \geq 2$ and $d_\infty \geq 2$ respectively;

(II) the map f admits an invariant Herman ring $\mathbb{H}$ with rotation number $\theta \in \Theta_{bdd}$;

(III) $\mathbb{H}$ separates 0 and ∞;

(IV) every critical point of f other than 0 and ∞ lies on the boundary of $\mathbb{H}$.

The space $\mathcal{H}$ encapsulates general Herman rings of the simplest configuration that can be obtained from Shishikura's surgery: they can be constructed out of two polynomials having unique invariant Siegel disks satisfying conditions similar to (IV). The existence and rigidity of maps in $\mathcal{H}$ of any prescribed combinatorics are guaranteed by a Thurston-type result by Wang [Wan12]. The following is our first main theorem.

Theorem A (*A priori bounds*)**.** *The boundary components of the Herman ring of every map in $\mathcal{H}$ are quasicircles with dilatation depending only on d_0, d_∞, and $\beta(\theta)$.*

The proof of this theorem is achieved in the **Near-Degenerate Regime**. We adapt the vocabulary developed by D. Dudko and Lyubich [DL22] and reduce the theorem to studying a family of near-degenerate surfaces. We prove by contradiction, apply tools such as the Quasi-Additivity Law and the Covering Lemma [KL05], and adapt the strategy in Kahn's seminal work [Kah06] on *a priori bounds* for infinitely renormalizable quadratic maps with bounded primitive combinatorics. A more comprehensive summary can be found in §1.2.1 and §1.3.

1.1.2 Existence and rigidity of Herman curves

Let us view $\mathcal{H}$ as a subspace of the space Rat_d of degree d rational maps endowed with the topology of uniform convergence on compact subsets. Denote the corresponding limit space by

$$\partial\mathcal{H} := \overline{\mathcal{H}} \backslash \mathcal{H} \subset \text{Rat}_d.$$

One consequence of Theorem A is that as maps in $\mathcal{H}$ approach $\partial\mathcal{H}$, the corresponding Herman rings must degenerate to a *Herman quasicircle*, i.e. a Herman curve that is also a quasicircle.

Corollary B. *$\partial\mathcal{H}$ is contained in the space $\mathcal{X}$ of degree d rational maps f such that*

(i) 0 *and* ∞ *are superattracting fixed points of f with local degrees $d_0 \geq 2$ and $d_\infty \geq 2$ respectively;*

(ii) *the function f admits a Herman quasicircle $\mathbf{H}$ of rotation number θ;*

(iii) $\mathbf{H}$ *separates* 0 *and* ∞*;*

(iv) *every critical point of f other than 0 and ∞ lies in $\mathbf{H}$;*

(v) *the conjugacy between $f|_{\mathbf{H}}$ and $R_\theta|_{\mathbb{T}}$ is quasisymmetric with dilatation depending only on d_0, d_∞ and $\beta(\theta)$.*

The combinatorics of a map f in $\mathcal{X}$ is encoded by the relative position of critical points along the Herman quasicircle of f and their inner and outer local degrees. All (topologically) admissible combinatorial data can be identified with points in the space $\mathcal{C} = \mathcal{C}_{d_0,d_\infty}$, a compact connected real orbifold of dimension $d-2$. (See Definition 3.1.5.)

Let us denote $f \sim g$ when two rational maps f and g are conjugate by a linear map $z \mapsto \lambda z$. Denote by $[f]$ the linear conjugacy class of a rational map f. The combinatorics $\mathrm{comb}(f) \in \mathcal{C}$ of a map in $\mathcal{X}$ is invariant under linear conjugacy.

Theorem C (Realization and combinatorial rigidity)**.** *The two spaces $\mathcal{X}$ and $\partial\mathcal{H}$ are equal. The map*

$$\mathcal{X}/_{\sim} \to \mathcal{C}, \quad [f] \mapsto \mathrm{comb}(f)$$

is a homeomorphism. In other words, given any prescribed combinatorics, there exists a rational map in $\mathcal{X}$ having a Herman quasicircle that realizes such combinatorics, and if two maps in $\mathcal{X}$ have the same combinatorics, then they are conformally conjugate.

Part of this theorem states that any combinatorics can be realized, and this is a consequence of *a priori bounds*. The rest of the theorem describes the combinatorial rigidity of $\mathcal{X}$, and this is proven by showing that every map f in $\mathcal{X}$ admits no *invariant line field* on its Julia set.

An *invariant line field* μ of a rational map f can be defined as a measurable Beltrami differential $\mu(z)\frac{d\bar{z}}{dz}$ such that $f^*\mu = \mu$ almost everywhere, $|\mu| = 1$ on a positive measure subset of $J(f)$, and $\mu = 0$ elsewhere. The absence of line fields implies the lack of non-trivial deformation space supported on the Julia set. A central rigidity conjecture [McM94; MS98] in rational dynamics states that *flexible Lattés maps* are the only rational maps that admit invariant line fields. This conjecture implies the Density of Hyperbolicity Conjecture.

Remark 1.1.2. Actually, our techniques hold in a much more general setting. In [Lim23b], we also show that our proof of the absence of invariant line fields is applicable to rational maps admitting multiple bounded type Siegel disks, Herman rings, and Herman curves. To maintain coherence, we omit the discussion of such generalizations here.

Theorem C states that the space of rational maps admitting bounded type Herman quasicircles of the simplest configuration forms the boundary of the moduli space of rational maps admitting bounded type Herman rings of the simplest configuration.

Prior to this dissertation, it was not known if there exists a Herman curve of a rational map that is not a round circle (or quasiconformally conjugate to such). Theorem C allows us to construct Herman curves of arbitrary asymmetric combinatorics.

1.1.3 Rigidity of critical quasicircle maps

Consider a critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$. (Refer to Definition 1.0.2.) The behaviour at the unique critical point on $\mathbf{H}$ can be encoded by two positive integers, namely the inner criticality d_0 and the outer criticality d_∞. The total local degree of f at the critical point is $d = d_0 + d_\infty - 1$ and it is at least 2. When the criticalities are specified, we call $f : \mathbf{H} \to \mathbf{H}$ a (d_0, d_∞)-*critical* quasicircle map.

By Theorem C, there exists a unique rational map $f = f_{d_0,d_\infty,\theta}$ in $\mathcal{X}$ admitting a unique critical point $z = 1$ on its Herman curve $\mathbf{H}$. By elementary computation, such a map f is of the form

$$F_c(z) := -c \, \frac{\sum_{j=d_0}^{d} \binom{d}{j} \cdot (-z)^j}{\sum_{j=0}^{d_0-1} \binom{d}{j} \cdot (-z)^j}$$

for some unique $c = F_c(1) \in \mathbb{C}^*$. See Figure 1.1 for some explicit examples. The map $f : \mathbf{H} \to \mathbf{H}$ will serve as our prototypical example of a (d_0, d_∞)-critical quasicircle map.

Beyond the realm of rational maps, it turns out that we still have a strong rigidity property for critical quasicircle maps. Given a constant $\alpha > 0$, we say that a map ϕ is *uniformly* $C^{1+\alpha}$*-conformal* on a set $S \subset \mathbb{C}$ if there are constants $C, \varepsilon > 0$ such that for every point z in S, the complex derivative $\phi'(z)$ at z exists and for $|t| < \varepsilon$,

$$\left| \frac{\phi(z+t) - \phi(z)}{t} - \phi'(z) \right| \leq C|t|^\alpha. \tag{1.1.1}$$

Theorem D ($C^{1+\alpha}$ rigidity)**.** *Any two* (d_0, d_∞)*-critical quasicircle maps* $f_1 : \mathbf{H}_1 \to \mathbf{H}_1$ *and* $f_2 : \mathbf{H}_2 \to \mathbf{H}_2$ *of the same bounded type rotation number are quasiconformally conjugate on the neighborhood of* $\mathbf{H}_1$ *and* $\mathbf{H}_2$*. Moreover, there is some* $\alpha > 0$ *such that the conjugacy is uniformly* $C^{1+\alpha}$*-conformal on* $\mathbf{H}_1$*.*

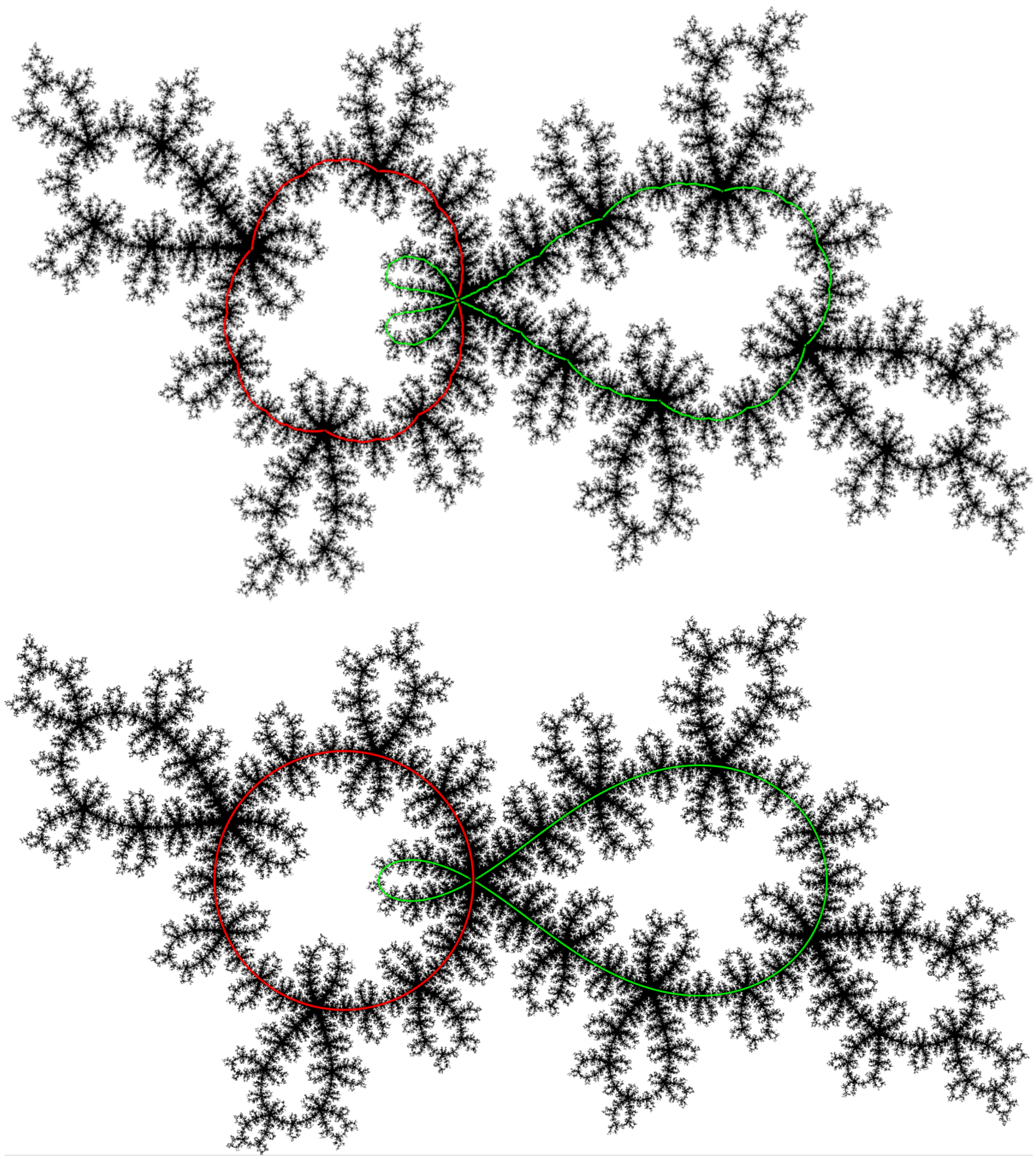

Figure 1.1: The Julia sets of

$$f(z) = bz^3 \frac{4 - z}{1 - 4z + 6z^2} \quad \text{and} \quad g(z) = cz^2 \frac{z - 3}{1 - 3z}.$$

The critical values $b \approx -1.144208 - 0.964454i$ and $c \approx -0.755700 - 0.654917i$ are picked such that $f : \mathbf{H} \to \mathbf{H}$ is a $(3,2)$-critical quasicircle map on some quasicircle $\mathbf{H}$, $g : \mathbb{T} \to \mathbb{T}$ is a $(2,2)$-critical circle map, and both have the golden mean rotation number. Both $\mathbf{H}$ and $\mathbb{T}$ are colored red, and their preimages are colored green.

In the proof, we study the renormalizations $\{\mathcal{R}^n f\}_{n\geq 1}$ of a critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$. They can be described as follows. Let $\{p_n/q_n\}_{n\geq 1}$ denote the best rational approxiations of the rotation number of f. For every $n \geq 1$, denote by I_n the shortest interval in $\mathbf{H}$ connecting the critical point c and $f^{q_n}(c)$. The n^{th} *pre-renormalization* of f is the commuting pair

$$(f^{q_n}|_{I_{n-1}}, f^{q_{n-1}}|_{I_n}),$$

which is the first return map of f to the interval $I_{n-1} \cup I_n \subset \mathbf{H}$. The n^{th} *renormalization* $\mathcal{R}^n f$ of f is the normalized commuting pair obtained by rescaling the n^{th} pre-renormalization to unit size. We can also define the renormalization of commuting pairs in a way such that $\mathcal{R}^k(\mathcal{R}^l f) = \mathcal{R}^{k+l} f$ for all $k, l \geq 0$.

The proof of Theorem D consists of two main ingredients. The first is *complex bounds* of renormalizations, which roughly states that the sequence $\{\mathcal{R}^n f\}$ is precompact. The second is an adaptation of McMullen's recipe [McM96], namely uniform twisting and deep points. McMullen's recipe was originally applied in the context of Feigenbaum Julia sets, but it has also been successfully applied in the study of rigidity of critical circle maps [FM99] as well as multicritical circle maps [GY21]. Let us list a number of important applications.

The presence of critical points generally destroys the smoothness of rotation curves. One consequence of Theorem D is that the corresponding quasicircle cannot be smooth except when it admits symmetric combinatorics, or equivalently, when the map is quasiconformally conjugate to a critical circle map.

Corollary E (Smoothness)**.** *Given a (d_0, d_∞)-critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$ of bounded type rotation number, the following are equivalent.*

(1) $\mathbf{H}$ *is $C^{1+\alpha}$ smooth;*

(1') $\mathbf{H}$ *is C^1 smooth at at least one point;*

(2) *the Hausdorff dimension of* $\mathbf{H}$ *is one;*

(3) $d_0 = d_\infty$.

See Figure 4.8 for an example of a C^1 smooth Herman curve that is not a round circle. Under combinatorial asymmetry, the dimension is also universal.

Corollary F (Universality of dimension)**.** *If two (d_0, d_∞)-critical quasicircle maps $f_1 : \mathbf{H}_1 \to \mathbf{H}_1$ and $f_2 : \mathbf{H}_2 \to \mathbf{H}_2$ have the same bounded type rotation number, then $\mathbf{H}_1$ and $\mathbf{H}_2$ have the same Hausdorff dimension, lower box dimension, and upper box dimension.*

Consider a critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$ with rotation number $\theta \in \Theta_{bdd}$. The dynamics of f determines the asymptotic geometry of the quasicircle $\mathbf{H}$ as follows. Denote by c the critical point of f and by $\{p_n/q_n\}_{n\in\mathbb{N}}$ the best rational approximations of θ. We define the n^{th} *scaling ratio* of f by

$$s_n(f) := \frac{f^{q_{n+1}}(c) - c}{f^{q_n}(c) - c}. \tag{1.1.2}$$

Corollary G (Universality of scaling ratios)**.** *If two (d_0, d_∞)-critical quasicircle maps $f_1 : \mathbf{H}_1 \to \mathbf{H}_1$ and $f_2 : \mathbf{H}_2 \to \mathbf{H}_2$ have the same bounded type rotation number, then asymptotically they have the same scaling ratios:*

$$\frac{s_n(f_2)}{s_n(f_1)} \to 1 \text{ exponentially fast as } n \to \infty.$$

Moreover, when θ is pre-periodic, the asymptotic geometry has the following remarkable property.

Theorem H (Self-similarity)**.** *Consider a (d_0, d_∞)-critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$ with a pre-periodic rotation number $\theta \in \Theta_{pre}$. Then, $\mathbf{H}$ is asymptotically self-similar about the critical point. The self-similarity factor is universal depending only on d_0, d_∞, and θ.*

On the other hand, $C^{1+\alpha}$ also allows us to study the dynamics of the renormalization operator. Let us fix a positive integer N, and denote by Θ_N the set of irrationals in $(0,1)$ whose terms in the continued fraction expansion are bounded above by N.

Theorem I (Renormalization horseshoe)**.** *There is a renormalization-invariant compact set $\mathcal{A}_N$ inside the space $\mathcal{CP}_N$ of normalized commuting pairs of fixed criticality (d_0, d_∞) and of rotation number in Θ_N with the following properties.*

(1) *The renormalization operator $\mathcal{R} : \mathcal{A}_N \to \mathcal{A}_N$ is topologically conjugate to the shift operator on the bi-infinite shift space of N symbols.*

(2) *For any ζ in $\mathcal{CP}_N$, the distance between $\mathcal{R}^n\zeta$ and $\mathcal{A}_N$ tends to 0 exponentially fast as $n \to \infty$.*

A precise version of the two theorems above can be found in Theorems 4.6.6, 4.6.7 and 4.6.8.

1.1.4 Hyperbolicity of renormalization

A Siegel map is a holomorphic map admitting an invariant quasiconformal closed Siegel disk with a critical point on the boundary. One of the most recent achievements in the renormalization theory of Siegel maps is the development of *pacman renormalization operator* by Dudko, Lyubich, and Selinger [DLS20]. Such an operator admits a hyperbolic fixed point whose stable manifold has codimension one and consists of Siegel maps with a fixed rotation number of periodic type. One remarkable feature of pacmen is that every pacman on the unstable manifold admits a global transcendental analytic extension. Techniques of transcendental dynamics were successfully adapted in [DL23] to study the escaping dynamics on the unstable manifold, which ultimately led to a progress in MLC and in finding new examples of positive area Julia sets.

For critical quasicircle maps, we develop a renormalization operator acting on the space of *coronas*, a doubly-connected version of pacmen. A corona is a holomorphic map $f: U \to V$ between two nested annuli $U \Subset V$ such that $f: U\backslash\gamma_0 \to V\backslash\gamma_1$ is a unicritical branched covering map where γ_1 is an arc connecting the two boundary components of V. The number of preimages of γ_1 on the boundary components of U determine the inner and outer criticalities d_0 and d_∞ of a corona; the total degree of f is equal to d. When the criticalities are specified, we call f a (d_0, d_∞)-critical corona. See Figure 5.1 for an illustration.

Similar to pacman renormalization, we define the corona renormalization operator as follows. First, we remove the quadrilateral bounded by γ_1 and $f(\gamma_1)$. The remaining space is a quadrilateral in which the first return map will be called a *pre-corona*. Gluing a pair of opposite sides of this quadrilateral gives us a new corona, which is called the *prime corona renormalization* $\mathcal{R}_{\text{prm}}f$ of f. A general corona renormalization operator $\mathcal{R}$ is an iterate of the prime corona renormalization.

We say that a (d_0, d_∞)-critical corona f is *rotational* with rotation number θ if it admits a Herman quasicircle $\mathbf{H}$ with rotation number θ. If rotational, $f: \mathbf{H} \to \mathbf{H}$ defines a (d_0, d_∞)-critical quasicircle map. The prime renormalization of a (d_0, d_∞)-critical rotational corona is again a (d_0, d_∞)-critical rotational corona, and the induced action on the rotation number is governed by

$$r_{\text{prm}}(\theta) = \begin{cases} \dfrac{\theta}{1-\theta}, & \text{if } 0 \leq \theta \leq \dfrac{1}{2}, \\ \dfrac{2\theta - 1}{\theta}, & \text{if } \dfrac{1}{2} \leq \theta \leq 1. \end{cases}$$

Every periodic type $\theta \in \Theta_{per}$ is a periodic point of r_{prm}.

Theorem J (Hyperbolicity)**.** *For any $\theta \in \Theta_{per}$, there exists a corona renormalization operator $\mathcal{R}: \mathcal{U} \to \mathcal{B}$ with the following properties.*

(1) *$\mathcal{U}$ is an open subset of a Banach analytic manifold $\mathcal{B}$ consisting of (d_0, d_∞)-critical coronas.*

(2) *$\mathcal{R}$ is a compact analytic operator with a unique fixed point f_* which is hyperbolic.*

(3) *The local stable manifold $\mathcal{W}^s_{\text{loc}}$ of f_* corresponds to the space of rotational coronas with rotation number θ in $\mathcal{B}$.*

(4) *The local unstable manifold $\mathcal{W}^u_{\text{loc}}$ is one-dimensional.*

Similar to [DLS20], the main step is justifying item (4), which will be accomplished via transcendental dynamics. Anti-renormalizations of a corona f on the local unstable manifold can be projected to a single dynamical plane and admit a maximal transcendental extension $\mathbf{F}$ called a *cascade* associated to f. A cascade can be described as a collection $\{\mathbf{F}^P\}_{P\in\mathbf{T}}$ of σ-proper maps parametrized by a dense semigroup $\mathbf{T} \subset (\mathbb{R}_{\geq 0}, +)$ such that $\mathbf{F}^P \circ \mathbf{F}^Q = \mathbf{F}^{P+Q}$. The second half of this paper is dedicated to the study of the dynamics of $\mathbf{F}$.

Similar to invariant line fields of rational maps, we can define an invariant line field of a cascade $\mathbf{F}$ to be a measurable Beltrami differential $\mu(z)\frac{d\bar{z}}{dz}$ such that $(\mathbf{F}^P)^* \mu = \mu$ almost everywhere for all P, $|\mu| = 1$ on a positive measure set, and $\mu = 0$ elsewhere. The existence of an invariant line field μ indicates the existence of a non-trivial deformation space for $\mathbf{F}$ associated to the support of μ. To justify (4), we prove a rigidity theorem for cascades $\mathbf{F}$.

Theorem K (Rigidity of escaping dynamics on $\mathcal{W}^u_{\text{loc}}$)**.** *Consider a cascade $\mathbf{F}$ associated to a corona f in $\mathcal{W}^u_{\text{loc}}$. The full escaping set*

$$\mathbf{I}(\mathbf{F}) := \left\{ z \in \mathbb{C} : \quad \textit{either } z \notin \bigcap_P \mathrm{Dom}\left(\mathbf{F}^P\right) \ \textit{ or } \mathbf{F}^P(z) \to \infty \ \textit{ as } P \to \infty \right\}$$

moves conformally away from the pre-critical points and supports no invariant line field. Consequently, if $\mathbf{F}$ has an attracting cycle, then the Julia set of $\mathbf{F}$ supports no invariant line field.

One may compare this theorem to Rempe's result [Rem09] on the rigidity of the escaping set of transcendental entire functions. Our methods allow for an analog of Theorem K in other settings, such as pacman and period-doubling renormalization fixed points. Ultimately,

$$\text{Theorem K} \implies \dim\left(\mathcal{W}^u_{\text{loc}}\right) \leq \text{number of critical orbits} \leq 1 \implies \text{Theorem J(4)}.$$

We would like to note some of the differences between our case and the pacmen case [DLS20; DL23].

Firstly, the existence of a non-attracting direction for pacman renormalization is straightforward. Unlike coronas, every pacman is designed to admit a natural fixed point α associated to it. For a Siegel pacman, the α-fixed point is the center of its Siegel disk. The multiplier λ

of α clearly induces a non-attracting eigenvalue at the pacman renormalization fixed point. In the corona case, this is not obvious, and we show it as an application of Theorem C.

Secondly, the proof of item (4) for pacmen does not require such a rigidity theorem. After obtaining the transcendental structure, it is immediate that λ induces a natural foliation $\{W^u(\lambda)\}_\lambda$ of the unstable manifold of the pacman renormalization fixed point. By applying the λ-lemma along parabolic leaves, the authors showed that

$$\dim \left(W^u \left(p/q\right)\right) \le \text{number of free critical orbits in } W^u \left(p/q\right) = 0$$

where $p/q \in \mathbb{Q}$ is sufficiently close to $\theta \in \Theta_{per}$.

Thirdly, the original aim of the study of the finite-time escaping set associated to the transcendental extension of pre-pacmen was to attain a puzzle structure, which was ultimately applied to understand the dynamics of maps on the unstable manifold and transfer the results to the quadratic family $\{z^2 + c\}_c$. In our case, the full escaping set $\mathbf{I}(\mathbf{F})$ is of interest because, together with the postcritical set, it is the measure-theoretic attractor of $\mathbf{F}$ on the Julia set.

Let us briefly discuss an immediate application of Theorem J. Given a critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$, we can define a Banach neighborhood $N(f)$ of f as follows. Pick a skinny annular neighborhood U of $\mathbf{H}$ such that f is holomorphic on a neighborhood of U, and pick a small $\varepsilon > 0$. Then, $N(f)$ is the space of unicritical holomorphic maps $g : U \to \mathbb{C}$ such that g extends continuously to the boundary of U and $\sup_{z\in U} |f(z) - g(z)| < \varepsilon$, equipped with the sup norm.

Corollary L (Structure of conjugacy classes)**.** *Consider a small Banach neighborhood $N(f)$ of a (d_0, d_∞)-critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$ with pre-periodic rotation number θ. The space S of maps in $N(f)$ which restrict to a (d_0, d_∞)-critical quasicircle map with rotation number θ forms an analytic submanifold of $N(f)$ of codimension at most one. The corresponding invariant quasicircle moves holomorphically over S.*

1.2 Historical notes

1.2.1 On the near-degenerate regime

The idea that compactness results are amenable for near-degenerate surfaces goes back to the work of W. Thurston on the geometry of 3-manifolds. (See, for instance, the Double Limit Theorem [Thu86].) In complex dynamics, the near-degenerate regime was successfully implemented in the proof of W. Thurston's characterization of postcritically finite rational maps [DH93].

In mid 2000's, Kahn [Kah06] introduced the near-degenerate regime to the Renormalization Theory of quadratic-like maps. Together with Lyubich, they set up fundamental tools, such as the Quasi-Additivity Law and the Covering Lemma [KL05], and attained substantial progress in the primitive case of the MLC conjecture [KL08; KL09a]. Other applications of the Covering Lemma include the extension of Yoccoz's results and puzzle-parapuzzle machinery to higher degrees [KL09b; Avi+09; KS09; ALS11]. (See also [Cla+22] for a detailed exposition.)

Recently, D. Dudko and Lyubich [DL22] transferred the near-degenerate regime to neutral dynamics of quadratic polynomials $e^{2\pi i\theta}z+z^2$: they constructed almost invariant *pseudo-Siegel disks* out of bounded type Siegel disks by filling in fjords at all scales, and showed that the top level pseudo-Siegel disks are quasidisks with uniform dilatation. Even though the above instances of the near-degenerate regime are unified by the same general principle, they have little in common on the technical level.

1.2.2 On critical circle maps

A *critical circle map* is a critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$ where $\mathbf{H}$ is simply the unit circle $\mathbb{T} \subset \mathbb{C}$. By symmetry, it is clear that in this case, the inner criticality d_0 must coincide with the outer criticality d_∞.

The renormalization theory of critical circle maps serves to justify the golden mean universality phenomena empirically observed in smooth families of critical circle maps by Feigenbaum et al. [FKS82] and Östlund et al. In both works, the golden mean universality was translated into a conjecture on the hyperbolicity of the renormalization operator on the space of critical commuting pairs. The conjecture was later generalized by various authors, in particular Lanford [Lan88] who introduced renormalization horseshoes to account for more complicated universalities. Below, we provide a brief historical summary of the development of the theory.

In [Far99], de Faria introduced the notion of *holomorphic commuting pairs* and proved the universality of scaling ratios and the existence of renormalization horseshoe for critical circle maps with bounded type rotation number. $C^{1+\alpha}$ rigidity was later established by de Faria and de Melo [FM99] for bounded type rotation number, and then by Khmelev and Yampolsky [KY06] for arbitrary irrational rotation number by studying parabolic bifurcations. Moreover, Yampolsky extended the horseshoe for all irrational rotation numbers in [Yam01], and brought Lanford's program to completion in [Yam02; Yam03] using *cylinder renormalization*.

Theorem 1.2.1 (Hyperbolicity of renormalization horseshoe [Yam03])**.** *The renormalization operator $\mathcal{R}$ in the space of critical commuting pairs admits a "horseshoe" attractor $\mathcal{A}$ on which* *its action is conjugated to the two-sided shift. Moreover, there exists an $\mathcal{R}$-invariant space of critical commuting pairs with the structure of an infinite dimensional smooth manifold, with respect to which $\mathcal{A}$ is a hyperbolic set with one-dimensional expanding direction.*

1.2.3 On Siegel maps

In this dissertation, we assume that both d_0 and d_∞ are always at least two. What happens to a critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$ if either d_0 or d_∞ is one? In the bounded type regime, this is equivalent to the statement that $\mathbf{H}$ is the boundary of a rotation domain. By Douady-Ghys surgery, it can be assumed that $\mathbf{H}$ is the boundary of a Siegel disk, that is, f is a *Siegel map*.

Stirnemann [Sti94] first gave a computer-assisted proof of the existence of a renormalization fixed point with a golden mean Siegel disk. McMullen [McM98] applied a measurable deep point argument to prove the existence of renormalization horseshoe for bounded type rotation number. In [AL22, §4], Avila and Lyubich established quasiconformal rigidity of bounded type Siegel maps; via McMullen's method, the regularity can be improved to $C^{1+\alpha}$.

Using the formalism of *almost commuting pairs*, Gaidashev and Yampolsky [Yam08; GY22] gave a computer-assisted proof of the golden mean hyperbolicity of renormalization of Siegel disks. As previously mentioned, Dudko, Lyubich, and Selinger [DLS20] constructed a compact analytic operator, called *pacman renormalization operator*, with a hyperbolic fixed point whose stable manifold has codimension one and consists of Siegel maps with a fixed rotation number of periodic type.

1.3 Organization

This dissertation is split into six chapters.

Chapter 2: Preliminaries

This chapter generally provides a range of preliminary background material for the main results. In Section §2.1, we cover in depth the dynamics of rigid irrational rotation R_θ, including sector renormalization and the induced cascades of translations. The map R_θ is a toy model of a general rotation curve of a holomorphic map. In Section §2.2, we discuss the fundamentals of extremal width and the main tools for the proof of *a priori bounds*, namely the Quasi-Additivity Law and the Covering Lemma [KL05]. In Section §2.3, we state and prove an upgraded version of Lyubich's Small Orbits Theorem [Lyu99, §2], which is a vital ingredient in our hyperbolicity theorem. The main addition here is the application of two invariant cones rather than just one.

Chapter 3: A priori bounds

Sections §3.1–3.7 are dedicated solely to the proof of Theorem A, which is done via the near-degenerate regime inspired by [Kah06; DL22]. In §3.1, we adapt the vocabulary of [DL22] and encode the near-degeneracy near an interval I on the boundary of a Herman ring as the extremal width $W_\tau(I)$ of some curve family, and the main step is to prove the Amplification Theorem 3.7.1, which roughly states that

$$W_\tau(I) = K \gg 1 \text{ for some interval I} \quad \Longrightarrow \quad W_\tau(J) \geq 2K \text{ for some interval J}$$

with constants independent of the modulus. The proof of the Amplification Theorem is captured in Sections §3.2–3.7. The heart of the argument resembles Kahn's *a priori bounds* [Kah06], although a few modifications are needed. For example, in Kahn's setting, little Julia sets are invariant under the first return map and the associated Hubbard tree has positive entropy; meanwhile, our intervals I are not precisely invariant, and the action of f on the Herman ring has zero entropy. A more technical and more detailed outline of the proof can be found in Section §3.1.5.

In Section §3.8, we show that *a priori bounds* provide sufficient pre-compactness for Herman rings of small moduli. This allows us to take limits of degenerating Herman rings and prove the realization part of Theorem C.

Chapter 4: Rigidity

We then move on to the rigidity problem in Chapter 4. In Section §4.1, we discuss the property of critical quasicircle maps $f : \mathbf{H} \to \mathbf{H}$ that an iterate of f locally behaves like a rotation (an *approximate rotation*) until it lands near the critical point. This property is the basis of our main analysis in later sections. In Section §4.2, we prove that every rational map f satisfying (i)–(v) does not admit any invariant line field supported on its Julia set $J(f)$. The main tool is Theorem 4.2.4, which is an analog of [McM94, Theorem 3.2]. By means of the standard pullback argument, we then complete the proof of Theorem C in Section §4.3.

The second part of Chapter 4 concerns with the study of renormalizations of a critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$. In Section §4.4.2, we argue that f is a conformal welding of a pair of *quasicritical circle maps*, which are a quasiregular analog of critical circle maps. Based on [AL22, §3], most results on critical circle maps, such as complex bounds and quasiconformal rigidity, hold for quasicritical circle maps of bounded type. In Section §4.4.3, we introduce the concept of *butterflies*, an analog of holomorphic commuting pairs, and transfer complex bounds for quasicritical circle maps to complex bounds in our setting (Theorems 4.4.16 and 4.4.18). By a pullback argument, we then show that complex bounds imply quasiconformal rigidity. To complete the proof of Theorem D, we show that our quasiconformal conjugacy is $C^{1+\alpha}$-conformal via McMullen's Dynamic Inflexibility Theorem [McM96, Theorem 9.15].

In Section §4.6, we prove a number of applications of $C^{1+\alpha}$ rigidity. Our universality results are immediate consequences of Theorem D. The proof of Corollary E uses an additional tool, which is Peter Jones' beta numbers (see Proposition 4.6.3). The construction of renormalization horseshoe is a standard tower rigidity argument. Lastly, Theorem H follows from self-similarity of the invariant quasicircle of each of the renormalization periodic points in the horseshoe.

Chapter 5: Hyperbolicity of Renormalization

This chapter concerns with the hyperbolicity of renormalization periodic points of critical quasicircle maps. In Section §5.1, we introduce the definition of *coronas* and *pre-coronas*. We define the corona renormalization operator and show that for any renormalizable corona f, we can always find a compact analytic operator $\mathcal{R}$ on a small Banach neighborhood of f. In Section §5.2, we analyze the structure of a rotational corona and prove that any critical quasicircle map can be renormalized to a rotational corona. By applying Theorem D, we also show that rotational coronas are quasiconformally rigid.

In Section §5.3, we construct a compact analytic corona renormalization operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ and a corona $f_* \in \mathcal{U}$ of periodic rotation number such that $\mathcal{R} f_* = f_*$. In Theorem 5.3.9, we prove that $\mathcal{R}$ and f_* satisfy items (2) and (3) in Theorem J, and that the dimension of the local unstable manifold $\mathcal{W}^u_{\text{loc}}$ is finite and positive. To show that $D\mathcal{R}_{f_*}$ has no neutral eigenvalues, we require an upgraded version of the Small Orbits Theorem 2.3.1. We then apply Theorem C to show that a repelling direction exists.

The rest of the chapter, namely §5.4–5.8, is dedicated to proving that $D\mathcal{R}_{f_*}$ has exactly one repelling eigenvalue. In Section §5.4, we show that for any f on the local unstable manifold, the maximal extension of the pre-corona associated to f is a commuting pair of σ-proper maps $\mathbf{F} = (\mathbf{f}_\pm : \mathbf{X}_\pm \to \mathbb{C})$. The general dynamical features of $\mathbf{F}$, i.e. escaping dynamics and Fatou-Julia theory, are described in Section §5.5.

In Section §5.6, we describe in detail the transcendental dynamics of the renormalization fixed point $\mathbf{F}_*$. We construct external rays and deduce its tree structure using their branch points, which are called *alpha-points*. These rays define dynamical wakes which form a puzzle structure partitioning the whole dynamical plane. Appropriately truncated wakes survive under perturbation, and we use them in §5.7 to study the motion of points z in the Julia set whose orbit $\mathbf{F}^P(z)$ remain close to ∞ for all P. In Section §5.8, we then prove Theorem K via an argument similar to [Rem09]. Lastly, we show that there exist hyperbolic cascades $\mathbf{F}$ arbitrarily close to $\mathbf{F}_*$. When $\mathbf{F}$ is hyperbolic, the Julia set of $\mathbf{F}$ is the union of $\mathbf{I}(\mathbf{F})$ and a zero measure set of non-escaping points, which implies that hyperbolic components on the unstable manifold must be one-dimensional.

Chapter 6: Questions and conjectures

In the final chapter, we discuss a number of questions and conjectures regarding Herman rings, Herman curves, critical quasicircle maps, and renormalization in some generality.

Chapter 2

Preliminaries

2.1 Rotational dynamics

In this section, we discuss the fundamental properties of the dynamics of irrational rotation. Subsections §2.1.1 and §2.1.2 set the foundation for the entire dissertation. In §2.1.3, we describe a general renormalization operator on irrational rotation, whereas in §2.1.4, we discuss the structure of renormalization cascades arising from irrational rotation.

2.1.1 Rigid rotation

Consider an irrational number $\theta \in (0,1)$. Let us identify $\mathbb{T}$ with the quotient $\mathbb{R}/\mathbb{Z}$, in which the rigid rotation R_θ by θ can be written as $R_\theta(x) = x + \theta$. For any pair of distinct points $x, y \in \mathbf{H}$, we denote by $[x, y]$ the shortest closed interval in $\mathbb{T}$ having endpoints x and y.

Let $\{p_n/q_n\}_{n\in\mathbb{N}}$ be the sequence of best rational approximations of θ. This sequence is determined by the recurrence relation

$$p_n = a_n p_{n-1} + p_{n-2} \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2}$$

where $p_0 = q_{-1} = 0$, $q_0 = p_{-1} = 1$, and $[0; a_1, a_2, \ldots]$ is the continued fraction expansion of θ. The q_n's are precisely the first return times for R_θ which alternate in the following fashion. For any $x \in \mathbb{T}$,

$$R_\theta^{q_1}(x) < R_\theta^{q_3}(x) < R_\theta^{q_5}(x) < \ldots < x < \ldots < R_\theta^{q_6}(x) < R_\theta^{q_4}(x) < R_\theta^{q_2}(x).$$

Definition 2.1.1. An interval $I \subset \mathbf{H}$ is a *level n combinatorial interval* if it is of the form $[x, R_\theta^{q_n}(x)]$ for some $x \in \mathbb{T}$.

Let us denote the length of a level n interval by

$$l_n := |p_n - q_n\theta|. \tag{2.1.1}$$

Proposition 2.1.2. *For any $c \in \mathbb{T}$ and $n \in \mathbb{N}$, the collection of combinatorial intervals*

$$\mathcal{P}_n(c) := \left\{[R_\theta^i(c), R_\theta^{q_n+i}(c)]\right\}_{i=0}^{q_{n+1}-1} \cup \left\{[R_\theta^{q_{n+1}+j}(c), R_\theta^j(c)]\right\}_{j=0}^{q_n-1}$$

forms a tiling of $\mathbb{T}$, that is, they have pairwise disjoint interiors and their union is $\mathbb{T}$. Moreover, $\mathcal{P}_{n+1}(c)$ is a refinement of $\mathcal{P}_n(c)$.

Recall that θ is of *bounded type* if there is a uniform bound on the terms a_n in its continued fraction expansion $[0; a_1, a_2, \ldots]$. If so, we denote the optimal bound by

$$\beta(\theta) := \max_{i \geq 1} a_i.$$

For any positive integer N, we define the set Θ_N to be the set of bounded type irrationals $\theta \in (0,1)$ satisfying $\beta(\theta) \leq N$. The bounded type assumption controls the rate of decrease of the lengths in (2.1.1).

Proposition 2.1.3. *If θ is in Θ_N, there exists a pair of constants $\tilde{C}, C > 1$ depending only on N such that for every positive integer n,*

$$\tilde{C} l_{n+1} \leq l_n \leq C l_{n+1}.$$

We will use the following lemma several times later.

Lemma 2.1.4. *Suppose θ is in Θ_N and S is a finite subset of $\mathbb{T}$. There is some constant $\varepsilon > 0$ depending only on N and $|S|$ such that for all $n \in \mathbb{N}$, every combinatorial interval $I \subset \mathbb{T}$ of level n contains a subinterval $J \subset I$ of length $|J| \geq \varepsilon l_n$ that is disjoint from $\bigcup_{i=0}^{q_{n+2}-1} R_\theta^i(S)$.*

Proof. By Proposition 2.1.2, for every $c \in S$, the finite orbit $\mathcal{O}_c = \{R_\theta^i(c)\}_{i=0,\ldots q_{n+2}-1}$ partitions $\mathbb{T}$ into intervals of lengths between l_{n+2} and l_n. By Proposition 2.1.3, the number of points in $\bigcup_{c \in S} \mathcal{O}_c$ that lie within I is at most some constant K depending only on N and $|S|$. Therefore, there is a subinterval of I of length at least l_n/K that satisfies the desired property. □

2.1.2 Rotation curves

Consider a rotation curve $\mathbf{H}$ of a holomorphic map f. By definition, there exists a topological conjugacy $\phi : \mathbf{H} \to \mathbb{T}$ between f and R_θ where θ is the rotation number. There is a unique normalized metric on $\mathbf{H}$ that is invariant under f, which we call the *combinatorial metric*. This can be constructed by pushing forward the Euclidean metric via ϕ^{-1}.

Let us assume that $\mathbf{H}$ is embedded in the Riemann sphere $\hat{\mathbb{C}}$. Label the two components of $\hat{\mathbb{C}} \backslash \mathbf{H}$ by Y^0 and Y^∞.

Definition 2.1.5. We say that a point $x \in \mathbf{H}$ is an *inner critical point* if for any point y in Y^0 sufficiently close to $f(x)$, the number d_0 of points in $f^{-1}(y) \cap Y^0$ near x is at least two. The quantity d_0 is called the *inner criticality* of x. Similarly, $x \in \mathbf{H}$ is called an *outer critical point* with *outer criticality* d_∞ if for any point y in Y^∞ sufficiently close to $f(x)$, the number d_∞ of points in $f^{-1}(y) \cap Y^\infty$ near x is at least two.

Consider the set $\mathscr{H} \subset \mathbb{R} \backslash \mathbb{Q}$ of *Herman numbers.* The set $\mathscr{H}$ has full Lebesgue measure and is characterized by a rather complicated arithmetic condition that was devised by Herman and Yoccoz [Her79; Yoc02] as the optimal condition for an analytic circle diffeomorphism to be analytically linearizable. Here, we only need the property that $\mathscr{H}$ contains the set of bounded type irrationals.

Proposition 2.1.6 (Trichotomy of rotation curves)**.** *Suppose $\theta \in \mathscr{H}$. Exactly one of the following holds.*

- (A) $\mathbf{H}$ *is an analytic curve contained in a rotation domain of f.*
- (B) $\mathbf{H}$ *is a boundary component of a rotation domain of f and contains a either an inner critical point or an outer critical point, but not both.*
- (H) $\mathbf{H}$ *is a Herman curve containing an inner critical point and an outer critical point.*

Proof. It is clear that if both inner and outer critical points are present, then $\mathbf{H}$ must be a Herman curve. Below, we will assume that $\mathbf{H}$ contains no inner critical points.

There is an annulus $W \subset Y^0$ such that $\mathbf{H}$ is one of the boundary components of W and f is univalent on W. Since $f|_{\mathbf{H}}$ is conjugate to a rotation, the image $Z := f(W)$ is again an annulus contained in Y^0 with $\mathbf{H}$ being one of its boundary components. Pick a conformal isomorphism $\psi : Y^0 \to \mathbb{D}$ and define the univalent map $F := \psi \circ f \circ \psi^{-1}$ from $\psi(W)$ to $\psi(Z)$. By Schwarz reflection, F extends continuously to a univalent map $F : W' \to Z'$ where W' and Z' are the smallest $\mathbb{T}$-symmetric annuli containing $\psi(W)$ and $\psi(Z)$ respectively.

The map F restricts to an analytic circle diffeomorphism with rotation number θ. Since θ is a Herman number, $F|_{\mathbb{T}}$ must be analytically linearizable, so F admits a $\mathbb{T}$-symmetric Herman ring. By pulling back this Herman ring via ψ, we obtain an invariant annulus $A^0 \subset W$ such that $\mathbf{H}$ is a boundary component of A^0 and $f|_{A^0}$ is analytically conjugate to R_θ. Denote by A the rotation domain of f containing A^0.

If $\mathbf{H}$ contains an outer critical point, then $f|_{\mathbf{H}}$ cannot be analytically conjugate to R_θ and $\mathbf{H}$ has to be a boundary component of A. Otherwise, by the same argument, there is an annulus $A^\infty \subset Y^\infty$ such that $\mathbf{H}$ is a boundary component of A^∞ and $f|_{A^\infty}$ is analytically conjugate to R_θ. Hence, $A^0 \cup \mathbf{H} \cup A^\infty$ lies in A and $\mathbf{H}$ is an invariant analytic curve. □

The trichotomy breaks when $\theta \notin \mathscr{H}$. For instance, there exist cubic rational maps admitting Herman curves of arbitrary non-Herman irrational rotation number containing no critical points. See [BF14, Proposition 6.6] and [Yan22].

Throughout this dissertation, any rotation curve $\mathbf{H}$ in consideration will be assumed to be a quasicircle. The following is a generalization of the Herman-Świątek theorem.

Theorem 2.1.7 ([Pet04]). *Suppose $\mathbf{H}$ is a quasicircle containing a critical point of f (hence either Ⓑ or Ⓗ). The rotation number θ is of bounded type if and only if there exists a quasiconformal map $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\phi(\mathbf{H}) = \mathbb{T}$ and $f = \phi^{-1} \circ R_\theta \circ \phi$ in $\mathbf{H}$.*

When the conjugacy ϕ is quasisymmetric, we can transfer what is known in the combinatorial metric back to $\mathbf{H}$ as a subset of $\hat{\mathbb{C}}$, equipped with the spherical metric.

Lemma 2.1.8. *Suppose θ is in Θ_N and the conjugacy $\phi : \mathbf{H} \to \mathbb{T}$ is K-quasisymmetric. For every point c on $\mathbf{H}$,*

(1) *the tilings $\mathcal{P}_n(c)$ have bounded geometry, that is, the diameters of any two adjacent tiles of the same level, or any two consecutive nested tiles, are comparable with a constant depending only on N and K;*

(2) *there are positive constants $C, \varepsilon_1, \varepsilon_2$ depending only on N and K such that $\varepsilon_1 < \varepsilon_2 < 1 < C$ and for every $n \geq 2$,*
$$C^{-1}\varepsilon_1^n \leq \frac{|f^{q_n}(c) - c|}{\operatorname{diam}(\mathbf{H})} \leq C\varepsilon_2^n.$$

2.1.3 Sector renormalization

Let us identify $\mathbb{T}$ with the standard unit circle in $\mathbb{C}$ with the induced intrinsic metric. Given two points x and y on $\mathbb{T}$, we denote by $[x, y] \subset \mathbb{T}$ the shortest closed interval with endpoints x and y. Consider the rotation
$$R_\theta : \mathbb{T} \to \mathbb{T}, z \mapsto e^{2\pi i\theta} z$$
for some fixed $\theta \in \mathbb{R}/\mathbb{Z}$. Let us fix a point $x \in \mathbb{T}$ and consider
$$X_- := [R_\theta^{-1}(x), x], \quad Y := [x, R_\theta(x)], \quad X_+ := \overline{\mathbb{T} \backslash (Y \cup X_-)}.$$
The first return map on $X_- \cup X_+$ is precisely the commuting pair
$$(R_\theta|_{X_+},\ R_\theta^2|_{X_-}),$$
Let us assume that $1 \neq Y$ and denote by ω the length of $X_- \cup X_+$. Then, the map $z \mapsto z^{1/\omega}$ projects the commuting pair to a new rotation $R_{r_{\mathrm{prm}}(\theta)}$ called the *prime renormalization* of R_θ. Note that $R_{r_{\mathrm{prm}}(\theta)}$ is independent of the initial choice of x.

Lemma 2.1.9 ([DLS20, Lemma A.1])**.** *We have*

$$r_{\mathrm{prm}}(\theta) = \begin{cases} \dfrac{\theta}{1-\theta}, & \text{if } 0 \le \theta \le \dfrac{1}{2}, \\ \dfrac{2\theta - 1}{\theta}, & \text{if } \dfrac{1}{2} \le \theta \le 1. \end{cases}$$

In general, we define a *sector renormalization* $\mathcal{R}(R_\theta)$ of R_θ as follows. First, consider a pair of intervals X_- and X_+ on $\mathbb{T}$ satisfying $X_- \cap X_+ = \{1\}$. Suppose the first return map on $X := X_- \cup X_+$, which we call a sector pre-renormalization, is a pair of the form

$$(R_\theta^{\mathbf{a}}|_{X_-},\ R_\theta^{\mathbf{b}}|_{X_+}), \tag{2.1.2}$$

for some positive integers $\mathbf{a}$ and $\mathbf{b}$ called the *renormalization return times* of $\mathcal{R}$. The map $z \mapsto z^{1/\omega}$, where ω is the length of X, glues the endpoints of X together and projects the pair (2.1.2) to a new rotation $R_\mu = \mathcal{R}(R_\theta)$.

Example 2.1.10. Recall that the Gauss map G sends an irrational $x = [0; a_1, a_2, \ldots]$ to another irrational $G(x) = [0; a_2, a_3, \ldots]$. Often, we are interested in the n^{th} *standard* renormalization operator sending R_θ to $R_{G^n\theta}$. This can be constructed as follows. Let $\{p_n/q_n\}_{n\ge1}$ denote the best rational approximations of θ. If we choose $X_- = [R_\theta^{q_{n-1}}(1), 1]$ and $X_+ = [1, R_\theta^{q_n}(1)]$, then the associated sector pre-renormalization $(R_\theta^{q_n}|_{X_-},\ R_\theta^{q_{n-1}}|_{X_+})$ projects onto the rotation $R_{G^n(\theta)}$.

Lemma 2.1.11 ([DLS20, Lemma A.2])**.** *Sector renormalization $\mathcal{R}$ is an iteration of the prime renormalization. In particular, $\mu = (r_{\mathrm{prm}})^m(\theta)$ for some $m \ge 1$, and R_θ is a fixed point of some sector renormalization if and only if $\theta \in \Theta_{per}$.*

Under the universal cover $\mathbb{R} \to \mathbb{T}, z \mapsto e^{-2\pi i z}$, the rotation R_θ can be lifted to the commuting pair of translations

$$T_{-\theta} : z \mapsto z - \theta, \qquad T_{1-\theta} : z \mapsto z + 1 - \theta.$$

The deck transformation $\chi := T_1$ is equal to $T_{1-\theta} \circ T_{-\theta}^{-1}$, and the original rotation R_θ can be recovered from the quotient map $T_{-\theta}/\langle\chi\rangle$.

Consider a general commuting pair of translations $(T_{-\mathbf{u}}, T_{\mathbf{v}})$ where $\mathbf{u}, \mathbf{v} \in \mathbb{R}_{\ge0}$. The prime renormalization $\mathcal{R}_{\mathrm{prm}}$ of $(T_{-\mathbf{u}}, T_{\mathbf{v}})$ is the new commuting pair $(T_{-\mathbf{u}_1}, T_{\mathbf{v}_1})$ where

$$(T_{-\mathbf{u}_1}, T_{\mathbf{v}_1}) := \begin{cases} (T_{-\mathbf{u}} \circ T_{\mathbf{v}},\quad T_{\mathbf{v}}\quad) & \text{if } \mathbf{u} \ge \mathbf{v}, \\ (\quad T_{-\mathbf{u}}\quad,\ T_{-\mathbf{u}} \circ T_{\mathbf{v}}) & \text{if } \mathbf{u} < \mathbf{v}. \end{cases} \tag{2.1.3}$$

Set $\chi := T_{\mathbf{v}} \circ T_{-\mathbf{u}}^{-1}$ and $\chi_1 = T_{\mathbf{v}_1} \circ T_{-\mathbf{u}_1}^{-1}$. The prime renormalization of pairs of translations is equivalent to that of rotations in the following sense.

Lemma 2.1.12. *If $T_{-\mathbf{u}}/\langle\chi\rangle \equiv R_\theta$, then*

$$\theta = \frac{\mathbf{v}}{\mathbf{u}+\mathbf{v}} \quad \textit{and} \quad T_{-\mathbf{u}_1}/\langle\chi_1\rangle \equiv R_{r_{\mathrm{prm}}(\theta)}.$$

2.1.4 Cascade of translations

By writing $(-\mathbf{u}, \mathbf{v})$ as a column vector, the transformation in (2.1.3) is represented by either $I^- := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ if $\mathbf{u} \geq \mathbf{v}$ or $I^+ := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ if $\mathbf{u} < \mathbf{v}$. Consider the region $\mathbb{R}_{\leq 0} \times \mathbb{R}_{\geq 0}$, which is split equally into two sectors by the diagonal line $\{x + y = 0\}$. The lower sector is mapped by I^- onto $\mathbb{R}_{\leq 0} \times \mathbb{R}_{\geq 0}$, whereas the upper sector is mapped by I^+ onto $\mathbb{R}_{\leq 0} \times \mathbb{R}_{\geq 0}$.

From now on, suppose θ is of periodic type. There exists some $m > 0$ such that $(r_{\mathrm{prm}})^m(\theta) = \theta$. Set $\mathbf{u} = \theta$ and $\mathbf{v} = 1 - \theta$. By (2.1.3), there is a unique matrix 2×2 matrix $\mathbf{M}$ of the form $I_1 I_2 \ldots I_m$, where $I_i \in \{I^+, I^-\}$ for all i, such that the m^{th} prime renormalization $(T_{-\mathbf{u}_1}, T_{\mathbf{v}_1}) := (\mathcal{R}_{\mathrm{prm}})^m(T_{-\mathbf{u}}, T_{\mathbf{v}})$ satisfies

$$\begin{pmatrix} -\mathbf{u}_1 \\ \mathbf{v}_1 \end{pmatrix} = \mathbf{M} \begin{pmatrix} -\mathbf{u} \\ \mathbf{v} \end{pmatrix}.$$

The matrix $\mathbf{M}$ is an element of the modular group $\mathrm{SL}_2(\mathbb{Z})$ mapping a sector in $\mathbb{R}_{\leq 0} \times \mathbb{R}_{\geq 0}$ onto $\mathbb{R}_{\leq 0} \times \mathbb{R}_{\geq 0}$. The condition $(r_{\mathrm{prm}})^m(\theta) = \theta$ implies that $\begin{pmatrix} -\mathbf{u}_1 \\ \mathbf{v}_1 \end{pmatrix}$ is a scalar multiple of $\begin{pmatrix} -\mathbf{u} \\ \mathbf{v} \end{pmatrix}$. We conclude that $\mathbf{M}$ has two eigenvalues $\mathbf{t} > 1$ and $1/\mathbf{t}$, and that

$$\begin{pmatrix} -\mathbf{u}_1 \\ \mathbf{v}_1 \end{pmatrix} = \frac{1}{\mathbf{t}} \begin{pmatrix} -\mathbf{u} \\ \mathbf{v} \end{pmatrix}.$$

We call $\mathbf{M}$ the *anti-renormalization matrix* associated with θ.

Observe that $\mathbf{M}$ has to be a matrix of positive integers and $\mathbf{t} \notin \mathbb{Q}$. For $n \in \mathbb{N}$, we write

$$\mathbf{u}_n := \mathbf{t}^{-n}\mathbf{u} \quad \text{and} \quad \mathbf{v}_n := \mathbf{t}^{-n}\mathbf{v}.$$

We then obtain a full pre-renormalization tower $\{(T_{-\mathbf{u}_n}, T_{\mathbf{v}_n})\}_{n \in \mathbb{Z}}$ where

$$(\mathcal{R}_{\mathrm{prm}})^m(T_{-\mathbf{u}_n}, T_{\mathbf{v}_n}) = (T_{-\mathbf{u}_{n+1}}, T_{\mathbf{v}_{n+1}}).$$

Given $(n, a, b) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, let us write

$$T^{(n,a,b)} := T^a_{-\mathbf{u}_n} \circ T^b_{\mathbf{v}_n} = T_{\mathbf{t}^{-n}(b\mathbf{v} - a\mathbf{u})}.$$

Lemma 2.1.13. *Given a pair of elements (n, a, b) and (n', a', b') of $\mathbb{Z} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$,*

$$T^{(n,a,b)} = T^{(n',a',b')} \quad \textit{if and only if} \quad (a\ b)\mathbf{M}^n = (c\ d)\mathbf{M}^{n'}.$$

Definition 2.1.14. We define the space $\mathbf{T}$ of *power-triples* to be the quotient of the semigroup $\mathbb{Z} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ under the equivalence relation $\sim$ where $(n, a, b) \sim (n-1, a', b')$ if and only if $(a' \; b') = (a \; b)\mathbf{M}$.

We will equip $\mathbf{T}$ with the binary operation $+$ defined by

$$(n, a, b) + (n, a', b') = (n, a + a', b + b').$$

With respect to $+$, $\mathbf{T}$ has a unique identity element $0 := (n, 0, 0)$. Thus, $(\mathbf{T}, +)$ still has the structure of a semigroup. According to Lemma 2.1.13, $\mathbf{T}$ acts freely on $\mathbb{R}$ as a cascade of translations $(T^P)_{P \in \mathbf{T}}$.

Lemma 2.1.15 ([DL23, Lemma 2.2])**.** *There is an embedding $\iota : \mathbf{T} \to \mathbb{R}$ such that $\iota(n-1, a, b) = \mathbf{t}^{-1}\iota(n, a, b)$. Identifying $\mathbf{T}$ with $\iota(\mathbf{T}) \subset \mathbb{R}$ equips $\mathbf{T}$ with*

(1) *a linear order $\geq$, which can be described as follows: $P \geq Q$ if and only if for sufficiently large $n \ll 0$, we can write $P = (n, a, b)$ and $Q = (n, a', b')$ where $a \geq a'$ and $b \geq b'$;*

(2) *subtraction, that is, if $P, T \in \mathbf{T}$ and $P \geq T$, then $P - T \in \mathbf{T}$;*

(3) *scalar multiplication by $\mathbf{t}$: $P = (n, a, b) \mapsto \mathbf{t}P = (n+1, a, b)$, which is an automorphism of $\mathbf{T}$.*

Moreover, for $P \in \mathbf{T}$, $n \in \mathbb{Z}$, and $x \in \mathbb{R}$,

$$T^P(x) = \mathbf{t}^n \cdot T^{\mathbf{t}^n P}\left(\mathbf{t}^{-n} x\right).$$

If T^P is a translation by $l > 0$, then $T^{\mathbf{t}^n P}$ is a translation by $\mathbf{t}^{-n} l$. The following observation is immediate.

Lemma 2.1.16 (Proper discontinuity)**.** *If $P \in \mathbf{T}_{>0}$ is small, then $|T^P(0)|$ is large.*

For all $P \in \mathbf{T}$, let us denote $b_P := T^{-P}(0)$. We say that b_P is *dominant* if every b_Q on $[0, b_P]$ satisfies $Q \geq P$. By proper discontinuity, we can enumerate all dominant points $\{b_{P_n}\}_{n \in \mathbb{Z}}$ such that $P_n < P_{n+1}$ for all n.

Lemma 2.1.17 ([DL23, Lemma 2.4])**.** *For every $i \in \mathbb{Z}$, there exist some $Q_i \in \mathbf{T}_{>0}$ and some integers m, n such that $n < m \leq i$ and T^{Q_i} maps $[b_{P_i}, b_{P_{i+1}}]$ to $[b_{P_n}, b_{P_m}]$.*

2.2 Near-degenerate regime

We summarize the properties of extremal width (also known as conformal modulus, the reciprocal of extremal length). Extremal width is a conformal invariant and it plays a vital role in describing near-degenerate Riemann surfaces. The tools in this section will be applied throughout Chapter 3 in the context of Herman rings.

2.2.1 Extremal width

Given a family $\mathcal{G}$ of curves on a Riemann surface S, we denote by $W(\mathcal{G})$ the extremal width of $\mathcal{G}$. We list without proof a number of fundamental results on extremal width. (See [Ahl06] and the appendix in [KL05] for details.)

Proposition 2.2.1 (Parallel Law)**.** *For any two curve families $\mathcal{G}_1$ and $\mathcal{G}_2$,*

$$W(\mathcal{G}_1 \cup \mathcal{G}_2) \le W(\mathcal{G}_1) + W(\mathcal{G}_2).$$

Equality is achieved when $\mathcal{G}_1$ and $\mathcal{G}_2$ have disjoint support.

We say that a curve family $\mathcal{G}$ *overflows* another curve family $\mathcal{H}$, denoted by $\mathcal{H} < \mathcal{G}$, if every curve in $\mathcal{G}$ contains a curve in $\mathcal{H}$ (curves in $\mathcal{G}$ are longer and fewer). We also say that $\mathcal{H}$ is a *restriction* of $\mathcal{G}$ if $\mathcal{G}$ overflows $\mathcal{H}$ but not any proper subfamily of $\mathcal{H}$ (curves in $\mathcal{G}$ are longer, but not more nor fewer).

Denote by $x \oplus y$ the harmonic sum $(x^{-1} + y^{-1})^{-1}$.

Proposition 2.2.2 (Series Law)**.** *Suppose a curve family $\mathcal{G}$ overflows two disjoint curve families $\mathcal{G}_1$ and $\mathcal{G}_2$. Then,*

$$W(\mathcal{G}) \le W(\mathcal{G}_1) \oplus W(\mathcal{G}_2).$$

The following proposition allows us to convert harmonic sums into friendlier expressions.

Proposition 2.2.3. *For any positive numbers $a_1, \ldots, a_n$,*

$$\bigoplus_{i=1}^{n} a_i \le \min\left\{a_1, \ldots, a_n, \frac{1}{n}\max_i a_i, \frac{1}{n^2}\sum_{i=1}^{n} a_i\right\}.$$

Extremal width is invariant under conformal maps. More generally, we have the following transformation rule.

Proposition 2.2.4. *Let $f : U \to V$ be a holomorphic map between two Riemann surfaces and $\mathcal{G}$ be a family of curves in U. Then,*

$$W(f(\mathcal{G})) \le W(\mathcal{G}).$$

If f is at most d to 1, then

$$W(\mathcal{G}) \le d \cdot W(f(\mathcal{G})).$$

A *(conformal) rectangle* P on a surface S is the image of a continuous map $\phi : [0,m] \times [0,1] \to \overline{S}$ that restricts to a conformal embedding in the interior. The vertical sides of a rectangle P are $\phi(\{0\} \times [0,1])$ and $\phi(\{m\} \times [0,1])$, and the horizontal sides of P are $\phi([0,m] \times \{0\})$ and $\phi([0,1] \times \{m\})$. A curve in P is called *vertical* if it connects the two horizontal sides of P. The *vertical foliation* of P is defined to be the collection of curves

$$\mathcal{F}(P) := \{\phi(\{t\} \times (0,1)) \mid t \in (0,m)\}.$$

The *width* of P is

$$W(P) := W(\mathcal{F}(P)) = m.$$

We say that P *crosses* a curve γ if every vertical curve in P intersects γ.

Proposition 2.2.5 (Non-Crossing Principle)**.** *If a pair of rectangles P_1 and P_2 on S has width $W(P_1), W(P_2) > 1$, then they never cross, i.e., there exist disjoint leaves $\gamma_1 \in \mathcal{F}(P_1)$ and $\gamma_2 \in \mathcal{F}(P_2)$.*

Suppose a rectangle P of width m has width greater than 8. The *buffers* of P are subrectangles of P of the form $\phi([0,\varepsilon) \times (0,1)$ or $\phi((m-\varepsilon, m] \times (0,1))$ for some $\varepsilon \le 4$. A direct consequence of the non-crossing principle is the following proposition.

Proposition 2.2.6 ([KL05, Lemma 2.14])**.** *Every pair of rectangles P_1 and P_2 of width greater than 8 admits subrectangles P_1^{new} and P_2^{new} obtained by removing some buffers such that P_1^{new} and P_2^{new} have disjoint vertical sides.*

When S has boundary, we say that a curve $\gamma : (0,1) \to S$ is *proper* if it has well-defined endpoints $\gamma(0)$ and $\gamma(1)$ contained in ∂S. For any disjoint subsets I and J of ∂S, we denote by $\mathcal{F}_S(I,J)$ and $W_S(I,J)$ the family of proper curves in S that connect I and J, and its width respectively. When S is a Jordan disk, the width $W_S(I,J)$ can be estimated as follows.

Proposition 2.2.7 (Log-Rule, [DL22, Lemma 2.5])**.** *Suppose S is a Jordan disk and suppose its boundary ∂S is partitioned into four intervals I_1, I_2, I_3, I_4, labelled cyclically. Denote by $|I_i|$ the harmonic measure of I_i in S about a point $x \in S$.*

(1) *If* $\min(|I_1|, |I_3|) \ge \min(|I_2|, |I_4|)$, *then*

$$W_S(I_1, I_3) \asymp \log \frac{\min\{|I_1|, |I_3|\}}{\min\{|I_2|, |I_4|\}} + 1;$$

(2) *Otherwise,*
$$W_S(I_1, I_3) \asymp \left(\log \frac{\min\{|I_2|, |I_4|\}}{\min\{|I_1|, |I_3|\}} + 1\right)^{-1}.$$

Given a compact subset I of S, we denote by $W(S, I)$ the extremal width of the family $\mathcal{F}(S, I)$ of proper curves in $S \backslash I$ connecting I and ∂S. We will formulate important near-degenerate tools from [KL05] in a way that is most suitable for our setting in Chapter 3.

Lemma 2.2.8 (Quasi-Additivity Law). *Suppose S is a topological disk in $\mathbb{C}$ and $A_1, \ldots, A_n$ be pairwise disjoint non-empty compact connected subsets of S. Let*
$$X := W\left(S, \bigcup_{i=1}^{n} A_i\right), \quad Y := \sum_{i=1}^{n} W(S, A_i), \quad Z_i := W\left(S \backslash \bigcup_{j \neq i} A_j, A_i\right) \text{ for } i = 1, \ldots, n.$$

Then, there exists some $K = K(n) > 0$ such that
$$Y \geq K \quad \Longrightarrow \quad \max\{X, Z_1, \ldots, Z_n\} \geq \frac{Y}{\sqrt{2n}}.$$

Lemma 2.2.9 (Covering Lemma). *Let $\Lambda \Subset \Lambda' \subset U$ and $B \Subset B' \subset V$ be two nests of simply connected domains and $f : (U, \Lambda', \Lambda) \to (V, B', B)$ be a branched covering map with degrees deg$(f : \Lambda' \to B') = d$ and deg$(f : U \to V) = D$. For all $\kappa > 1$, there is some $\mathbf{K} = \mathbf{K}(\kappa, D) > 0$ such that if $W(U, \Lambda) = K > \mathbf{K}$, then either*
$$W(B', B) > \kappa K, \qquad \text{or} \qquad W(V, B) > (2\kappa d^2)^{-1} K.$$

2.2.2 Canonical lamination

Consider an open hyperbolic Riemann surface S with a finite number of boundary components. We allow the presence of finitely many punctures, which are separate from the ideal boundary ∂S. We will survey the fundamental properties of the *canonical lamination* $\mathcal{F}_{\text{can}}(S)$ of S following Kahn's work [Kah06]. The canonical lamination captures the near-degeneracy of S induced by components of ∂S that are very close to one another. Let us first sketch the construction.

Let $\pi : \mathbb{D} \to S$ be the universal cover of S. Since ∂S is non-empty, the limit set $\Lambda \subset \partial \mathbb{D}$ of $\pi_1(S)$ is a Cantor set. For every component $\tilde{I} \subset$ of $\mathbb{D} \backslash \Lambda$, π extends continuously to a universal covering $\tilde{I} \to I$ for some component I of ∂S. Two proper curves γ_0 and γ_1 are *properly homotopic* in S if there is a homotopy γ_t, $t \in [0, 1]$ between γ_0 and γ_1 such that each γ_t is also a proper curve in S. An *arc* in S is a proper homotopy class of proper curves in S.

Consider a non-trivial arc α in S connecting two (not necessarily distinct) components I and J of ∂S. Let $\tilde{\alpha}$ be a lift of α under π; it connects $\tilde{I}$ and $\tilde{J}$, which are some lifts of I and J respectively. Let us identify $\mathbb{D}$ with the structure of a conformal rectangle with horizontal sides $\tilde{I}$ and $\tilde{J}$. Kahn observed that removing buffers of width 1 gives us a subrectangle that can be pushed forward by π to a new conformal rectangle $\mathcal{R}_{\text{can}}(S;\alpha)$ with horizontal sides contained in I and J.

The *canonical arc diagram* $\mathcal{A}_{\text{can}}(S)$ is the set of non-trivial arcs α in S such that the canonical rectangle $\mathcal{R}_{\text{can}}(S;\alpha)$ is non-empty. The removal of buffers in the construction ensures that these rectangles are pairwise disjoint. The cardinality of $\mathcal{A}_{\text{can}}(S)$ is at most a constant depending only on the Euler characteristic of S.

We define the *thick-thin decomposition* and the *canonical lamination* of S by

$$\text{TTD}(S) \coloneqq \bigcup_{\alpha\in\mathcal{A}_{\text{can}}(S)} \mathcal{R}_{\text{can}}(S;\alpha) \quad \text{and} \quad \mathcal{F}_{\text{can}}(S) \coloneqq \bigcup_{\alpha\in\mathcal{A}_{\text{can}}(S)} \mathcal{F}_{\text{can}}(S;\alpha),$$

respectively, where $\mathcal{F}_{\text{can}}(S;\alpha)$ is the vertical foliation of the canonical rectangle $\mathcal{R}_{\text{can}}(S;\alpha)$. Every leaf of $\mathcal{F}_{\text{can}}(S;\alpha)$ is represented by $\alpha \in \mathcal{A}_{\text{can}}(S)$. If a proper arc α is not in $\mathcal{A}_{\text{can}}(S)$, we set $\mathcal{F}_{\text{can}}(S;\alpha)$ to be the empty lamination.

Below, we list without proof a number of fundamental properties of the canonical lamination. Firstly, it is maximal in the following sense.

Proposition 2.2.10 ([Kah06, Lemma 3.2])**.** *For any proper family $\mathcal{F}$ of curves in S represented by a single arc α,*

$$W(\mathcal{F}) - 2 \leq W\left(\mathcal{F}_{\text{can}}(S;\alpha)\right).$$

In other words, up to an additive constant, curves in $\mathcal{F}$ are vertical curves inside of the rectangle $\mathcal{R}_{\text{can}}(S;\alpha)$.

Consider two hyperbolic Riemann surfaces U and V with boundary. The fact that the thick-thin decomposition is defined via the universal cover yields the following property.

Proposition 2.2.11 ([Kah06, Lemma 3.3])**.** *For any holomorphic covering map $f : U \to V$ of finite degree,*

$$\text{TTD}(U) = f^*\text{TTD}(V) \quad \textit{and} \quad \mathcal{F}_{\text{can}}(U) = f^*\mathcal{F}_{\text{can}}(V).$$

When $U \subset V$, the restriction of $\mathcal{F}_{\text{can}}(V)$ onto U results in a proper lamination in U. By Proposition 2.2.10, the width of this restriction will be bounded above by the canonical lamination of U after some buffers are removed. This can be formulated more precisely as follows.

Proposition 2.2.12 ([Kah06, Lemma 3.10]). *When $U \subset V$, there exists a sublamination $\mathcal{L} \subset \mathcal{F}_{\text{can}}(V)$ such that*

$$W(\mathcal{F}_{\text{can}}(V)) - C \leq W(\mathcal{L})$$

for some constant $C > 0$ depending only on the Euler characteristic of U with the following property. For every leaf γ of $\mathcal{L}$, every component of $\gamma \cap U$ is either

(1) *a homotopically trivial proper curve in U, or*

(2) *a vertical curve in $\mathcal{R}_{\text{can}}(U;\alpha)$ for some $\alpha \in \mathcal{A}_{\text{can}}(U)$.*

In application, the Riemann surface S we consider in §3.5–3.6 is of the form $U \backslash K$ where $U \subset \mathbb{C}$ is a disk and K is a non-empty compact subset of U. We say that a proper curve in $U \backslash K$ is *horizontal* if both of its endpoints are on K, and *vertical* if it connects a point on K and a point on ∂U. We define the *canonical horizontal* (resp. *vertical*) *lamination* $\mathcal{F}^h_{\text{can}}(U,K)$ (resp. $\mathcal{F}^v_{\text{can}}(U,K)$) on $U \backslash K$ to be the lamination consisting of all horizontal (resp. vertical) leaves of $\mathcal{F}_{\text{can}}(U \backslash K)$. Similarly, we define the *horizontal* and *vertical thick-thin decomposition* $\text{TTD}^h(U,K)$ and $\text{TTD}^v(U,K)$ of $U \backslash K$ respectively.

Let us fix a holomorphic map $f : U \to V$, where $U, V \subset \mathbb{C}$ are domains in $\hat{\mathbb{C}}$ and assume that it admits a rotation curve $\mathbf{H} \subset U$. Let $\phi : \mathbf{H} \to \mathbb{T}$ be a conjugacy between $f|_{\mathbf{H}}$ and the irrational rotation $R_\theta|_{\mathbb{T}}$. Note that ϕ is unique up to post-composition with any rotation.

2.3 Small Orbits Theorem

This section will applied in Chapter 5 §5.3 as a vital ingredient in the proof of Theorem J.

Consider a complex Banach space $\mathcal{B}$. Given a linear operator $L : \mathcal{B} \to \mathcal{B}$, denote the corresponding set of eigenvalues by $\text{spec}(L)$. We say that an eigenvalue $\lambda \in \text{spec}(L)$ is *attracting* if $|\lambda| < 1$, *neutral* if $|\lambda| = 1$, and *repelling* if $|\lambda| > 1$.

Theorem 2.3.1 (Small Orbits Theorem). *Let $\mathcal{R} : (\mathcal{U}, 0) \to (\mathcal{B}, 0)$ be a compact analytic operator on a neighborhood $\mathcal{U}$ of 0 in a complex Banach space $(\mathcal{B}, \|\cdot\|)$. If the differential $D\mathcal{R}_0 : \mathcal{B} \to \mathcal{B}$ has a neutral eigenvalue, then $\mathcal{R}$ has slow small orbits, that is, for any neighborhood $\mathcal{V}$ of 0, there is a forward orbit $\{\mathcal{R}^n g\}_{n\in\mathbb{N}}$ in $\mathcal{V}$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{R}^n g\| = 0.$$

In the absence of repelling eigenvalues of $D\mathcal{R}_0$, the theorem above was proven by Lyubich in [Lyu99, §2]. The original Small Orbits Theorem was a vital ingredient in the proof of hyperbolicity of quadratic-like renormalization horseshoe [Lyu99; Lyu02] and more recently the proof of hyperbolicity of pacman renormalization fixed points [DLS20].

Below we will generalize Lyubich's proof. The key addition is the application of two invariant cones, namely the center-stable cone $\mathcal{C}^{cs}$ and the center-unstable cone $\mathcal{C}^{cu}$.

Proof. Let $\mathcal{R}$ be as in the hypothesis. Denote the unit disk in $\mathbb{C}$ by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. We present the Banach space $\mathcal{B}$ as a direct sum

$$\mathcal{B} = E^s \oplus E^c \oplus E^u,$$

where subspaces E^s, E^c, E^u are invariant under $D\mathcal{R}_0$ and

$$\operatorname{spec}(D\mathcal{R}_0|_{E^s}) \subset \mathbb{D}, \quad \operatorname{spec}(D\mathcal{R}_0|_{E^c}) \subset \partial\mathbb{D}, \quad \operatorname{spec}(D\mathcal{R}_0|_{E^u}) \subset \mathbb{C}\backslash\overline{\mathbb{D}}.$$

Note that the spectrum can only accumulate at 0 because $\mathcal{R}$ is a compact operator. In particular, the subspace $E^c \oplus E^u$ must be finite dimensional. We will assume that each of the three subspaces have positive dimension. (Else, we are reduced to [Lyu99, §2].)

For $h \in \mathcal{B}$, we will write $h = h^s + h^c + h^u$, where for $a \in \{s, c, u\}$, h^a is the projection of h onto the subspace E^a. We will also denote $h^{cs} := h^c + h^c$ and $h^{cu} := h^c + h^u$. There exist an adapted norm $\|\cdot\|$ on $\mathcal{B}$ and some positive constants $\mu_s, \mu_{cs}, \mu_{cu}, \mu_u$ such that $\mu_s < 1 < \mu_u$, $\mu_s < \mu_{cu}$, $\mu_{cs} < \mu_u$, and

$$\begin{aligned}
&\|D\mathcal{R}_0 h\| \leq \mu_s \|h\| && \text{for all } h \in E^s,\\
&\|D\mathcal{R}_0 h\| \leq \mu_{cs} \|h\| && \text{for all } h \in E^{cs},\\
&\|D\mathcal{R}_0 h\| \geq \mu_{cu} \|h\| && \text{for all } h \in E^{cu},\\
&\|D\mathcal{R}_0 h\| \geq \mu_u \|h\| && \text{for all } h \in E^u.
\end{aligned}$$

The proof below will involve two fixed constants $\alpha > 1$ and $\delta > 0$ where δ is small. We consider a pair of cone fields C^{cu} and C^{cs} given by

$$C^{cu}_f = \{h \in T_f\mathcal{U} : \alpha\|h^s\| \leq \|h^{cu}\|\} \quad \text{and} \quad C^{cs}_f = \{h \in T_f\mathcal{U} : \alpha\|h^u\| \leq \|h^{cs}\|\} \tag{2.3.1}$$

for every $f \in \mathcal{U}$. For $a \in \{s, c, u\}$, we denote by $D^a = D^a(\delta)$ the open ball of radius δ centered at 0 in E^a. Let

$$\mathcal{D} := D^s \times D^c \times D^u$$

the corresponding open polydisk centered at 0 in $\mathcal{B}$. The boundary of $\mathcal{D}$ can be decomposed as follows:

$$\partial^s\mathcal{D} := \partial D^s \times D^c \times D^u, \qquad \partial^c\mathcal{D} := D^s \times \partial D^c \times D^u, \qquad \partial^u\mathcal{D} := D^s \times D^c \times \partial D^u.$$

Claim 1. For sufficiently small $\delta > 0$, the following properties hold.

1. If $f \in \overline{\mathcal{D}}$, then $\mathcal{R}f \notin \partial^s \mathcal{D}$;

2. If $f \in \partial^u \mathcal{D}$, then $\mathcal{R}f \notin \overline{\mathcal{D}}$;

3. The cone field C^{cu} is forward invariant: if $f, \mathcal{R}f \in \mathcal{D}$, then

$$D\mathcal{R}_f(C^{cu}_f) \subset C^{cs}_{\mathcal{R}f};$$

4. The cone field C^{cs} is backward invariant: if $f, \mathcal{R}f \in \mathcal{D}$, then

$$(D\mathcal{R}_f)^{-1}(C^{cs}_{\mathcal{R}f}) \subset C^{cs}_f.$$

Proof. Fix a small constant $\varepsilon > 0$. We can assume that δ is sufficiently small depending on ε such that the difference

$$Gf := \mathcal{R}f - D\mathcal{R}_0 f$$

has C^1 norm on $\overline{\mathcal{D}}$ bounded by ε, that is, for all $f \in \overline{\mathcal{D}}$ and $h \in T_f\mathcal{U}$,

$$\|Gf\| \le \varepsilon \|f\|, \quad \text{and } \|DG_f h\| \le \varepsilon \|h\|.$$

When f lies in $\overline{\mathcal{D}}$,

$$\|(\mathcal{R}f)^s\| \le \|D\mathcal{R}_0|_{E^s}(f^s)\| + \|(Gf)^s\| \le \mu_s \|f^s\| + \varepsilon \|f\|.$$

Assuming $\mu_s + 3\varepsilon < 1$, we then have $\|(\mathcal{R}f)^s\| < \delta$. Additionally, when $\|f^u\| = \delta$,

$$\|(\mathcal{R}f)^u\| \ge \|D\mathcal{R}_0|_{E^u}(f^u)\| - \|(Gf)^u\| \ge \mu_u \delta - \varepsilon \|f\|.$$

Assuming $\mu_u - 3\varepsilon > 1$, we then have $\|(\mathcal{R}f)^u\| > \delta$. Hence, (1) and (2) hold.

Suppose both f and $\mathcal{R}f$ are in $\mathcal{D}$. For every $h \in C^{cu}_f$, we have

$$\begin{aligned}
\|(D\mathcal{R}_f h)^{cu}\| &= \|D\mathcal{R}_0|_{E^c \oplus E^u}(h^{cu}) + (DG_f(h))^{cu}\| \\
&\ge \mu_{cu} \|h^{cu}\| - \varepsilon \|h\| \\
&\ge \left(\mu_{cu} - \left(1 + \frac{1}{\alpha}\right)\varepsilon\right) \|h^{cu}\|,
\end{aligned}$$

and

$$\begin{aligned}
\alpha \|(D\mathcal{R}_f h)^s\| &= \alpha \|D\mathcal{R}_0|_{E^s}(h^s) + (DG_f(h))^s\| \\
&\le \alpha \left(\mu_s \|h^s\| + \varepsilon \|h\|\right) \\
&\le \left(\mu_s + (\alpha + 1)\varepsilon\right) \|h^{cu}\|.
\end{aligned}$$

We can take ε to be small enough such that $\alpha \|(D\mathcal{R}_f h)^s\| \le \|(D\mathcal{R}_f h)^{cu}\|$ and thus $D\mathcal{R}_f h \in C^{cu}_{\mathcal{R}f}$. The proof that the cone field C^{cs} is backward invariant works in a similar way. □

Let us consider the perturbation $\mathcal{R}_\lambda := \lambda \cdot \mathcal{R}$ for $0 < \lambda < 1$. When λ is sufficiently close to 1, $\mathcal{R}_\lambda$ still satisfies all the properties listed in Claim 1. The following claim is a consequence of Lemma 2.3.2, which we will elaborate later separately.

Claim 2. There exists some point $f_\lambda \in \partial^c \mathcal{D}$ such that the orbit $\{\mathcal{R}_\lambda^n f_\lambda\}_{n\in\mathbb{N}}$ lies entirely inside of $\overline{\mathcal{D}}$ and $\mathcal{R}_\lambda^n f_\lambda \to 0$.

Since $\mathcal{R}$ is compact, there exist an increasing sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive numbers and some $g \in \overline{\mathcal{D}}$ such that as $n \to \infty$, $\lambda_n \to 1$ and $\mathcal{R}_{\lambda_n} f_{\lambda_n} \to g$. Clearly, for all $n \in \mathbb{N}$, the n^{th} iterate $g_n := \mathcal{R}^n g$ lies in $\overline{\mathcal{D}}$.

As f_λ is in $\partial^c \mathcal{D}$, f_λ is also inside of the cone $\hat{C}_0^{cu} = \{\|h^s\| \le \|h^{cu}\|\}$. Similar to the proof of Claim 1 (3), $\hat{C}_0^{cu}$ is forward invariant under $\mathcal{R}_\lambda$ for $\lambda \le 1$. Hence, for every $n \in \mathbb{N}$, $\|g_n^s\| \le \|g_n^{cu}\|$. This implies that for every $n \in \mathbb{N}$,

$$g_{n+1}^{cu} = D\mathcal{R}_0|_{E^c \oplus E^u}(g_n^{cu}) + O\left(\|g_n^{cu}\|^2\right). \tag{2.3.2}$$

At last, we will show that the orbit of g is a slow small orbit. Indeed, suppose for a contradiction that

$$\liminf_{n\to\infty} \frac{1}{n} \log \|g_n\| < -c_0 \tag{2.3.3}$$

for some constant $c_0 > 0$. Note that this property holds for every norm that is equivalent to $\|\cdot\|$. Pick some $c_1 \in [0, c_0)$. There exists an adapted norm $\|\cdot\|$ equivalent to the original one such that the operator norm of $D\mathcal{R}_0|_{E^c\oplus E^u}^{-1}$ is at most e^{c_1}. By (2.3.2), for sufficiently small $\delta > 0$, there is some $c_2 \in (0, c_0)$ such that

$$\|g_{n+1}^{cu}\| \ge e^{-c_2} \|g_n^{cu}\| \qquad \text{for all } n \in \mathbb{N}.$$

This contradicts (2.3.3). $\square$

It remains to prove Claim 2, which will follow directly from the lemma below. Again, we suppose $\mathcal{B}$ can be decomposed into $E^s \oplus E^c \oplus E^u$ and consider the cone fields C^{cu} and C^{cs} defined in (2.3.1). We consider a small neighborhood $\mathcal{U} \subset \mathcal{B}$ of some polydisk $\mathcal{D}$ centered at 0. For any $r > 0$, we denote the open disk $\{z \in \mathbb{C} : |z| < r\}$ by $\mathbb{D}_r$.

Lemma 2.3.2. *Let $\mathcal{R} : (\mathcal{U}, 0) \to (\mathcal{B}, 0)$ be a compact analytic operator such that the differential $D\mathcal{R}_0$ preserves the decomposition $\mathcal{B} = E^s \oplus E^c \oplus E^u$ and satisfies the following properties.*

(1) *Hyperbolicity: There exists some $0 < r < 1$ such that*

$$\operatorname{spec}(D\mathcal{R}_0|_{E^s}) \subset \mathbb{D}_r, \quad \operatorname{spec}(D\mathcal{R}_0|_{E^c}) \subset \mathbb{D}\backslash\mathbb{D}_r, \quad \operatorname{spec}(D\mathcal{R}_0|_{E^u}) \subset \mathbb{C}\backslash\overline{\mathbb{D}}.$$

(2) *Boundary behaviour: If $f \in \overline{\mathcal{D}}$, then $\mathcal{R}f \notin \partial^s \mathcal{D}$. If $f \in \partial^u \mathcal{D}$, then $\mathcal{R}f \notin \overline{\mathcal{D}}$.*

(3) *Invariant cone fields: Whenever* $f, \mathcal{R}f \in \overline{\mathcal{D}}$,

$$D\mathcal{R}_f(C^{cu}_f) \subset C^{cu}_{\mathcal{R}f}, \qquad (D\mathcal{R}_f)^{-1}(C^{cs}_{\mathcal{R}f}) \subset C^{cs}_f.$$

Then, there exists some $f \in \partial^c\mathcal{D}$ *such that* $\{\mathcal{R}^n f\}_{n\in\mathbb{N}} \subset \overline{\mathcal{D}}$ *and* $\|\mathcal{R}^n f\| \to 0$ *as* $n \to \infty$.

Proof. By the compactness of $\mathcal{R}$, the subspace $E^c \oplus E^u$ is finite dimensional. Let $d_c := \dim(E^c)$ and $d_u := \dim(E^u)$. By (1), the stable manifold

$$\mathcal{A} = \{f \in \overline{\mathcal{D}} : \{\mathcal{R}^n f\}_{n\in\mathbb{N}} \subset \overline{\mathcal{D}} \text{ and } \|\mathcal{R}^n f\| \to 0\}$$

exists and is a forward invariant analytic submanifold of codimension d_u.

Let us assume for a contradiction that $\mathcal{A}$ is disjoint from $\partial^c\mathcal{D}$.

Claim 1. The set $\mathcal{A}^o := \mathcal{A} \cap \mathcal{D}$ is a forward invariant open submanifold of $\mathcal{A}$.

Proof. The only non-trivial property to prove here is forward invariance. Suppose $f \in \mathcal{A}^o$. As $f \in \mathcal{A}$, then $\mathcal{R}^n f \in \overline{\mathcal{D}}$ for all $n \geq 1$. By (2), $\mathcal{R}f$ cannot lie in $\partial^s\mathcal{D} \cup \partial^u\mathcal{D}$. By the assumption, $\mathcal{R}f$ cannot lie in $\partial^c\mathcal{D}$ either. Thus, $\mathcal{R}f \in \mathcal{D}$. □

Claim 2. The set $\partial^c\mathcal{A} := \overline{\mathcal{A}} \backslash (\mathcal{A}^o \cup \partial^s\mathcal{D})$ is also forward invariant.

Proof. Suppose for a contradiction that there is some $f \in \partial^c\mathcal{A}$ such that $\mathcal{R}f \in \mathcal{A}^o \cup \partial^s\mathcal{D}$. By (2), $\mathcal{R}f$ must lie in $\mathcal{A}^o$, which implies that $f \in \mathcal{A} \cap (\partial^c\mathcal{D} \cup \partial^u\mathcal{D})$. However, this is impossible because f does not lie in $\partial^c\mathcal{D}$ by our main assumption, nor in $\partial^u\mathcal{D}$ due to (2). □

Claim 3. The tangent space $T_f\mathcal{A}^o$ at every point f in $\mathcal{A}^o$ is contained in C^{cs}_f.

Proof. Let $f \in \mathcal{A}^o$. As $\mathcal{A}^o$ is tangent to the subspace $E^s \oplus E^c$ at 0, for all sufficiently high n, $\mathcal{R}^n f$ is sufficiently close to 0 and so the tangent space $T_{\mathcal{R}^n f}\mathcal{A}^o$ lies within $C^{cs}_{\mathcal{R}^n f}$. By backward invariance of C^{cs} in (3), the tangent space of $\mathcal{A}^o$ at f also lies within C^{cs}_f. □

Let us consider the family $\mathcal{G}$ of all immersed analytic d_c-dimensional submanifolds Γ of $\mathcal{A}^o$ containing 0 with the following properties.

(a) The tangent space $T_f\Gamma$ at every point $f \in \Gamma$ lies in the cone C^{cu}_f;

(b) The accumulation set $\overline{\Gamma}\backslash\Gamma$ lies in $\partial^c\mathcal{A}$.

Note that $\mathcal{G}$ is non-empty: it contains $\mathcal{A}^o \cap (E^c \oplus E^u)$ because, by Claim 3, the intersection between $\mathcal{A}^o$ and the subspace $E^c \oplus E^u$ is transversal. Another consequence of Claim 3 is the following claim.

Claim 4. For every $\Gamma \in \mathcal{G}$ and $h \in T_f\Gamma$, $\|h^c\| \asymp \|h\|$. In particular, the projection $P : \Gamma \to D^c$ is non-singular.

Proof. Let $h \in T_f\Gamma$. By Property (a) and Claim 3, $\alpha\|h^s\| \le \|h^{cu}\|$ and $\alpha\|h^u\| \le \|h^{cs}\|$. By triangle inequality, these imply that $(\alpha - 1)\max\{\|h^s\|, \|h^u\|\} \le \|h^c\|$ and consequently $\|h^c\| \le \|h\| \le \frac{\alpha+1}{\alpha-1}\|h^c\|$. □

Recall that the Kobayashi norm of a tangent vector $v \in T_f\Gamma$ at a point f on a complex manifold Γ is defined as

$$\|h\|_\Gamma := \inf\{\|w\|_{\mathbb{D}} \,:\, D\phi_f(w) = h \text{ for some holomorphic map } \phi : (\mathbb{D}, 0) \to (\Gamma, f)\}$$

where $\|w\|_{\mathbb{D}}$ denotes the Poincaré metric of $w \in T_0\mathbb{D}$ on the unit disk $\mathbb{D}$. We will supply every $\Gamma \in \mathcal{G}$ with the Kobayashi metric.

Claim 5. There is some $K > 0$ such that for every $\Gamma \in \mathcal{G}$ and $h \in T_0\Gamma$, $\|h\|_\Gamma \le K\|h\|$.

Proof. By Claim 4, there is some $\delta > 0$ such that for every $\Gamma \in \mathcal{G}$, the component $\Gamma(\delta)$ of $\Gamma \cap D^c(\delta)$ containing 0 is a graph of an analytic map $D^c(\delta) \to D^s \times D^u$. Therefore, for any $h \in T_0\Gamma$,

$$\|h\|_\Gamma \le \|h\|_{\Gamma(\delta)} = \|h^c\|_{D^c(\delta)}.$$

Clearly, $\|h^c\|_{D^c(\delta)} \asymp \|h^c\|$ (with bounds depending only on δ). By Claim 4, this yields the desired inequality $\|h\|_\Gamma \le K\|h\|$ for some K independent of Γ. □

By Property (3) and Claim 2, the map $\mathcal{R}$ induces a well-defined graph transform

$$\mathcal{R}_* : \mathcal{G} \to \mathcal{G}, \quad \Gamma \mapsto \mathcal{R}\Gamma.$$

Note that $\mathcal{R} : \Gamma \to \mathcal{R}\Gamma$ is a proper non-singular map, hence a holomorphic covering map. Therefore, for every $\Gamma \in \mathcal{G}$, $n \in \mathbb{N}$, and non-zero tangent vector $h \in T_0\Gamma$,

$$\|h\|_\Gamma = \|(D\mathcal{R}^n)_0(h)\|_{\mathcal{R}_*^n\Gamma}.$$

By Claim 5,

$$\|h\|_\Gamma \le K\|(D\mathcal{R}^n)_0(h)\|.$$

However, by (1), $\|(D\mathcal{R}^n)_0(h)\|$ tends to 0 as $n \to \infty$. This yields a contradiction. □

2.4 Notation

Throughout this dissertation, we fix a pair of positive integers $d_0, d_\infty \geq 2$ and an irrational number $\theta \in (0,1)$ with continued fraction expansion θ. Unless otherwise stated, we will always assume that θ is of bounded type.

In our analysis, we will often use the following notation:

- ▷ $x \oplus y := (x^{-1} + y^{-1})^{-1}$ for any $x, y > 0$;
- ▷ $g = O(h)$ when $h > 0$ and $|g| \leq \alpha h$ for some implicit constant $\alpha > 0$;
- ▷ $g \succ h$ when $g, h > 0$ and $g \geq \alpha h$ for some implicit constant $\alpha > 0$;
- ▷ $g \asymp h$ when $g \succ h$ and $h \succ g$.

Number theory.

- ▷ $\beta(\theta) = \sup_{n\geq 1} a_n$ if $\theta \in (0,1)\backslash\mathbb{Q}$ has continued fraction expansion $\theta = [0; a_1, a_2, \ldots]$;
- ▷ $\Theta_N := \{\theta \in (0,1)\backslash\mathbb{Q} : \beta(\theta) \leq N\}$, the set of bounded type irrationals with bound $\leq N$;
- ▷ $\Theta_{bdd} := \bigcup_{N\geq 1} \Theta_N$, the set of bounded type irrationals;
- ▷ Θ_{pre} := the set of pre-periodic irrationals (quadratic irrationals) in $(0,1)$;
- ▷ Θ_{per} := the set of periodic irrationals in $(0,1)$.

Euclidean geometry. The Euclidean norm on the complex plane $\mathbb{C}$ is denoted by $|\cdot|$.

- ▷ $\mathbb{D}(z, \varepsilon) = \{w \in \mathbb{C} : |w - z| < \varepsilon\}$;
- ▷ $\mathbb{A}(r, R) = \{z \in \mathbb{C} : r < |z| < R\}$;
- ▷ $\operatorname{dist}(A, B)$ = Euclidean distance between two subsets A and B of $\mathbb{C}$.

Given a pointed topological disk (U, x), we define the following.

- ▷ $r_{\text{in}}(U, x) := \operatorname{dist}(x, \partial U)$, the inner radius of U about x;
- ▷ $r_{\text{out}}(U, x) := \inf\{\varepsilon > 0 : U \subset \mathbb{D}(x, \varepsilon)\}$, the outer radius of U about x.

For any $C \geq 1$, we say that (U, x) has *C-bounded shape* if $r_{\text{out}}(U, x) \leq C r_{\text{in}}(U, x)$. We say that (U, x) has *bounded shape* if it has C-bounded shape for some implicit constant $C > 1$.

Hyperbolic geometry. Given a hyperbolic Riemann surface Ω, denote by $d_\Omega(\cdot,\cdot)$ the corresponding hyperbolic metric of Ω.

▹ $\mathbb{D}_\Omega(z,\varepsilon) = \{w \in \Omega : d_\Omega(w,z) < \varepsilon\}$;

▹ $\operatorname{dist}_\Omega(A,B)$ = hyperbolic distance between two subsets A and B of Ω.

Given a pointed topological disk (U,x) in Ω,

▹ $r_{\text{in},\Omega}(U,x) := \operatorname{dist}_\Omega(x,\partial U)$ the hyperbolic inner radius of U about x;

▹ $r_{\text{out},\Omega}(U,x) := \inf\{\varepsilon > 0 : U \subset \mathbb{D}_\Omega(x,\varepsilon)\}$ the hyperbolic inner radius of U about x.

Conformal geometry. The main conformal invariants in consideration are the following.

▹ $\operatorname{mod}(A)$ = the conformal modulus of an annulus A;

▹ $W(\mathcal{F})$ = the extremal width of a curve family $\mathcal{F}$.

Given a compact subset K of a Riemann surface U with boundary, we will use the following notation.

▹ $\mathcal{F}(U,K)$ = the family of proper curves in $U\backslash K$ connecting ∂U and K;

▹ $\mathcal{F}^h_{\text{can}}(U,K)$ = the set of leaves of the canonical lamination of $U\backslash K$ that are *horizontal* (both endpoints are on K);

▹ $\mathcal{F}^v_{\text{can}}(U,K)$ = the set of leaves of the canonical lamination of $U\backslash K$ that are *vertical* (connects ∂U and K);

▹ $W(U,K)$ = the extremal width of $\mathcal{F}(U,K)$.

For any pair of disjoint sets I and J in $\hat{\mathbb{C}}$, we say that I and J are *well separated* if there exists an annulus A of modulus $\operatorname{mod}(A) \asymp 1$ separating I and J. For any set I contained in a domain D, we say that I is *well contained* in D if I and ∂D are well separated.

Chapter 3

A Priori Bounds

Let us consider the family $\mathcal{H} = \mathcal{H}_{d_0,d_\infty,\theta}$ of all degree d rational maps f such that

(I) 0 and ∞ are superattracting fixed points of f with local degrees $d_0 \geq 2$ and $d_\infty \geq 2$ respectively;

(II) the map f admits an invariant Herman ring $\mathbb{H}$ with a bounded type rotation number θ;

(III) $\mathbb{H}$ separates 0 and ∞;

(IV) every critical point of f other than 0 and ∞ lies on the boundary of $\mathbb{H}$.

In this chapter, we will establish a priori bounds for $\mathcal{H}$ and apply it to study the "boundary" $\mathcal{X} := \overline{\mathcal{H}} \backslash \mathcal{H}$.

3.1 Setting up the stage

3.1.1 Herman rings

The following procedure allows one to obtain Siegel disks out of invariant curves.

Theorem 3.1.1 (Douady-Ghys surgery)**.** *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map, $Y \subset \hat{\mathbb{C}}$ be a quasidisk such that ∂Y is forward invariant and $f|_{\partial Y}$ is quasisymmetrically conjugate to an irrational rotation R_θ of the circle $\mathbb{T}$. There exists a K-quasiconformal map $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and a rational map F such that*

(1) *$F = \phi \circ f \circ \phi^{-1}$ on $\hat{\mathbb{C}} \backslash \phi(Y)$, and*

(2) *F has a Siegel disk of the same rotation number θ containing $\phi(Y)$.*

Moreover, K depends only on the dilatation of the conjugacy between $f|_{\partial Y}$ and R_θ.

The original idea of the surgery procedure was by Ghys in [Ghy84], but the formulation above follows from [BF14, §7.2]. The essence of the surgery procedure is to replace the dynamics $f|_Y$ with a rotation. More precisely, we replace $f|_Y$ with $\psi^{-1} \circ R_\theta \circ \psi$, where $\psi : Y \to \mathbb{D}$ is a quasiconformal extension of the quasisymmetric conjugacy between $f|_{\partial Y}$ and $R_\theta|_{\partial \mathbb{D}}$, and straighten the new map via the measurable Riemann mapping theorem.

As explained in the introduction, Douady-Ghys surgery plays an essential role in deducing the regularity of the boundary of Siegel disks with bounded type rotation number. The most general version of this result is the following theorem.

Theorem 3.1.2 ([Zha11]). *Let f be a rational map of degree $d \geq 2$. If f has an invariant Siegel disk Z with bounded type rotation number θ, then the boundary ∂Z is a $K(d, \beta(\theta))$-quasicircle containing at least one critical point.*

In [Shi87, §6], Shishikura originally discovered a way to convert Herman rings into Siegel disks (and vice versa) through quasiconformal surgery. We will formulate this procedure as a straightforward application of Douady-Ghys surgery and combine it with Zhang's theorem to obtain the following corollary.

Corollary 3.1.3. *Let f be a rational map of degree $d \geq 3$ having an invariant Herman ring $\mathbb{H}$ with bounded type rotation number θ and modulus $\operatorname{mod}(\mathbb{H}) \geq \mu > 0$. Then,*

1. *every boundary component of $\mathbb{H}$ is a K-quasicircle containing at least one critical point;*
2. *there is an L-quasiconformal map $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that is conformal in $\mathbb{H}$ and conjugates $f|_{\overline{\mathbb{H}}}$ and the rigid rotation R_θ on the annulus $\overline{\mathbb{A}(1, e^{2\pi \operatorname{mod}(\mathbb{H})})}$.*

Moreover, the dilatations K and L depend only on d, $\beta(\theta)$, and μ.

Proof. Along the core curve γ of $\mathbb{H}$, $f|_\gamma$ is K'-quasisymmetrically conjugate to $R_\theta|_{\mathbb{T}}$ for some $K' = K'(\mu)$. Pick a boundary component H of $\mathbb{H}$ and let D be the component of $\hat{\mathbb{C}} \backslash \gamma$ containing H. Apply Douady-Ghys surgery along γ to obtain a degree $\leq d-1$ rational map F having an invariant Siegel disk Z and an L'-quasiconformal map $\psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that maps H to ∂Z and restricts to a conjugacy between $f|_D$ and $F|_{\psi(D)}$, where L' depends on μ. Then, the corollary follows from applying Zhang's theorem to F. □

In this chapter, we would like to remove the dependency on the modulus μ for rational maps in $\mathcal{H}$. Such rational maps can be constructed through Shishikura's quasiconformal surgery [Shi87] (see also [BF14, §7.3]) from two polynomials P_0 and P_∞ of degree d_0 and d_∞ respectively where both P_0 and P_∞ have invariant Siegel disks Z_0 and Z_∞ of rotation numbers $1-\theta$ and θ respectively and satisfy a condition similar to (IV). The surgery involves removing a proper invariant sub-disk of each Z_0 and Z_∞, gluing the two remaining Riemann surfaces along the boundary of the sub-disks and applying the measurable Riemann mapping theorem to obtain some $f \in \mathcal{H}$ that mimics the dynamics of both P_0 and P_∞ outside of the removed disks.

Denote by Y^0 and Y^∞ the connected components of $\hat{\mathbb{C}} \backslash \overline{\mathbb{H}}$ containing 0 and ∞ respectively. The hypothesis assumes that ∂Y^0 contains a unique critical point c_0 and ∂Y^∞ contains a unique critical point c_∞. The covering structure of f is well understood.

Proposition 3.1.4. *The preimage $f^{-1}(\mathbf{H})$ of $\mathbf{H}$ is of the form*

$$\mathbf{H} \cup \bigcup_{i=1}^{d_0-1} A_i^0 \cup \bigcup_{j=1}^{d_\infty-1} A_j^\infty$$

where for each $\bullet \in \{0, \infty\}$ and $i \in \{1, \ldots, d_\bullet - 1\}$,

(1) *$A_i^\bullet$ is a closed topological annulus in $\overline{Y^\bullet}$;*

(2) *$A_i^\bullet \cap \mathbf{H} = \{c\}$ for some critical point c;*

(3) *if $j \neq i$, $A_i^\bullet \cap A_j^\bullet$ is either empty or $\{c\}$ for some critical point c;*

(4) *f is univalent in the interior of $A_i^\bullet$.*

Proof. For each $\bullet \in \{0, \infty\}$, the boundary $\partial Y^\bullet$ is a quasicircle along which f is conjugate to the irrational rotation. We can perform Douady-Ghys surgery[1] to replace f on the disk $D_\bullet := \hat{\mathbb{C}} \backslash (\mathbf{H} \cup Y^\bullet)$ with a rotation and obtain a rational map $P_\bullet$ that satisfies the following properties:

- ▹ $P_\bullet$ admits an invariant Siegel disk $Z_\bullet \subset \mathbb{C}$, which is a quasidisk;
- ▹ there is a quasiconformal map $\phi_\bullet : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that restricts to a conjugacy between $f|_{\hat{\mathbb{C}} \backslash D_\bullet}$ and $P_\bullet|_{\hat{\mathbb{C}} \backslash Z_\bullet}$;
- ▹ $\phi_\bullet(\bullet) = \infty$, and thus $P_\bullet$ has a superattracting fixed point at ∞ with local degree $d_\bullet$.

Clearly, for each $\bullet$, $P_\bullet$ must have degree at least $d_\bullet$. The critical points of $P_\bullet$ aside from ∞ must lie on $\partial Z_\bullet$. Moreover, the sum of the numbers of critical points of P_0 and P_∞ is equal to the number of critical points of f, which is $2(d_0 + d_\infty - 2)$. As such, P_0 and P_∞ must be polynomials of degrees d_0 and d_∞ respectively.

For each $\bullet$, the maximum modulus principle implies that the preimage of $Z_\bullet$ under $P_\bullet$ must be of the form $Z_\bullet \cup E_1^\bullet \cup \ldots, E_{d_\bullet - 1}^\bullet$ for some $d_\bullet - 1$ pairwise disjoint open disks $E_i^\bullet$'s

[1]A combinatorial proof avoiding the surgery procedure is possible, but we will leave it as an exercise to the keen reader.

where for each i, $P_\bullet$ is univalent in $E_i^\bullet$ and the closure $\overline{E_i^\bullet}$ intersects $\overline{Z_\bullet}$ precisely at one point, which is a critical point of $P_\bullet$. Therefore, the preimage of $\mathbf{H}$ under f is of the form

$$\mathbf{H} \cup \bigcup_{\bullet \in \{0,\infty\}} \bigcup_{i=1}^{d_\bullet - 1} \left(\phi_\bullet^{-1}(\overline{E_i^\bullet}) \cap f^{-1}(\mathbf{H}) \right).$$

Then, the proposition follows immediately. □

Denote by $\mathcal{C} = \{c_0, c_\infty\}$ the set of free critical points of f. For any $n \geq 1$, we refer to the closure of a component of $f^{-n}(\overline{\mathbb{H}}) \backslash f^{-(n-1)}(\overline{\mathbb{H}})$ as a *bubble* of *generation* n. By Proposition 3.1.4, every bubble B of generation n is a closed annulus admitting a unique point on the outer boundary of B that lies on the pre-critical set $f^{-(n-1)}(\mathcal{C})$. This unique point will be called the *root* of B. In particular, every bubble of generation 1 is precisely one of the $A_i^\bullet$'s above and it is rooted at a unique critical point. (See Figure 3.1.) We say that a bubble attached to $\mathbf{H}$ is an *inner* bubble if it lies in $\overline{Y^0}$ and an *outer* bubble if it lies in $\overline{Y^\infty}$.

3.1.2 Combinatorial data

We shall formally define combinatorics of Herman ring $\mathbb{H}$ of $f \in \mathcal{H}$ as follows. For any $n \in \mathbb{N}$, the n^{th} symmetric product $\mathrm{SP}^n(\mathbb{T})$ of the unit circle $\mathbb{T}$ is the quotient of the n-dimensional torus $\mathbb{T}^n$ under the symmetric group S_n acting by permutation. Elements of $\mathrm{SP}^n(\mathbb{T})$ are precisely unordered n-tuples of elements of $\mathbb{T}$.

Definition 3.1.5. Define $\mathcal{C}_{m,n}$ to be the quotient space of $\mathrm{SP}^{m-1}(\mathbb{T}) \times \mathrm{SP}^{n-1}(\mathbb{T})$ modulo the action of $\mathbb{T}$ by any rigid rotation, endowed with the quotient topology.

Let $\phi : \mathbf{H} \to \overline{\mathbb{A}(1,R)}$, where $R = e^{2\pi \mathrm{mod}(\mathbf{H})}$, denote a linearization of $f|_{\mathbf{H}}$. Let $(c_1^0, \ldots, c_{d_0-1}^0)$ and $(c_1^\infty, \ldots, c_{d_\infty-1}^\infty)$ denote the tuples of inner and outer critical points of f counting multiplicity.

Definition 3.1.6. The *combinatorics* of $f \in \mathcal{H}$ is the element $\mathrm{comb}(f)$ in $\mathcal{C} = \mathcal{C}_{d_0,d_\infty}$ induced by the pairs of tuples $\left(\phi(c_1^0), \ldots, \phi(c_{d_0-1}^0)\right)$ and $\left(\frac{\phi(c_1^\infty)}{R}, \ldots, \frac{\phi(c_{d_0-1}^\infty)}{R}\right)$.

Note that $\mathrm{comb}(f)$ is well-defined because ϕ is unique up to post-composition with rigid rotation.

Zhang [Zha08] proved that bounded type Siegel disks of any prescribed combinatorics are realized by a unique rational map as long as outside the closure of the Siegel disk, the postcritical set is finite and there are no Thurston obstructions. Using methods similar to Shishikura's quasiconformal surgery, Wang [Wan12] extended Zhang's result to rational maps with an invariant Herman ring where outside the closure of the Herman ring, the postcritical set is finite and there are no Thurston obstructions. In particular, every map f in $\mathcal{H}$ is uniquely determined by the conformal modulus of its Herman ring $\mathbb{H}$ and the combinatorial data on $\partial\mathbb{H}$.

Theorem 3.1.7 ([Wan12]). *For any $\mu > 0$ and $\sigma \in \mathcal{C}$, there is a rational map $f \in \mathcal{H}$ such that its Herman ring has modulus μ and combinatorics σ. Moreover, such f is unique up to conformal conjugacy.*

Towards the end of the chapter, we will also consider the space $\mathcal{X}$ of degree d rational maps f such that

(i) 0 and ∞ are superattracting fixed points of f with local degrees $d_0 \geq 2$ and $d_\infty \geq 2$ respectively;

(ii) the function f admits a Herman quasicircle $\mathbf{H}$ of rotation number θ;

(iii) $\mathbf{H}$ separates 0 and ∞;

(iv) every critical point of f other than 0 and ∞ lies in $\mathbf{H}$;

(v) the conjugacy between $f|_{\mathbf{H}}$ and $R_\theta|_{\mathbb{T}}$ is quasisymmetric with dilatation depending only on d_0, d_∞ and $\beta(\theta)$.

Consider a map $f \in \mathcal{X}$. Denote by $\mathbf{H}$ the Herman quasicircle of f, and by Y^0 and Y^∞ the connected components of $\hat{\mathbb{C}} \backslash \mathbf{H}$ containing 0 and ∞ respectively.

Topologically, maps in $\mathcal{X}$ are dynamically identical to maps in $\mathcal{H}$ with the exception that quasiconformal Herman rings of positive moduli are replaced with quasicircles (zero modulus). Similar to Proposition 3.1.4, the strict preimage of the Herman quasicircle $\mathbf{H}$ of f is a union of quasicircles (bubbles of generation one) attached to critical points on $\mathbf{H}$. Bubbles of arbitrary generation are obtained by taking iterated preimages of $\mathbf{H}$.

Proposition 3.1.8. $J(f) = \overline{\cup_{k=0}^{\infty} f^{-k}(\mathbf{H})}$.

Let $\phi : \mathbf{H} \to \mathbb{T}$ be the quasisymmetric conjugacy between $f|_{\mathbf{H}}$ and R_θ. By pushing forward inner and outer critical points of f under ϕ, we again obtain a well-defined element $\operatorname{comb}(f) \in \mathcal{C}$.

Definition 3.1.9. The *combinatorics* of $f \in \mathcal{X}$ is the element $\operatorname{comb}(f)$ in $\mathcal{C}$.

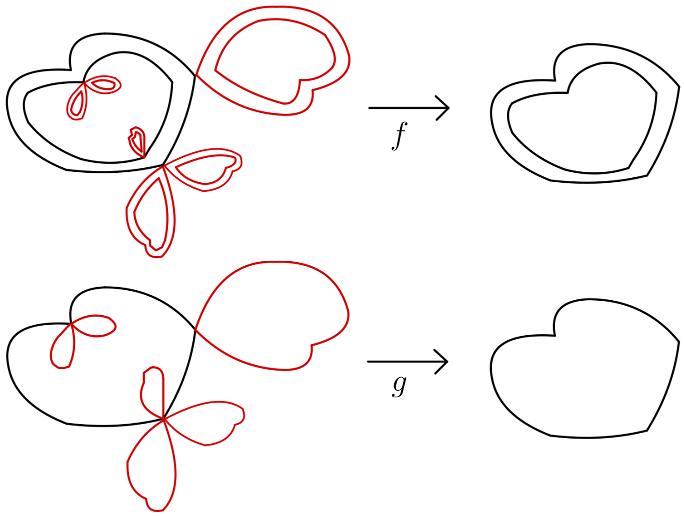

Figure 3.1: Bubbles of generation 1 for $f \in \mathcal{H}$ above and $g \in \mathcal{X}$ below.

3.1.3 Encoding degeneration

Consider the following general setup. Let $f : \mathbf{H} \to \mathbf{H}$ be a homeomorphism on a closed annulus $\mathbf{H} \subset \hat{\mathbb{C}}$. Suppose f is topologically conjugate via $\phi : \mathbf{H} \to A$ to the rigid rotation $R_\theta(z) = e^{2\pi i\theta}z$ on a closed round annulus $A = \{1 \leq |z| \leq R\}$. Via the projection $\psi : A \to \mathbb{T}, z \mapsto \frac{1}{2\pi}\arg(z)$, we can equip $\mathbf{H}$ with the pullback under $\psi \circ \phi$ of the Euclidean metric on $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, called the *combinatorial pseudometric* of $\mathbf{H}$.

A closed set $I \subset \mathbf{H}$ is called a *piece* in $\mathbf{H}$ if it is of the form $(\psi \circ \phi)^{-1}(I')$ for some closed interval $I' \subset \mathbb{T}$. Define the *combinatorial length* $|I|$ of a piece I to be the diameter of I with respect to the combinatorial pseudometric.

For any two distinct points $x, y \in \mathbf{H}$, we denote by $[x, y]$ the unique combinatorially shortest piece that contains both x and y. Note that if $\psi(\phi(x)) = \psi(\phi(y))$, then $[x, y]$ is a radial segment in $\mathbf{H}$ with zero combinatorial length.

Let $\{p_n/q_n\}$ be the sequence of best rational approximations of θ.

Definition 3.1.10. A *combinatorial piece* of *level* n is a piece of the form $[x, f^{q_n}(x)]$ for some $x \in \mathbf{H}$.

Recall from Proposition 2.1.2 that rotational behaviour induces a nest of tilings on $\mathbf{H}$. For any $x \in \mathbf{H}$ and $n \in \mathbb{N}$, consider the n^{th} *renormalization tiling* induced by $x \in \mathbf{H}$:

$$\mathbf{H} = \bigcup_{i=0}^{q_{n+1}-1} f^i\left([x, f^{q_n}(x)]\right) \cup \bigcup_{j=0}^{q_n-1} f^j\left([f^{q_{n+1}}(x), x]\right).$$

All the pieces in the expression above have pairwise disjoint interiors, and all the level $n+1$ combinatorial pieces above are pairwise disjoint. Keeping only the level n pieces from the

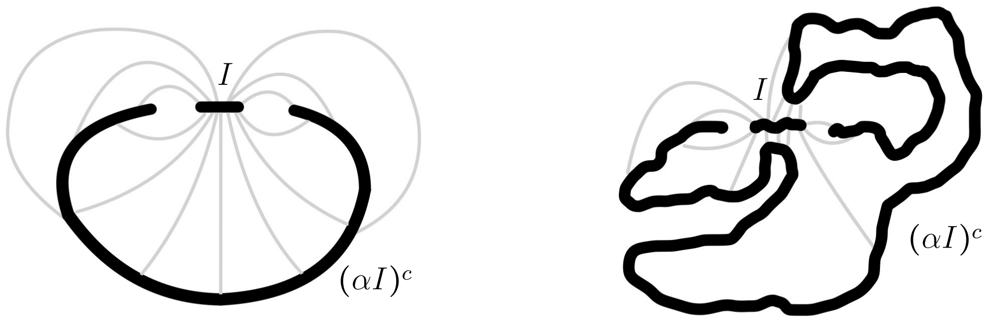

Figure 3.2: On the left, I has small α-width. On the right, I has large α-width.

renormalization tiling gives us an *almost tiling* whose gaps have length l_{n+1}. We will also often apply the weaker fact that for any $n \geq 3$, the orbit $\{f^i(x)\}_{i=0,\dots,q_n}$ partitions $\mathbf{H}$ into pieces of length between l_n and l_{n-2}.

For every $\alpha \geq 3$ and piece $I \subset \mathbf{H}$ of length $|I| < \frac{1}{\alpha}$, we will use the following notation:

▷ I^c = the closure of $\mathbf{H} \backslash I$;

▷ αI = the combinatorial rescaling of I by the factor of α, that is, the unique piece in $\mathbf{H}$ of length $\alpha|I|$ having the same combinatorial mid-segment as I;

▷ $\mathcal{F}_\alpha(I)$ = the set of proper curves in $\hat{\mathbb{C}} \backslash (I \cup (\alpha I)^c)$ connecting I and $(\alpha I)^c$;

▷ $W_\alpha(I)$ = the *α-width* of I, that is, the extremal width of $\mathcal{F}_\alpha(I)$.

When $R = 1$, $\mathbf{H}$ is a Jordan curve, the combinatorial pseudometric is a metric on $\mathbf{H}$, and every piece in $\mathbf{H}$ is a genuine interval. Additionally, when the conjugacy ϕ is quasiconformal, we have the following. (Compare with [DL22, Lemma 11.3].)

Proposition 3.1.11. *Let $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a quasiconformal map that maps a quasicircle $\mathbf{H}$ onto the unit circle $\mathbb{T}$. Equip $\mathbf{H}$ with the combinatorial metric induced by ϕ.*

(1) *For every $\alpha \geq 3$, there is a constant $\mathbf{K}$ depending on α and the dilatation of ϕ such that every interval $I \subset \mathbf{H}$ of combinatorial length $|I| < (2\alpha)^{-1}$ satisfies $W_\alpha(I) \leq \mathbf{K}$.*

(2) *Conversely, if there are some constants $\alpha \geq 3$, $\varepsilon \in (0,1)$, and $\mathbf{K} > 0$ such that $W_\alpha(I) \leq \mathbf{K}$ for every interval $I \subset \mathbf{H}$ of combinatorial length at most ε, then the dilatation of $\mathbf{H}$ depends only on α, ε and $\mathbf{K}$.*

Proof. Pick any $\alpha \geq 3$ and any interval $I \subset \mathbf{H}$ of length $|I| < (2\alpha)^{-1}$. On the circle, $\phi(I)$ has width $W_\alpha(\phi(I)) \leq M$ for some constant $M = M(\alpha) > 1$. Therefore, on $\mathbf{H}$, the interval I has width $W_\alpha(I) \leq kM$, where k denotes the dilatation of ϕ, and so (1) holds.

To show the converse, we first claim that every interval I of length at most ε must satisfy $W_3(I) \le \alpha\mathbf{K}$. If otherwise, then we could partition I into $\lfloor \alpha \rfloor$ pieces $I_1, \ldots, I_{\lfloor \alpha \rfloor}$ of equal combinatorial length. Since each $(\alpha I_i)^c$ contains $(3I)^c$,

$$\sum_{i=1}^{\lfloor \alpha \rfloor} W_\alpha(I_i) \ge \sum_{i=1}^{\lfloor \alpha \rfloor} W(I_i, (3I)^c) \ge W_3(I) > \alpha\mathbf{K}.$$

Then, at least one of the pieces I_j satisfies $W_\alpha(I_j) > \mathbf{K}$, which is a contradiction.

Assume without loss of generality that $\mathbf{H}$ separates 0 and ∞. For any $\bullet \in \{0, \infty\}$, we denote by $Y^\bullet$ the component of $\hat{\mathbb{C}} \backslash \mathbf{H}$ containing $\bullet$. For any interval $J \subset \mathbf{H}$, let $\mathrm{m}_\bullet(J)$ denote the harmonic measure of J on $Y^\bullet$ about $\bullet$ and let $W_3^\bullet(J)$ denote the width of the family of proper curves in $Y^\bullet$ connecting J and $(3J)^c$. Since $W_3(I) \le \alpha\mathbf{K}$, then $W_3^\bullet(I) \le \alpha\mathbf{K}$ for $\bullet \in \{0, \infty\}$.

Denote by L and R the two connected components of $\overline{3I \backslash I}$. For $\bullet \in \{0, \infty\}$, by Proposition 2.2.7,

$$\mathrm{m}_\bullet(I) < M \cdot \min\{\mathrm{m}_\bullet(L), \mathrm{m}_\bullet(R)\}$$

for some $M = M(\alpha\mathbf{K}) \ge 1$. Thus, any two neighboring combinatorial intervals I and J of equal combinatorial length satisfy

$$M^{-1}\mathrm{m}_\bullet(J) < \mathrm{m}_\bullet(I) < M\mathrm{m}_\bullet(J).$$

As such, the inner and outer harmonic measures are quasisymmetrically equivalent to the combinatorial measure, and consequently to each other as well. By conformal welding, this implies (2). □

In application, in order to bound the dilatation of a quasicircle $\mathbf{H}$, it is sufficient to obtain a bound on the α-width of sufficiently deep intervals in $\mathbf{H}$ and for some $\alpha \ge 3$. Degeneration is encoded by the presence of an interval with very large α-width.

3.1.4 Setup and notation

Throughout this chapter, dependence on d_0, d_∞ and $\beta(\theta)$ will always be implicit. Throughout Sections §3.3–3.7, we will fix a rational map f in $\mathcal{H}$ and denote its Herman ring by $\mathbb{H}$. We always assume that the modulus μ of $\mathbb{H}$ is sufficiently small: $\mu \ll 1$. (Otherwise, a priori bounds can be obtained from Corollary 3.1.3.)

Let us define $\mathbf{H}$ in two different ways:

- ⊙ $\mathbf{H}$ is the closure of the Herman ring $\mathbb{H}$ of f;

$\circledcirc$ $\mathbf{H}$ is the outer boundary component of the Herman ring $\mathbb{H}$ of f. (The treatment for the inner boundary is analogous.)

In Section §3.2, only $\odot$ is considered.

In both cases, we let

$$Y^\bullet := \text{the connected component of } \hat{\mathbb{C}} \backslash \mathbf{H} \text{ containing } \bullet, \text{ for } \bullet \in \{0, \infty\}.$$

For any piece $I \subset \mathbf{H}$,

- ▷ $|I|$:= the combinatorial length of $I \subset \mathbf{H}$;
- ▷ I^c := the closure of $\mathbf{H} \backslash I$.

Moreover, for any $\alpha \in (1, |I|^{-1})$,

- ▷ αI := the piece in $\mathbf{H}$ of length $\alpha|I|$ that shares the same mid-segment as I;
- ▷ $\mathcal{F}_\alpha(I) := \mathcal{F}(\hat{\mathbb{C}} \backslash (\alpha I)^c, I)$;
- ▷ $W_\alpha(I) := W(\hat{\mathbb{C}} \backslash (\alpha I)^c, I)$, a conformal invariant measuring the (near-)degeneracy at I.

When $W_\alpha(I) \geq K$ for some $K > 1$, we say that I is $[K, \alpha]$-*wide*.

Fix the constant[2] $\tau := 10$. Local degeneration will be represented by two quantities, namely the τ-*degeneration* $W_\tau(I) \gg 1$ and the λ-*degeneration* $W_\lambda(I) \gg 1$ at a piece $I \subset \mathbf{H}$ for some large parameter $\lambda \gg \tau$. We will take λ to be sufficiently large for our analysis to work, and emphasize whenever other constants depend on λ throughout Sections §3.3–3.6. One particular parameter that will appear frequently is $\mathbf{n}_\lambda$ defined below.

Definition 3.1.12. For any $\lambda > 1$, denote by $\mathbf{n}_\lambda$ the smallest integer such that for any combinatorial piece $I \subset \mathbf{H}$ of level $\geq \mathbf{n}_\lambda$, the pieces $2\lambda I$, $2\lambda f(I)$, and $2\lambda f^2(I)$ are pairwise disjoint.

In Case $\circledcirc$, we impose the additional assumption that any interval $I \subset \mathbf{H}$ we consider is always at the *Siegel scale*, i.e. $|I| \leq \mu$.

Lemma 3.1.13. *In Case $\circledcirc$, for any interval $I \subset \mathbf{H}$, the width of curves in the Herman ring $\mathbb{H}$ connecting I and the inner boundary component H^0 is at most* 5.

[2]The reader may wish to assign a different value for τ as long as it is a sufficiently large integer.

Proof. For any interval J in the outer boundary component $\mathbf{H}$, let $\tilde{J}$ denote the corresponding piece in $\overline{\mathbb{H}}$ such that $J = \tilde{J} \cap \mathbf{H}$. It comes with a canonical structure of a conformal rectangle with horizontal sides $\tilde{J} \cap \partial\mathbb{H}$.

At the Siegel scale, it is sufficient to prove the lemma for any interval I of length $|I| = \mu$. Let L and R denote the two intervals in $\mathbf{H}$ adjacent to I that have the same length μ. Then, $W(\widetilde{L}) = W(\widetilde{R}) = W(\tilde{I}) = 1$. The family $\mathcal{F}^0$ of curves in $\mathbb{H}$ connecting I and H^0 is contained in the union $\mathcal{F}_1 \cup \mathcal{F}_2$, where $\mathcal{F}_1$ consists of vertical curves of $\widetilde{3I}$ and $\mathcal{F}_2 := \mathcal{F}^0 \backslash \mathcal{F}_1$. Observe that $W(\mathcal{F}_1) = W(\widetilde{3I}) = 3$. Since every curve in $\mathcal{F}_2$ must cross either of the two rectangles $\widetilde{L}$ and $\widetilde{R}$, then by Proposition 2.2.5, $W(\mathcal{F}_2) \le 2$. Therefore, $W(\mathcal{F}^0) \le 5$. □

Intervals at the Siegel scale are conformally far from the inner boundary component H^0 of the Herman ring $\mathbb{H}$. As such, this situation is comparable to that of an interval on the boundary of a Siegel disk, in which the width between I and the inner component, which is the singleton consisting of the center, is 0.

The arguments we present in Sections §3.2–3.6 will mainly address Case ⦿ using only the combinatorial and dynamical properties of $\mathbf{H}$. The modulus μ will not play any major role until Sections §3.7–3.8. Most of the arguments in Sections §3.3–3.6 apply to Case ⊚ with a few adjustments presented as separate remarks.

3.1.5 Outline

Let us provide an outline of the proof of Theorem A. The key to *a priori bounds* is the Amplification Theorem 3.7.1 which states that the existence of a $[K,\tau]$-wide piece $I \subset \mathbf{H}$ implies the existence of a $[2K,\tau]$-wide piece, where K is sufficiently large (depending only on d_0, d_∞, and $\beta(\theta)$). Our analysis is split into two cases.

Herman scale $|I| > \mu$, (the **main case**, roughly ⦿)

Siegel scale $|I| \le \mu$. (roughly Case ⊚)

In the Siegel scale, this theorem is similar to (and was inspired by) [DL22, Theorem 8.1] in the context of quadratic Siegel disks. In the Herman scale, the techniques in [DL22], especially [DL22, Snake Lemma 2.12], are not applicable because, unlike in the Siegel scale, the geometry on both sides of I is unknown. In Sections §3.2–3.6, we develop the fundamental results needed to prove the Amplification Theorem.

In Section §3.2, we discuss the bubble-wave argument, a mechanism (Proposition 3.2.2) that generates large τ-width at a *shallow* level, i.e. when $|I| \asymp 1$. The main idea is to use the fact that *bubbles*, i.e. preimages of $\mathbf{H}$, up to a certain generation that are attached to $\mathbf{H}$ have controlled harmonic measure about either 0 or ∞ (Claim 2 in the proof of Lemma 3.2.3).

Section §3.3 discusses some ways to spread degeneration. Let $\frac{p_n}{q_n}$ be the best rational approximations of θ. A piece $I \subset \mathbf{H}$ is a *combinatorial piece* of level n if it has endpoints x and $f^{q_n}(x)$ for some point $x \in \mathbf{H}$. A level n *almost tiling* $\mathcal{I}$ is a collection of pieces with disjoint interiors of the form $\{f^i(I)\}_{0 \leq i \leq q_{n+1}-1}$ for some level n combinatorial piece I. By applying the Covering Lemma 2.2.9, we show in Proposition 3.3.2 that for any $\Xi > 1$, $\lambda \gg 10$, and $K \gg_{\Xi,\lambda} 1$, the existence of a $[K, \lambda]$-wide combinatorial piece implies the existence of either a $[\Xi K, 10]$-wide piece or an almost tiling $\mathcal{I}$ consisting of $[\xi K, \lambda]$-wide pieces for some $\xi = \xi(\Xi) > 0$.

The main result in Section §3.4 is Theorem 3.4.1 which states that for $K \gg_\lambda 1$, the existence of an almost tiling consisting of $[K, \lambda]$-wide pieces implies the existence of a $[\Pi_\lambda K, 10]$-wide piece where $\Pi_\lambda \to \infty$ as $\lambda \to \infty$. The proof is split into two cases: the deep case, where the level of the almost tiling is high, and the shallow case, where the level is low. The deep case is an application of the Quasi-Additivity Law, whereas the shallow case is handled using the bubble-wave argument.

In Section §3.5, we prove in Theorem 3.5.1 that for $\lambda \gg 10$ and $K \gg_\lambda 1$, the existence of a $[K, 10]$-wide combinatorial piece induces the existence of a piece that is either $[2K, 10]$-wide or $[\chi K, \lambda]$-wide, where $0 < \chi < 1$ is independent of λ. Since the shallow case can again be handled via the bubble-wave argument, we are left with the deep case. Our main strategy is to adapt Kahn's push-forward argument in [Kah06, §7] to our setting. A key ingredient in the original push-forward argument is the positivity of the core entropy corresponding to primitive renormalization, which stands in contrast to the lack of entropy of the rotational action of f on $\mathbf{H}$. Section §3.6 is dedicated to developing a replacement for Kahn's entropy argument, namely Proposition 3.6.2. Due to technical considerations, we supply a more detailed outline in §3.6.1.

Finally, the proof of the Amplification Theorem 3.7.1 is an application of Theorems 3.4.1 and 3.5.1. In short, we will eventually pick λ to be large enough such that the constant Π_λ beats the constant χ. This is captured in Figure 3.13.

3.2 Bubble-wave argument

In Sections §3.4 and §3.5, we will encounter degeneration witnessed by a combinatorial piece I that is either *shallow*, i.e. has level bounded above by some constant, or *deep*, i.e. not shallow. In the shallow case, we will need to rule out the presence of wide waves. Waves are defined as follows.

Definition 3.2.1. For $\bullet \in \{0, \infty\}$ and a piece $A \subset \mathbf{H}$, we say that a curve γ *protects A from $\bullet$*

if it is a proper curve in $Y^\bullet$ such that $A \cap \partial Y^\bullet$ is contained on the boundary of the connected component of $Y^\bullet \backslash \gamma$ that does not contain $\bullet$. We say that a lamination Ω is a *wave* if it is a proper lamination in $Y^\bullet$ for some $\bullet \in \{0, \infty\}$ such that there exists a piece A that is protected from $\bullet$ by every leaf. If $\bullet = 0$, it is called an *inner wave*; if $\bullet = \infty$, it is called an *outer wave*. The *(combinatorial) length* $|\Omega|$ of a wave Ω is the maximum combinatorial length $|A|$ of pieces A protected by Ω.

In this section, our aim is to convert a wide wave into τ-degeneration which increases with the length and width of the wave and is witnessed by a combinatorial piece of a controlled level.

Proposition 3.2.2 (Wide waves $\longrightarrow$ τ-degeneration)**.** *There exists an absolute constant $m \in \mathbb{N}$ such that the following holds. For every $n \in \mathbb{N}$ and $\alpha \geq 1$, there exists some $\mathbf{K} = \mathbf{K}(n) > 1$ such that if*

$$\textit{there exists a wave } \Omega \textit{ of length } |\Omega| \geq \alpha l_n \textit{ and width } W(\Omega) \geq \mathbf{K},$$

then

$$\textit{there exists a level } n+m \textit{ combinatorial piece } J \textit{ with } W_\tau(J) \succ \alpha W(\Omega).$$

The main idea of the proof is to use the interaction between waves and bubbles. The key step is Claim 2 in the proof of Lemma 3.2.3 below, in which we deduce that most of the wave at the shallow scale should pass through bubbles up to a certain generation. From there, we use these bubbles to split up the wave into wider ones and achieve a multiplicative factor depending on α.

3.2.1 Amplifying waves

We first argue that a combinatorially long wide wave induces an even wider wave of smaller but controlled length. Refer to Figure 3.3.

Lemma 3.2.3. *There exists an absolute constant $m' \in \mathbb{N}$ such that the following holds. For every $n \in \mathbb{N}$ and $\alpha \geq 1$, there exists some $\mathbf{K} = \mathbf{K}(n) > 1$ such that if*

$$\textit{there exists a wave } \Omega \textit{ of length } |\Omega| \geq \alpha l_n \textit{ and width } W(\Omega) \geq \mathbf{K},$$

then

$$\textit{there is another wave } \Omega' \textit{ of length } |\Omega'| \geq l_{n+m'} \textit{ and width } W(\Omega') \geq 2\alpha\, W(\Omega).$$

Proof. Pick $n \in \mathbb{N}$ and $\alpha \geq 1$. We shall first introduce two absolute constants $m'', m' \in \mathbb{N}$ satisfying

$$10\, l_{n+m''} < l_n, \text{ and} \tag{3.2.1}$$

$$2\, l_{n+m'} \leq l_{n+m''+2}. \tag{3.2.2}$$

Set $t := q_{n+m''+2}$.

Suppose Ω is an outer wave of length $\geq \alpha l_n$ and width $\geq \mathbf{K}$. Denote by A the longest piece protected by Ω. For every outer critical point $c \in \mathbf{H}$, denote by $\mathcal{O}_c := \{f|_{\mathbf{H}}^{-i}(c)\}_{i=0,\dots,t-1} \cap A$ the set of preimages of c up to time $t-1$ that lie on A, and by $\mathcal{B}_c$ the set of outer bubbles B such that B is rooted at some point $f|_{\mathbf{H}}^{-i}(c)$ in $\mathcal{O}_c$ where $i+1$ is the generation of B.

Claim 1. There exist $k \geq 5\alpha$ distinct pieces $P_{n_1}, \dots, P_{n_k}$ length $\geq l_{n+m'}$.

Proof. Since $|A| \geq \alpha l_n$, it is sufficient to show that the claim is true for $k \geq 5|A|/l_n$. Suppose otherwise. Then, the number of pieces P_i of length $< l_{n+m'}$ is more than $N - 5|A|/l_n$ and the rest have length between $l_{n+m'}$ and $l_{n+m''}$. In particular,

$$|A| < \left(N - \frac{5|A|}{l_n}\right) l_{n+m'} + \frac{5|A|}{l_n} l_{n+m''}.$$

By (3.2.1), this simplifies to

$$1 < \left(\frac{2N}{|A|} - \frac{10}{l_n}\right) l_{n+m'}.$$

By 2.1.2, for every critical point c, adjacent points in $\mathcal{O}_c$ have distance at least $l_{n+m''+2}$, so $\mathcal{O}_c$ has cardinality at most $|A|/l_{n+m''+2}$. Since f has less than d outer critical points, we deduce that $N < d|A|/l_{n+m''+2}$. As such,

$$1 < \frac{2d\, l_{n+m'}}{l_{n+m''+2}} - \frac{10\, l_{n+m'}}{l_n}.$$

However, this implies that $2d\, l_{n+m'} > l_{n+m''+2}$, which contradicts (3.2.2). $\square$

Next, we will remove parts of the wave that skip some bubbles. We claim that such removal is harmless.

Claim 2. The width of leaves in Ω that are disjoint from a bubble $B \in \mathcal{B}$ is at most some constant depending only on n.

Proof. Denote by Y_t^∞ the connected component of $f^{-t}(Y^\infty)$ that contains ∞. The map $f^t : Y_t^\infty \to Y^\infty$ is a degree d_∞^t covering map branched only at ∞. For every bubble $B \in \mathcal{B}$, the harmonic measure κ of its outer boundary $B \cap \partial Y_t^\infty$ in Y_t^∞ about ∞ is equal to d_∞^{-g}, where g is the generation of B. Since $g \leq t = q_{n+m''+2}$, then $\kappa \geq d_\infty^{-q_{n+m''+2}}$. Therefore, the claim follows from Proposition 2.2.7. $\square$

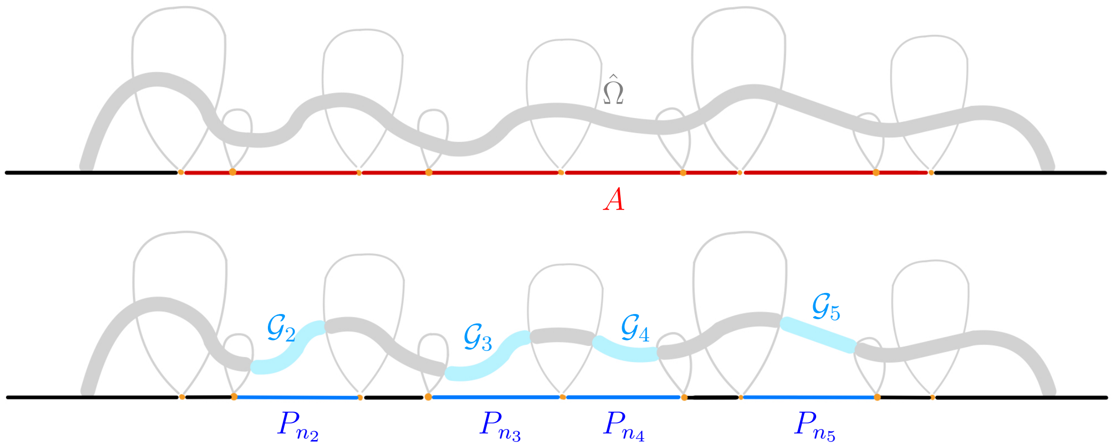

Figure 3.3: The outer wave $\hat{\Omega}$ and the laminations $\mathcal{G}_i$'s connecting bubbles attached to the endpoints of P_{n_i}'s.

By Claim 2, we can take $\mathbf{K}$ to be sufficiently high depending on n and assume that the sublamination $\hat{\Omega}$ consisting of leaves of Ω that consecutively intersect every bubble in $\mathcal{B}$ has width

$$W\left(\hat{\Omega}\right) \geq \frac{2}{3} W(\Omega). \tag{3.2.3}$$

There exist pairwise disjoint proper laminations $\mathcal{G}_2, \mathcal{G}_3, \ldots, \mathcal{G}_{k-1}$ in Y_t^∞ such that each $\mathcal{G}_i$ is a *restriction* of $\hat{\Omega}$ (refer to §2.2.1) and connects two bubbles in $\mathcal{B}$ attached to the two endpoints of P_{n_i}. See Figure 3.3. Suppose the widest one is $\mathcal{G}_s$ for some s. Since $\mathcal{G}_2, \ldots, \mathcal{G}_{k-1}$ are pairwise disjoint, by Propositions 2.2.2 and 2.2.3,

$$W\left(\hat{\Omega}\right) \leq W(\mathcal{G}_2) \oplus \ldots \oplus W(\mathcal{G}_{k-1}) \leq \frac{1}{k-2} W(\mathcal{G}_s). \tag{3.2.4}$$

By (3.2.3), (3.2.4), and the assumption that $\alpha \geq 1$,

$$W(\mathcal{G}_s) \geq (k-2) W\left(\hat{\Omega}\right) \geq (5\alpha - 2) \cdot \frac{2}{3} W(\Omega) \geq 2\alpha\, W(\Omega).$$

Let $g \in \mathbb{N}$ be the maximum generation of the two bubbles that the endpoints of leaves of $\mathcal{G}_s$ lie on. The image $\Omega' := f^g(\mathcal{G}_s)$ is the wave we are looking for. Indeed, it has the same width as $\mathcal{G}_s$, which is at least $2\alpha\, W(\Omega)$, and Ω' has length at least $l_{n+m'}$ because of Claim 1 and the fact that Ω' protects the piece $f^g(P_{n_s})$. □

3.2.2 Wide waves yield τ-degeneration

By an inductive argument, we can now obtain a τ-degeneration out of a wide wave.

Proof of Proposition 3.2.2. Pick $n \in \mathbb{N}$ and $\alpha \geq 1$. Let m' and $\mathbf{K}$ be the constants from Lemma 3.2.3 and set $m \in \mathbb{N}$ to be the smallest integer such that $\frac{\tau-1}{2} l_{n+m} \leq l_{n+m'}$. Let Ω be a wave of combinatorial length $\geq \alpha l_n$ and width $W(\Omega) \geq \mathbf{K}$.

Claim. Either there exists a level $n+m$ combinatorial piece J satisfying $W_\tau(J) \succ \alpha W(\Omega)$, or for every $t \geq 1$, there exists a wave Ω_t of length $\geq l_n$ and width

$$W(\Omega_t) \geq \left(\frac{3}{2}\right)^t \alpha W(\Omega). \tag{3.2.5}$$

Proof. We will proceed by induction. Suppose there exists a wave Ω_t of length $\geq l_n$ satisfying (3.2.5) for some $t \in \mathbb{N}$. We will also include the initial case $t = 0$, in which $\Omega_0 := \Omega$ has length $\geq \alpha l_n$ and width $W(\Omega)$. By Lemma 3.2.3, there is a wave Ω'_t of width

$$W(\Omega'_t) \geq \begin{cases} 2\, W(\Omega_t) & \text{if } t \geq 1, \\ 2\alpha\, W(\Omega) & \text{if } t = 0, \end{cases} \tag{3.2.6}$$

protecting a piece J_t of length $l_{n+m'}$. Note that by (3.2.5) and (3.2.6), we have

$$W(\Omega'_t) \geq 2\alpha\, W(\Omega). \tag{3.2.7}$$

Let I_{t+1} be the level n combinatorial piece that shares the same combinatorial mid-segment as J_t. We present Ω'_t as $\Omega_{t+1} \cup \Omega''_t$ where Ω_{t+1} is the set of leaves of Ω'_t that protect I_{t+1} and Ω''_t is the set of leaves that land on $I_{t+1} \backslash J_t$.

If $W(\Omega_{t+1}) \geq \frac{3}{4} W(\Omega'_t)$, then by combining this with (3.2.5) and (3.2.6), the wave Ω_{t+1} satisfies (3.2.5) and we are done. Suppose instead that

$$W(\Omega''_t) > \frac{1}{4} W(\Omega'_t). \tag{3.2.8}$$

There exists a level $n+m$ combinatorial piece $J \subset \overline{I_{t+1} \backslash J_t}$ such that amongst every level $n+m$ subpiece of $\overline{I_{t+1} \backslash J_t}$, the width of leaves of Ω'_t landing on J is the widest. Our choice of m guarantees that leaves of Ω''_t that land on J lie in $\mathcal{F}_\tau(J)$, yielding

$$W_\tau(J) \geq \frac{|J|}{|I_{t+1} \backslash J_t|} W(\Omega''_t) \succ W(\Omega''_t). \tag{3.2.9}$$

Therefore, by combining (3.2.7), (3.2.8), and (3.2.9), we obtain $W_\tau(J) \succ \alpha W(\Omega)$. □

The proposition holds because if otherwise, the claim above would give us an infinite sequence of waves Ω_t of uniformly bounded length and exponentially increasing width, which contradicts the compactness of $\mathbf{H}$. □

3.3 Spreading degeneration

Recall from Proposition 2.1.2 that for any level n piece I, the corresponding pieces I, $f(I)$, $f^2(I), \ldots, f^{q_{n+1}-1}(I)$ have pairwise disjoint interior.

Definition 3.3.1. The *level n almost tiling* $\mathcal{I}$ generated by a level n combinatorial piece $I \subset \mathbf{H}$ is the collection of iterated pieces $\{f^i(I)\}_{i=0,\ldots,q_{n+1}-1}$.

In this section, we will spread a given λ-degeneration to an almost tiling consisting of pieces that are all comparably λ-degenerate relative to the original. Recall the threshold parameter $\mathbf{n}_\lambda$ defined in §3.1.4.

Proposition 3.3.2. *For any $\Xi > 1$ and $\lambda > \tau$, there are some $\mathbf{K} = \mathbf{K}(\Xi, \lambda) > 1$ and $\xi = \xi(\Xi) > 0$ such that if there is a $[K, \lambda]$-wide level n combinatorial piece I where $n \geq \mathbf{n}_\lambda$ and $K \geq \mathbf{K}$, then either*

(1) *there is a $[\Xi K, \tau]$-wide combinatorial piece of level n, or*

(2) *there is a level n almost tiling consisting of $[\xi K, \lambda]$-wide pieces.*

In the proof, we will apply the Covering Lemma (Lemma 2.2.9) to spread λ-degeneration around $\mathbf{H}$. We will introduce *cuts* (Lemma 3.3.6) to bound the degree of the appropriate branched covering in terms of λ.

3.3.1 Spreading τ-degeneration

We will first discuss what we can do with τ-degeneration. This can be seen as a special case of Proposition 3.3.2 when $\lambda = \tau$.

Proposition 3.3.3. *There are absolute constants $0 < \varepsilon < 1$ and $\mathbf{K} > 1$ such that for any $[K, \tau]$-wide combinatorial piece $I \subset \mathbf{H}$ of level n where $n \geq \mathbf{n}_\tau$ and $K \geq \mathbf{K}$, every piece in the almost tiling generated by $f^2(I)$ is $[\varepsilon K, \tau]$-wide.*

We will apply Proposition 2.2.4 as the main tool to compare the τ-widths of a piece I and its iterate $f^i(I)$. This motivates us to first estimate the degree of f^i near τI, which we can deduce in a more general way as follows.

Lemma 3.3.4. *Suppose $f^a : U \to U'$ is a branched covering map between two open disks U and U' in $\mathbb{C}^*$ where $a \leq q_{n+k}$ and $\overline{U'} \cap \mathbf{H}$ is a piece of length ρl_n for some positive integers k and n, and some constant $\rho \geq 1$. Then,*

$$deg(f^a : U \to U') \leq M$$

for some $M = M(k, \rho) > 1$.

Proof. For $t = 0, 1, \dots, a$, let $U_t := f^t(U)$. Observe that each $\overline{U_t} \cap \mathbf{H}$ must be a piece of length ρl_n. Let $C > 1$ be the constant from Proposition 2.1.3, then $l_n \leq C^k l_{n+k}$. Since $a \leq q_{n+k}$, for every critical point $c \in \mathbf{H}$, there are at most $C^k \rho$ values of $t \in \{0, 1, \dots, a\}$ such that U_t contains c. Since f has $d-1$ free critical points counting multiplicity, the number of different pairs (c, t) such that U_t contains a free critical point c is at most $C^k \rho (d-1)$. Therefore, the degree of $f^a : U \to U'$ is at most $2^{C^k \rho (d-1)}$. □

Remark 3.3.5. In Case ◎ (as outlined in §3.1.4), in order for the lemma above to work, we shall assume additionally that U and U' are disjoint from the connected component $\hat{Y}^0$ of $\hat{\mathbb{C}} \backslash \mathbb{H}$ containing 0 so that every critical value of the mapping $f^a : U \to U'$ lies on the outer boundary $\mathbf{H}$.

Next, we have to pick the disk U containing τI carefully. In particular, we would like to restrain the local degree of an iterate f^i on U so that it is independent of i.

Lemma 3.3.6 (Cuts)**.** *For any piece I such that I, $f(I)$, and $f^2(I)$ are pairwise disjoint, there exist some $t \in \{0, 1, 2\}$ and a pair of closed rays $\gamma_0 \subset \overline{Y^0}$ and $\gamma_\infty \subset \overline{Y^\infty}$ connecting a point in $(f^t(I))^c$ to 0 and ∞ respectively such that the width of curves in $\hat{\mathbb{C}} \backslash (\mathbf{H} \cup \gamma_0 \cup \gamma_\infty)$ connecting $f^t(I)$ and $\gamma_0 \cup \gamma_\infty$ is at most* 10.

Proof. Since I, $f(I)$, and $f^2(I)$ are pairwise disjoint, there is some $t \in \{0, 1, 2\}$ such that for $\bullet \in \{0, \infty\}$, the harmonic measure of $f^t(I) \cap \partial Y^\bullet$ in $Y^\bullet$ about $\bullet$ is less than $\frac{1}{2}$. Then, [GM05, Chapter IV Theorem 5.2] guarantees the existence of a pair of such rays γ_0 and γ_∞ where the width of curves in Y^0 (resp. Y^∞) connecting $f^t(I)$ and γ_0 (resp. γ_∞) is at most 5. □

The rays γ_0 and γ_∞ satisfying the above will be called *cuts* for the piece $f^t(I)$. These cuts will help us define the appropriate disks.

Proof of Proposition 3.3.3. Let $I \subset \mathbf{H}$ be a $[K, \tau]$-wide combinatorial piece of level $n \geq \mathbf{n}_\tau$ and let $I_s := f^s(I)$ for any $s \geq 0$. Pick any integer $a \in [2, q_{n+1} + 1]$. We can assume that there exist cuts γ_0 and γ_∞ for τI_a. (Otherwise, replace I_a with I_{a-i} for some $i \in \{1, 2\}$ and apply Proposition 2.2.4.)

Let U' denote the open disk $\hat{\mathbb{C}} \backslash ((\tau I_a)^c \cup \gamma_0 \cup \gamma_\infty)$, and let U be the connected component of $f^{-a}(U')$ containing I. By Proposition 2.2.4,

$$K \leq W(U, I) \leq \deg(f^a : U \to U') \cdot W(U', I_a).$$

By Lemma 3.3.4, the inequality implies $W(U', I_a) \succ K$. Curves in $\mathcal{F}(U', I_a)$ connect I_a to either $(\tau I_a)^c$ or the cuts $\gamma_0 \cup \gamma_\infty$. The width of those landing at $\gamma_0 \cup \gamma_\infty$ is at most 10, so when $K \geq \mathbf{K}$ and $\mathbf{K}$ is sufficiently high, we have $W_\tau(I_a) \succ K$. □

Remark 3.3.7. In Case ⊙, we shall modify the proof above by replacing the topological disk U' with $U' \backslash \hat{Y}^0$. The removal of $\hat{Y}^0$ is necessary in order to apply Lemma 3.3.4 (see Remark 3.3.5), and harmless because the width of curves in $\mathcal{F}(U', I_a)$ that land on $\hat{Y}^0$ is negligible due to Lemma 3.1.13.

3.3.2 Spreading λ-degeneration

Even though the proof of the previous lemma can also be applied to λ-degeneration, the corresponding multiplicative factor would depend on λ. We will employ a different spreading approach by applying the Covering Lemma as follows. (See [DL22, §8.1] in the case of quadratic Siegel disks.)

Proof of Proposition 3.3.2. Let $I \subset \mathbf{H}$ be a $[K, \lambda]$-wide combinatorial piece of level $n \geq \mathbf{n}_\lambda$, where $K \geq \mathbf{K}$, and let $I_s := f^s(I)$ for any $s \geq 0$.

Pick an integer $a \in [2, q_{n+1} + 1]$. Since $n \geq \mathbf{n}_\lambda$, by Lemma 3.3.6, there exist cuts γ_0 and γ_∞ for λI_b for some $b \in \{a-2, a-1, a\}$. Then, consider the iterate $f^b : (U, \Lambda, I) \to (V, B, I_b)$ where

- ▹ $V := \hat{\mathbb{C}} \backslash ((\lambda I_b)^c \cup \gamma_0 \cup \gamma_\infty)$;
- ▹ $B := V \backslash (\tau I_b)^c$;
- ▹ $U :=$ the connected component of $f^{-b}(V)$ containing I;
- ▹ $\Lambda :=$ the connected component of $f^{-b}(B)$ containing I.

By Lemma 3.3.4,

$$\deg(f^b : \Lambda \to B) \leq M(\tau), \quad \text{and} \quad \deg(f^b : U \to V) \leq M(\lambda).$$

Fix the constant $\Xi > 1$. Since ∂U contains $(\lambda I)^c$, we have $W(U, I) \geq K$. By Lemma 2.2.9, for sufficiently high $\mathbf{K}$ depending on Ξ and λ, either

$$W(B, I_b) > (d^2 \Xi + 1) K \quad \text{or} \quad W(V, I_b) > \xi_1 K,$$

where $\xi_1 \in (0,1)$ depends only on Ξ. By Lemma 3.3.6, the width of curves in $\mathcal{F}_\lambda(I_b)$ landing at the cuts $\gamma_0 \cup \gamma_\infty$ is at most 10. Therefore, for sufficiently high $\mathbf{K}$, either

$$W_\tau(I_b) \geq d^2 \Xi K \quad \text{or} \quad W_\lambda(I_b) \geq \xi_2 K,$$

for some $\xi_2 \in (0,1)$ depending only on Ξ. After pushing forward by f^{a-b}, we conclude that the piece I_a is either $[\Xi K, \tau]$-wide or $[\xi K, \lambda]$-wide, where $\xi = d^{-2} \xi_2$. Therefore, if there is no $2 \leq a \leq q_{n+1} + 1$ such that I_a is $[\Xi K, \tau]$-wide, then I_2 generates an almost tiling consisting of $[\xi K, \lambda]$-wide pieces. □

Remark 3.3.8. In Case ⊚, the proof above needs to be modified by replacing the disk V with $V\backslash\hat{Y}^0$, similar to Remark 3.3.7.

3.4 Trading λ-degeneration for a τ-degeneration

Given a λ-degeneration, the previous section tells us how to spread and obtain an almost tiling of λ-degenerate pieces. Next, we would like to convert such an almost tiling into a much larger τ-degeneration with a multiplicative factor that grows with λ. The main result of this section is the following theorem.

Theorem 3.4.1. *For all sufficiently large λ, there are parameters $\mathbf{m}_\lambda, \mathbf{K}, \Pi_\lambda > 1$ all depending on λ where $\lim_{\lambda\to\infty} \Pi_\lambda = +\infty$ such that if*

$$\text{there is an almost tiling } \mathcal{I} \text{ consisting of } [K,\lambda]\text{-wide pieces of level } n \geq \mathbf{n}_\lambda$$

where $K \geq \mathbf{K}$, then

$$\text{there is a } [\Pi_\lambda K, \tau]\text{-wide combinatorial piece } J \text{ of level } n' \geq \mathbf{n}_\lambda$$

where $|n' - n| \leq \mathbf{m}_\lambda$.

The proof will be split into two cases:

deep case $n \geq \mathbf{n}_\lambda + \mathbf{m}_\lambda$;

shallow case $\mathbf{n}_\lambda \leq n < \mathbf{n}_\lambda + \mathbf{m}_\lambda$.

The threshold level $\mathbf{n}_\lambda$ is essential because we will apply the theorem above inductively in Section §3.7.

Remark 3.4.2. By assuming sufficiently small modulus (depending on λ), we can ensure that in Case ⊚, intervals at the Siegel scale (see §3.1.4) are deep. Therefore, the shallow case can be ignored in Case ⊚.

3.4.1 Deep case

A deep almost tiling can be handled through a straightforward application of the Quasi-Additivity Law (Lemma 2.2.8).

Proof of Theorem 3.4.1 in the deep case. Assume $\lambda \gg \tau^2$ and set $N := \lfloor \frac{\lambda}{3\tau^2} \rfloor$. Suppose there is a level n almost tiling $\mathcal{I}$ consisting of $[K,\lambda]$-wide pieces where $K \geq \mathbf{K}$. There exists a sequence $\{I_j\}_{j=1,\dots,N}$ of distinct pieces in the almost tiling $\mathcal{I}$, labelled in consecutive order, such that for every $j \in \{1, 2, \dots, N-1\}$, I_j and I_{j+1} have controlled combinatorial distance:

$$(\tau - 1)l_n < \operatorname{dist}(I_j, I_{j+1}) \le \tau l_n.$$

This ensures that τI_j and $\cup_{i \ne j} I_i$ are always disjoint but not too far apart.

Let P be the unique shortest piece containing $\bigcup_{j=1}^N I_j$. We set $\mathbf{m}_\lambda$ to be the largest integer less than n such that $|P| \ge l_{n-\mathbf{m}_\lambda}$. Our choice of N ensures that each λI_j contains τP. Consider the disk $S \coloneqq \hat{\mathbb{C}} \setminus \bigcup_{j=1}^N (\lambda I_j)^c$. Following [Ahl06], we will use the notation $\mathcal{H} < \mathcal{G}$ to denote that $\mathcal{G}$ *overflows* $\mathcal{H}$. (See §2.2.1.) Then, for every $j \in \{1, \dots, N\}$,

$$\mathcal{F}(S, I_j) < \mathcal{F}_\lambda(I_j), \quad \mathcal{F}_\tau(I_j) < \mathcal{F}\left(S \setminus \bigcup_{i \ne j} I_i, I_j\right), \quad \mathcal{F}_\tau(P) < \mathcal{F}\left(S, \bigcup_{i=1}^N I_i\right).$$

We are under the assumption that for each j,

$$W(S, I_j) \ge W_\lambda(I_j) \ge K.$$

For sufficiently large $\mathbf{K}$, we can apply Lemma 2.2.8 and obtain

$$\max\{W_\tau(P), W_\tau(I_1), \dots, W_\tau(I_N)\} \ge \frac{1}{\sqrt{2N}} \sum_{j=1}^N W(S, I_j) \ge \sqrt{\frac{N}{2}} K.$$

Since $N \asymp \lambda$, we conclude that either

$$W_\tau(P) \succ \sqrt{\lambda} K, \quad \text{or} \quad W_\tau(I_j) \succ \sqrt{\lambda} K$$

for some j. If the former, there exists a combinatorial subpiece $J \subset P$ of level $n - \mathbf{m}_\lambda$ and τ-width $W_\tau(J) \succ \sqrt{\lambda} K$. □

Notice that, in the proof above, the piece J is longer than the original piece I. This justifies the need for a different approach when n is shallow.

3.4.2 Shallow case

The main ingredient in the shallow case is the bubble-wave argument. Given a wide lamination, we are split into two different situations: either it forms combinatorially long wide waves or it intersects $\mathbf{H}$ frequently in a snake-like pattern (see Figure 3.5). Both cases will produce a large τ-degeneration.

Proof of Theorem 3.4.1 in the shallow case. Fix $\mathbf{K}$, and suppose there is a $[K, \tau]$-wide piece I of shallow level n (e.g. by picking any piece from the almost tiling $\mathcal{I}$ in the hypothesis) where $K \ge \mathbf{K}$.

If there is a wave Ω of width $\geq K/10$ and length $\geq \sqrt[3]{\lambda} l_n$, then by Proposition 3.2.2, there is a level $n+m$ a combinatorial piece J such that

$$W_\tau(J) \succ \sqrt[3]{\lambda} \cdot \frac{K}{10}$$

and we are done, assuming $\mathbf{K}$ is sufficiently high depending on $\mathbf{n}_\lambda + \mathbf{m}_\lambda$, and λ is sufficiently large such that $\mathbf{m}_\lambda \geq m$. As such, we proceed under the following assumption.

No-wide-wave assumption: Every wave of length $\geq \sqrt[3]{\lambda} l_n$ has width $\leq K/10$.

Let us decompose $(\lambda I)^c$ into $T^+ \cup T \cup T^-$, where T^+ and T^- are the left and rightmost pieces of $(\lambda I)^c$ of length $\sqrt[3]{\lambda} l_n$, and $T := (\lambda I)^c \backslash (T^+ \cup T^-)$. Let $\mathcal{F}$ be the set of leaves of the canonical radial foliation of the conformal annulus $\hat{\mathbb{C}} \backslash (I \cup (\lambda I)^c)$ that never restrict to curves protecting $T^\dagger$ from $\bullet$ for any $\bullet \in \{0, \infty\}$ and $\dagger \in \{+, -\}$.

Claim 1. Every leaf of $\mathcal{F}$ connects I and $T^+ \cup T^-$.

Proof. If a radial leaf lands on T, then it must restrict to a subcurve that protects either T^+ or T^-. □

We can decompose $\mathcal{F}$ into $\mathcal{F}^+ \cup \mathcal{F}^-$ according to whether leaves land on T^+ or T^-. From the no-wide-wave assumption, the width $W(\mathcal{F})$ of $\mathcal{F}$ is at least $6K/10$. Without loss of generality, assume that $\mathcal{F}^+$ is wider, so then

$$W\left(\mathcal{F}^+\right) \geq \frac{3}{10} K. \tag{3.4.1}$$

Recall the notion of conformal rectangles and buffers from §2.2.1. We say that a lamination $\mathcal{L}$ is *rectangular* if it is a sublamination of the vertical foliation of a conformal rectangle R. Moreover, a sublamination of a rectangular lamination $\mathcal{L}$ is a *buffer* of $\mathcal{L}$ if it is the set of leaves of $\mathcal{L}$ that lie in a buffer of R.

Claim 2. $\mathcal{F}^+$ is rectangular[3].

Proof. By construction, the set $\tilde{\mathcal{F}}^+$ of leaves in the radial foliation of $\hat{\mathbb{C}} \backslash (I \cup (\lambda I)^c)$ that land on T^+ forms a single conformal rectangle. By Claim 1, $\mathcal{F}^+$ is a sublamination of $\tilde{\mathcal{F}}^+$. □

Consider a finite sequence of distinct pieces $I_0 := I, I_1, \ldots, I_N$ labelled in consecutive order such that for each $j = 1, \ldots, N$,

(i) I_j is a subpiece of λI located between I and T^+;

[3] In fact, $\mathcal{F}^+$ is the vertical foliation of a conformal rectangle. An approach similar to Claim 3 can be used to prove this.

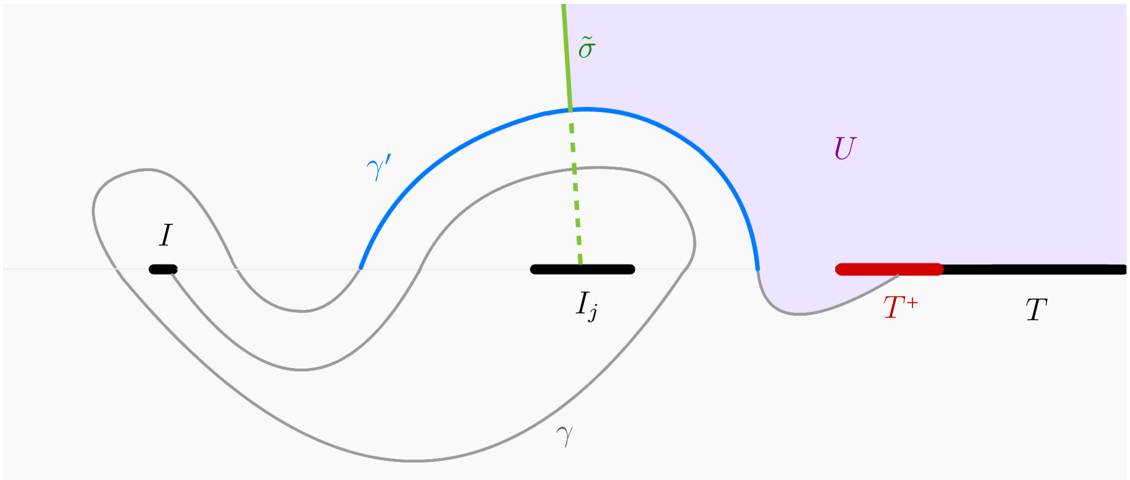

Figure 3.4: Domain U defined by the ray σ and the leaf γ.

(ii) $|I_j| = \sqrt[3]{\lambda} l_n$;

(iii) I_j is of distance at least $\frac{\tau-1}{2} l_n$ away from I_i for all $i \neq j$.

We pick N to be the maximum possible integer such that (i)-(iii) holds.

Claim 3. For any $\bullet \in \{0, \infty\}$ and $j \in \{1, \ldots, N\}$, the set $\mathcal{F}^+_{j,\bullet}$ of leaves of $\mathcal{F}^+$ that contain a subcurve protecting I_j from $\bullet$ is a buffer of $\mathcal{F}^+$.

Proof. Every leaf of $\mathcal{F}^+$ cannot contain a subcurve protecting $(\lambda I)^c$, because otherwise it would protect T^+. As such, for each $\bullet \in \{0, \infty\}$, there exists a ray $\sigma^\bullet$ in $Y^\bullet$ that is disjoint from $\mathcal{F}^+$ and connects $\bullet$ and T. Let $D := \hat{\mathbb{C}} \backslash \big((\lambda I)^c \cup \overline{\sigma^0 \cup \sigma^\infty} \big)$.

Suppose a leaf γ of $\mathcal{F}^+$ contains a subcurve protecting I_j from $\bullet$; label by γ' the corresponding subcurve of γ that is furthest from I_j. Pick any simple curve σ in $D \cap Y^\bullet$ connecting $\bullet$ and a point on I_j. Then, σ intersects γ' and contains a subcurve $\tilde{\sigma}$ that connects $\bullet$ and a point $w \in \gamma' \cap \sigma$ and is disjoint from γ away from w. Clearly, $\tilde{\sigma}$ splits the disk $D \backslash (I \cup \gamma)$ into two components, one of which, labelled by U, has closure that is disjoint from I. See Figure 3.4.

The leaf γ splits $\mathcal{F}^+$ into two rectangular sublaminations on opposite sides of γ. One of the sublaminations, labelled by $\mathcal{F}^+_\gamma$, has support that intersects U. Since every leaf of $\mathcal{F}^+_\gamma$ must land on I and avoid γ, then every leaf of $\mathcal{F}^+_\gamma$ must intersect σ. As σ is arbitrary, this implies that every leaf of $\mathcal{F}^+_\gamma$ contains a subcurve protecting I_j from $\bullet$. Then, the claim follows from the fact that $\mathcal{F}^+_\gamma$ is a buffer of $\mathcal{F}^+$. $\square$

Let $\mathcal{G}$ be the sublamination of $\mathcal{F}^+$ consisting of all leaves that intersect all the I_j's in consecutive order (I_{j-1} before I_j). If a leaf γ of $\mathcal{F}^+$ is not in $\mathcal{G}$, then it must contain a

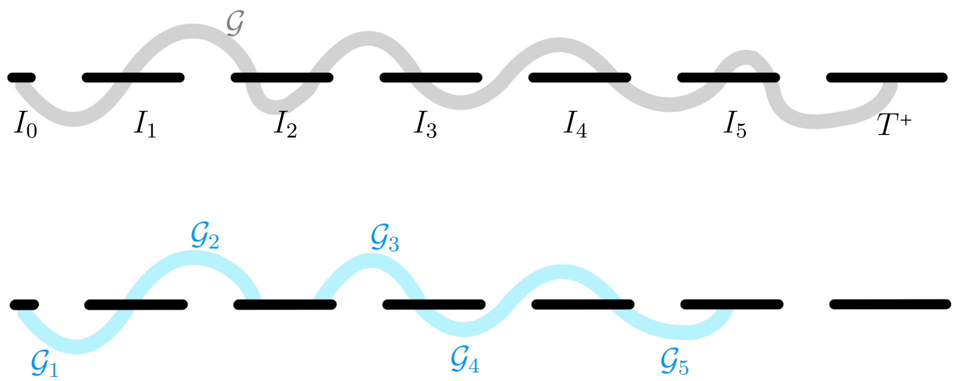

Figure 3.5: The sublamination $\mathcal{G} \subset \mathcal{F}^+$ intersects all I_i's in order.

subcurve protecting some I_j. Therefore, by Claim 3, there exist pairs $j, k \in \{1, \dots, N\}$ and $\sharp, \flat \in \{0, \infty\}$ such that $\mathcal{F}^+ \backslash \mathcal{G}$ is contained in a union of two maximal buffers $\mathcal{F}^+_{j,\sharp}$ and $\mathcal{F}^+_{k,\flat}$. In particular, $\mathcal{F}^+ \backslash \mathcal{G}$ overflows a union of two waves of length at least $\sqrt[3]{\lambda} l_n$. By (3.4.1) and the no-wide-wave wave assumption,

$$W(\mathcal{G}) \geq \frac{K}{10}.$$

There exist pairwise disjoint laminations $\mathcal{G}_1, \dots, \mathcal{G}_N$ such that each $\mathcal{G}_j$ is a restriction of $\mathcal{G}$ and connects I_{j-1} and I_j. See Figure 3.5 for illustration. Suppose $\mathcal{G}_s$ has the largest width amongst all the $\mathcal{G}_i$'s. By Propositions 2.2.2 and 2.2.3,

$$W(\mathcal{G}_s) \geq \frac{N}{10} K.$$

Since the gaps between the I_j's are at least $\frac{\tau - 1}{2} l_n$ in length, there must be a level n combinatorial subpiece $J \subset I_s$ such that

$$W_\tau(J) \geq \frac{1}{\sqrt[3]{\lambda}} W(\mathcal{G}_s).$$

For sufficiently high λ, the maximum possible value of N satisfies $N \asymp \sqrt[3]{\lambda^2}$. By combining the two inequalities above, we have

$$W_\tau(J) \geq \frac{N}{10\sqrt[3]{\lambda}} K \asymp \sqrt[3]{\lambda} K,$$

and we are done. $\square$

3.5 Amplifying τ-degeneration

In this section, we work our way towards the amplification of a τ-degeneration. More precisely, we aim to find a way to promote a τ-degeneration in **H** into either a significantly larger τ-

degeneration or a comparable λ-degeneration. Unlike the previous section, the multiplicative factor will be independent of λ.

Theorem 3.5.1. *There are absolute constants* $\chi \in (0,1)$ *and* $\mathbf{m} \in \mathbb{N}$ *such that for sufficiently large* λ, *there is some* $\mathbf{K} = \mathbf{K}(\lambda) > 1$ *such that if*

there is a $[K,\tau]$*-wide combinatorial piece* I *of level* $n \geq \mathbf{n}_\lambda$

where $K \geq \mathbf{K}$, *then*

there is a combinatorial piece J *of level* $n' \geq \mathbf{n}_\lambda$, *where* $|n'-n| \leq \mathbf{m}$,
that is either $[\chi K, \lambda]$*-wide or* $[2K,\tau]$*-wide.*

Similar to the previous section, we shall split the proof into two cases:

deep case $n \geq \mathbf{n}_\lambda + \mathbf{m}$;

shallow case $\mathbf{n}_\lambda \leq n < \mathbf{n}_\lambda + \mathbf{m}$.

Remark 3.5.2. By assuming sufficiently small modulus, we can again ensure that in Case ◎, intervals at the Siegel scale (see §3.1.4) are deep.

3.5.1 Shallow case

The shallow case can again be handled using the bubble-wave argument almost the exact same way as our treatment in §3.4.2. Any repeated details will be spared.

Proof of Theorem 3.5.1 in the shallow case. Fix $\mathbf{K}$ and suppose I is a $[K,\tau]$-wide level n combinatorial piece where $K \geq \mathbf{K}$ and n is shallow. Fix a pair of positive integers m' and m''; both are independent of λ and will be determined later.

Assume that m'' is high enough such that $l_{n+m'} \geq \tau l_{n+m'+m''}$. If there is a wave of width $\geq K/10$ and combinatorial length $\geq l_{n+m'}$, then by Proposition 3.2.2, there is a level $n+m+m'+m''$ combinatorial piece J such that

$$W_\tau(J) \succ \frac{l_{n+m'}}{l_{n+m'+m''}} \cdot \frac{K}{10} \succ \tilde{C}^{m''} K,$$

for some absolute constants $\tilde{C} > 1$ and $m \in \mathbb{N}$. By picking a sufficiently high m'' such that J is $[2K,\tau]$-wide, and by picking the threshold increment $\mathbf{m} \in \mathbb{N}$ such that $\mathbf{m} \geq m + m' + m''$, we are done. As such, we will proceed under the following assumption.

No-wide-wave assumption: Every wave of length $\geq l_{n+m'}$ has width $\leq K/10$.

Let T^+ and T^- be the leftmost and rightmost level $n+m'$ combinatorial subpieces of $(\tau I)^c$. Let $\mathcal{F}$ be the set of leaves of the canonical radial foliation of the conformal annulus $\hat{\mathbb{C}} \backslash (I \cup (\tau I)^c)$ that never restrict to curves protecting $T^\dagger$ from $\bullet$ for any $\bullet \in \{0, \infty\}$ and $\dagger \in \{+,-\}$. Since leaves of $\mathcal{F}$ must connect I and $T^+ \cup T^-$, we can decompose $\mathcal{F}$ into $\mathcal{F}^+ \cup \mathcal{F}^-$ according to whether leaves land on T^+ or T^-. Without loss of generality, assume that $\mathcal{F}^+$ is wider.

Consider a finite sequence of distinct combinatorial pieces $I_0 := I, I_1, \ldots, I_N$ labelled in consecutive order such that for each $j = 1, \ldots, N$,

(i) I_j is a subpiece of τI located between I and T^+;

(ii) I_j is of level $n+m'$;

(iii) I_j is of distance at least $\frac{\tau-1}{2} l_{n+m'}$ away from I_i for all $i \neq j$.

We pick N to be the maximum possible integer such that (i)-(iii) holds.

Similar to the argument in Section §3.4.2, the no-wide-wave assumption implies that there exists some $s \in \{1, \ldots, N\}$ and a lamination $\mathcal{G}_s$ connecting I_{s-1} and I_s such that

$$W(\mathcal{G}_s) \geq \frac{N}{10} K.$$

The I_j's are constructed such that $(\tau I_s)^c$ contains every I_i for $i \neq s$. In particular, $\mathcal{G}_s$ overflows $\mathcal{F}_\tau(I_s)$ and thus the piece I_s is $[\frac{N}{10}K, \tau]$-wide. Since $N > \tilde{C}^{m'}$, we can pick m' to be just high enough such that $N \geq 20$. Hence, I_s is a $[2K, \tau]$-wide combinatorial piece of level $n+m'$. □

3.5.2 Deep case

In the deep case, our approach below is inspired by Kahn's push-forward argument in [Kah06, §7]. The proof below contains a series of reductive lemmas before we finally adapt the push-forward argument at the very end.

Proof of Theorem 3.5.1 in the deep case. Suppose there is a $[K, \tau]$-wide combinatorial piece in $\mathbf{H}$ of some deep level n with $K \geq \mathbf{K}$. By Proposition 3.3.3, we have a level n almost tiling $\mathcal{I}$ consisting of $[\varepsilon K, \tau]$-wide pieces for some absolute constant $0 < \varepsilon < 1$.

Lemma 3.5.3 (Localization of τ-degeneration)**.** *There are absolute constants ρ, m_0, $m_* \in \mathbb{N}_{>1}$, where $\rho \gg \tau$, such that for sufficiently large $\mathbf{K}$ and for $n \geq m_*$, either*

(1) *there is a $[2K, \tau]$-wide combinatorial piece of level between $n - m_0$ and n, or*

(2) *there is some $L^* \in \mathcal{I}$ such that the width of curves in $\mathcal{F}_\tau(L^*)$ that land on $\rho L^* \cap (\tau L^*)^c$ is greater than $\varepsilon K/2$.*

Roughly speaking, if (2) does not hold, then we apply the Quasi-Additivity Law to the family $\mathcal{F}_\rho(I)$ (for a fixed ρ) to obtain (1) in a way that is similar to Section §3.4.1.

Proof. Fix m_*. There exists a finite sequence of distinct pieces $I_1, \ldots, I_N$ in $\mathcal{I}$ labelled in consecutive order such that any pair of adjacent pieces I_j and I_{j+1} have bounded combinatorial distance:

$$(\tau - 1)l_n \leq \operatorname{dist}(I_j, I_{j+1}) \leq \tau l_n.$$

This condition ensures that for each j, τI_j and $\cup_{i \neq j} I_i$ are disjoint but not too far apart. The integer $N \geq 2$ will be specified later, but nonetheless it must be bounded above by some constant depending on m_*.

Let P be the unique shortest piece containing $\bigcup_{j=1}^N I_j$. We set $m_0 = m_0(N)$ to be the largest integer such that $|P| \geq l_{n-m_0}$. Also, set $\rho = \rho(N)$ to be the smallest integer such that for every j, ρI_j contains τP. Let $S := \hat{\mathbb{C}} \backslash \bigcup_{j=1}^N (\rho I_j)^c$. We will again use the notation $\mathcal{H} < \mathcal{G}$ to denote that $\mathcal{G}$ overflows $\mathcal{H}$. Then, for every $j \in \{1, \ldots, N\}$,

$$\mathcal{F}(S, I_j) < \mathcal{F}_\rho(I_j), \quad \mathcal{F}_\tau(I_j) < \mathcal{F}\left(S \backslash \bigcup_{i \neq j} I_i, I_j\right), \quad \mathcal{F}_\tau(P) < \mathcal{F}\left(S, \bigcup_{i=1}^N I_i\right).$$

Suppose (2) does not hold. For each j, the width of curves connecting I_j and $(\rho I_j)^c$ exceeds $\varepsilon K/2$ and consequently,

$$W(S, I_j) \geq W_\rho(I_j) \geq \frac{\varepsilon K}{2}.$$

For sufficiently large $\mathbf{K}$, we can apply Lemma 2.2.8 and obtain

$$\max\{W_\tau(P), W_\tau(I_1), \ldots, W_\tau(I_N)\} \geq \frac{1}{\sqrt{2N}} \sum_{j=1}^N W(S, I_j) \succ \sqrt{N} K.$$

Suppose the maximum τ-width is attained by $J' \in \{P, I_1, \ldots, I_N\}$. Then, there is a combinatorial subpiece $J \subset J'$ of width $W_\tau(J) \succ \sqrt{N} K$ and level between $n - m_0$ and n. Finally, pick N (and ultimately m_*) to be just high enough such that J is $[2K, \tau]$-wide. This leads to (1). □

Let λ be sufficiently large such that $\lambda > \rho$ and $\mathbf{n}_\lambda \geq m_0 + m_*$. Then, by the lemma, it is sufficient to consider case (2). In this case, there is a connected component R^* of $\rho L^* \cap (\tau L^*)^c$ such that the family $\mathcal{F}(L^*, R^*)$ of curves connecting L^* and R^* has width

$$W\left(\mathcal{F}(L^*, R^*)\right) \geq \frac{\varepsilon}{4} K. \tag{3.5.1}$$

Lemma 3.5.4. *There exist $t \in \{0, 1, 2\}$ and a closed set $G \subset \hat{\mathbb{C}}$ such that*

(1) $U^0 := \hat{\mathbb{C}}\setminus\big((\lambda f^t(L^*))^c \cup G\big)$ *is a topological disk containing* $U^0 \cap \mathbf{H} = \lambda f^t(L^*)$*;*

(2) *for all* $j \in \mathbb{N}$*, every critical value of* f^j *in* U^0 *lies inside* $U^0 \cap \mathbf{H}$*;*

(3) *the width of curves in* $\hat{\mathbb{C}}\setminus(\mathbf{H} \cup G)$ *connecting* $\lambda f^t(L^*)$ *and* G *is* $O(1)$*.*

Proof. By Lemma 3.3.6, there exist $t \in \{0,1,2\}$ and a pair of closed rays $\gamma_0 \subset \overline{Y^0}$ and $\gamma_\infty \subset \overline{Y^\infty}$ connecting a point in $(\lambda f^t(L^*))^c$ to 0 and ∞ respectively such that the family of curves in $\hat{\mathbb{C}}\setminus(\mathbf{H} \cup \gamma_0 \cup \gamma_\infty)$ connecting $\lambda f^t(L^*)$ and $\gamma_0 \cup \gamma_\infty$ has width at most 10. In Case ⦿, the desired closed set is $G := \gamma_0 \cup \gamma_\infty$. In Case ◎, we can set $G := \gamma_0 \cup \gamma_\infty \cup \hat{Y}^0$, where $\hat{Y}^0$ is the connected component of $\hat{\mathbb{C}}\setminus\mathbb{H}$ containing 0. See Lemma 3.1.13 and Remark 3.3.7. ◻

Set U^0 to be the disk in Lemma 3.5.4 and set

$$L := f^t(L^*), \quad R := f^t(R^*).$$

For every $j \in \mathbb{N}$, define the corresponding lifts under f^j:

(a) $L^j :=$ the connected component of $f^{-j}(I)$ intersecting $\mathbf{H}$;

(b) $R^j :=$ the connected component of $f^{-j}(L)$ intersecting $\mathbf{H}$;

(c) $\Upsilon^j := L^j \cup R^j$;

(d) $U^j :=$ the connected component of $f^{-j}(U^0)$ containing Υ^j.

(e) $\mathcal{F}^j :=$ the canonical horizontal lamination $\mathcal{F}^h_{\text{can}}(U^j, \Upsilon^j)$ of $U^j\setminus\Upsilon^j$.

Lemma 3.5.5 (Width of $\mathcal{F}^0$)**.** *There exists an absolute constant* $\varepsilon_1 > 0$ *such that for sufficiently large* $\mathbf{K}$*, either*

(1) $W_\lambda(I) \succ K$*, or*

(2) $W(\mathcal{F}^0) \geq \varepsilon_1 K$*.*

Proof. Let $\mathcal{S}$ denote the family of curves connecting L and R. Since $\mathcal{S}$ contains $f^t(\mathcal{F}(L^*, R^*))$, then by Proposition 2.2.4 and inequality (3.5.1),

$$W(\mathcal{S}) \geq \frac{\varepsilon}{4d^2} K.$$

Let $\mathcal{G}$ be the set of curves in $\mathcal{S}$ that lie within U^0. If more than half of curves in $\mathcal{S}$ are in $\mathcal{G}$, then

$$W(\mathcal{G}) > \frac{1}{2} W(\mathcal{S}) \geq \frac{\varepsilon}{8d^2} K.$$

By Proposition 2.2.10, this inequality yields (2). Otherwise, at least half of curves in $\mathcal{S}$ intersect either $(\lambda L)^c$ or G. Therefore, by Lemma 3.5.4 (3),

$$W_\lambda(I) \geq \frac{1}{2} W(\mathcal{S}) - O(1) \geq \frac{\varepsilon}{8d^2} K - O(1).$$

For sufficiently high $\mathbf{K}$, this yields (1). $\square$

To proceed, it is then sufficient to consider case (2) of the lemma above.

Fix a positive integer $r > n$ that is to be determined later. We would like to show that since every piece is almost invariant under f^{q_r}, a definite amount of canonical horizontal leaves of $\mathcal{F}^0$ should restrict to vertical curves in $U^{q_r} \backslash \Upsilon^{q_r}$. To do this, some technical adjustments are required.

Let us define

$$\hat{L} := L \cup f^{q_r}(L), \quad \hat{R} := R \cup f^{q_r}(R), \quad \hat{U}^0 := U^0 \backslash f^{q_r}\left((\lambda L)^c\right).$$

The thickened pieces $\hat{L}$, $\hat{R}$ and the new domain $\hat{U}^0$ are combinatorially very close to L, R, and U^0 respectively. They come with new separation constants $1 < \hat{\tau} \ll \hat{\rho} \ll \hat{\lambda}$ such that $\overline{\hat{U}^0 \cap \mathbf{H}} = \hat{\lambda}\hat{L}$, $\hat{R}$ is a component of $\overline{\hat{\rho}\hat{L} \backslash \hat{\tau}\hat{L}}$, and $\hat{\tau} \asymp \tau$, $\hat{\rho} \asymp \rho$ and $\hat{\lambda} \asymp \lambda$.

Similar to (a)–(e), we denote for every $j \in \mathbb{N}$ the corresponding lifts:

- (â) $\hat{L}^j :=$ the connected component of $f^{-j}\left(\hat{L}\right)$ intersecting $\mathbf{H}$;
- ($\hat{\text{b}}$) $\hat{R}^j :=$ the connected component of $f^{-j}\left(\hat{R}\right)$ intersecting $\mathbf{H}$;
- ($\hat{\text{c}}$) $\hat{\Upsilon}^j := \hat{L}^j \cup \hat{R}^j$;
- ($\hat{\text{d}}$) $\hat{U}^j :=$ the connected component of $f^{-j}\left(\hat{U}^0\right)$ containing $\hat{\Upsilon}^j$;
- (ê) $\hat{\mathcal{F}}^j := \mathcal{F}^h_{\text{can}}\left(\hat{U}^j, \hat{\Upsilon}^j\right)$.

Note the following relations:

$$\hat{\Upsilon}^0 \supset \Upsilon^0, \quad \partial\hat{U}^0 \supset \partial U^0, \quad \hat{U}^0 \subset U^0, \tag{3.5.2}$$

$$\hat{U}^{q_r} \backslash \hat{\Upsilon}^{q_r} \subset U^0 \backslash \Upsilon^0. \tag{3.5.3}$$

The relationship between $\hat{\mathcal{F}}^0$ and $\mathcal{F}^0$ is not trivial. Nonetheless, the following lemma states that we can reduce the problem to the case where the widths of $\hat{\mathcal{F}}^0$ and $\mathcal{F}^0$ are comparable.

Lemma 3.5.6 (Comparability between $\hat{\mathcal{F}}^0$ and $\mathcal{F}^0$)**.** *There is an absolute constant $\varepsilon_2 > 1$ such that for sufficiently large $\lambda \gg \rho$ and $\mathbf{K}$, either*

(1) *there is a level r combinatorial piece J such that either $W_\tau(J) \geq 2K$ or $W_\lambda(J) > K$, or*

(2) $\frac{1}{2}W(\mathcal{F}^0) \leq W\left(\hat{\mathcal{F}}^0\right) \leq \varepsilon_2 W(\mathcal{F}^0)$.

In the proof, we will show that either such a piece J in (1) can be found from the symmetric difference between $\partial U^0 \cup \Upsilon^0$ and $\partial \hat{U}^0 \cup \hat{\Upsilon}^0$, or (2) holds.

Proof. Suppose (1) does not hold. Let us present the canonical horizontal lamination $\hat{\mathcal{F}}_0$ as the union $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ where:

- ▷ $\mathcal{S}_1$ = set of leaves in $\hat{\mathcal{F}}_0$ that has an endpoint on $P_1 := \overline{\hat{L}\backslash L}$;
- ▷ $\mathcal{S}_2$ = set of leaves in $\hat{\mathcal{F}}_0$ that has an endpoint on $P_2 := \overline{\hat{R}\backslash R}$;
- ▷ $\mathcal{S}_0 = \hat{\mathcal{F}}_0 \backslash (\mathcal{S}_1 \cup \mathcal{S}_2)$.

Note that P_1 and P_2 are combinatorial pieces of level r. See Figure 3.6.

For each $i \in \{1,2\}$, $\mathcal{S}_i$ restricts to a sublamination of $\mathcal{F}_\tau(P_i)$ because the combinatorial distance between $\hat{L}$ and $\hat{R}$ is greater than $\frac{\tau-1}{2} l_r$. We can assume that $W(\mathcal{S}_i) < 2\varepsilon_1^{-1} W(\mathcal{F}^0)$ because otherwise, by Lemma 3.5.5 (2), we would have

$$W_\tau(P_i) \geq W(\mathcal{S}_i) \geq 2\varepsilon_1^{-1} W\left(\mathcal{F}^0\right) \geq 2K$$

and this would yield (1) instead. Meanwhile, since every leaf of $\mathcal{S}_0$ is a horizontal curve in $U^0 \backslash \Upsilon^0$, then by Proposition 2.2.10,

$$W(\mathcal{S}_0) \leq W\left(\mathcal{F}^0\right) + O(1).$$

Therefore,

$$W\left(\hat{\mathcal{F}}^0\right) \leq \sum_{i=0}^{2} W(\mathcal{S}_i) < (4\varepsilon_1^{-1} + 1) W\left(\mathcal{F}^0\right) + O(1).$$

For sufficiently large $\mathbf{K}$, the inequality above yields the upper bound in (2).

Next, to obtain the lower bound in (2), we will consider the set $\mathcal{S}_3$ of leaves of $\mathcal{F}^0$ that intersect the level r combinatorial interval $P_3 := \overline{\partial \hat{U}^0 \backslash \partial U^0}$. By (3.5.2), every horizontal leaf of $\mathcal{F}^0$ either intersects P_3 or restricts to a horizontal curve in $\hat{U}^0 \backslash \hat{\Upsilon}^0$. As such,

$$W\left(\mathcal{F}^0\right) \leq W\left(\hat{\mathcal{F}}^0\right) + W(\mathcal{S}_3) + O(1) \tag{3.5.4}$$

Observe that $\mathcal{S}_3$ overflows $\mathcal{F}_\lambda(P_3)$. We can assume that $W(\mathcal{S}_3) < \frac{1}{3} W(\mathcal{F}^0)$ because otherwise

$$W_\lambda(P_3) \geq \frac{1}{3} W\left(\mathcal{F}^0\right) \succ K.$$

which would again yield (1). By applying this assumption to (3.5.4),

$$\frac{2}{3} W\left(\mathcal{F}^0\right) \leq W\left(\hat{\mathcal{F}}^0\right) + O(1).$$

Hence, for sufficiently high $\mathbf{K}$, we immediately obtain the lower bound in (2). □

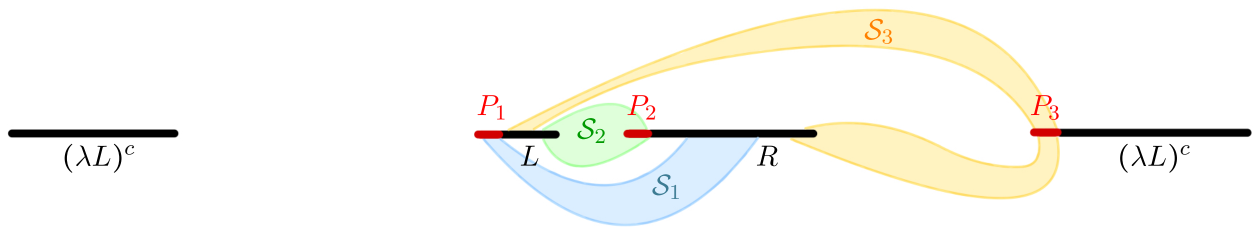

Figure 3.6: $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$.

It is sufficient to proceed under the assumption that Lemma 3.5.6 (2) holds. In this case, $W(\hat{\mathcal{F}}^0) \geq \frac{\varepsilon_1}{2} K$.

Before finishing the proof of the theorem, the following important lemma is needed. It states that either τ-degeneration doubles or there is significant loss in horizontal width after a certain number of pullbacks.

Lemma 3.5.7 (Loss of horizontal width)**.** *There exist absolute constants $m_1, m_2 \in \mathbb{N}$ such that for sufficiently large $\lambda \gg \rho$ and $\mathbf{K}$, either*

(1) *there is a $[2K, \tau]$-wide combinatorial piece of level $n + m_1$, or*

(2) $W\left(\hat{\mathcal{F}}^{q_{n+m_2}}\right) < (2\varepsilon_2)^{-1} W\left(\hat{\mathcal{F}}^0\right)$.

This lemma is a replacement for Kahn's entropy argument in [Kah06, §6.3], and it will directly follow from Proposition 3.6.2 in the next section. See Remark 3.6.3. Let us finally set $r = n + m_2$ and assume that Lemma 3.5.7 (2) holds. We are now in position to adapt the push-forward argument.

From the embedding in (3.5.3), horizontal leaves in $U^0 \backslash \Upsilon^0$ restrict to either horizontal or vertical curves in $\hat{U}^{q_r} \backslash \hat{\Upsilon}^{q_r}$. In other words,

$$W\left(\mathcal{F}^0\right) \leq W\left((\mathcal{F}^v_{\text{can}}(\hat{U}^{q_r}, \hat{\Upsilon}^{q_r})\right) + W\left(\hat{\mathcal{F}}^{q_r}\right) + O(1).$$

By Lemma 3.5.6 (2) and Lemma 3.5.7 (2),

$$W\left(\mathcal{F}^0\right) \leq W\left((\mathcal{F}^v_{\text{can}}(\hat{U}^{q_r}, \hat{\Upsilon}^{q_r})\right) + \frac{1}{2} W\left(\mathcal{F}^0\right) + O(1).$$

For sufficiently high $\mathbf{K}$, the inequality simplifies to

$$W\left((\mathcal{F}^v_{\text{can}}(\hat{U}^{q_r}, \hat{\Upsilon}^{q_r})\right) \geq \frac{1}{3} W\left(\mathcal{F}^0\right).$$

By Lemma 3.5.5 (2), this implies that $W\left(\hat{U}^{q_r}, \hat{J}^{q_r}\right) \succ K$ for some $\hat{J} \in \left\{\hat{L}, \hat{R}\right\}$.

Consider the iterate

$$f^{q_r} : \left(\hat{U}^{q_r}, \hat{U}^{q_r}_\tau, \hat{J}^{q_r}\right) \to \left(\hat{U}^0, \hat{U}^0_\tau, \hat{J}\right)$$

where $\hat{U}^0_\tau := \hat{U}^0 \backslash \left(\tau \hat{J}\right)^c$ and $\hat{U}^{q_r}_\tau$ is the pullback of $\hat{U}^0_\tau$ under f^{q_r} containing $\hat{J}^{q_r}$. By Lemma 3.3.4, the degree of f^{q_r} on $\hat{U}^{q_r}_\tau$ remains independent of λ. By Lemma 2.2.9, for sufficiently high $\mathbf{K}$, either

$$W\left(\hat{U}^0, \hat{J}\right) > K, \quad \text{or} \quad W\left(\hat{U}^0_\tau, \hat{J}\right) \geq (2C+1)K,$$

where C is the constant from Proposition 2.1.3. In either case, Lemma 3.5.4 asserts that the width of curves that land on G is negligible. Hence, for sufficiently large $\mathbf{K}$, there exists a combinatorial subpiece $J \subset \hat{J}$ such that either

$$W_\lambda(J) > K, \quad \text{or} \quad W_\tau(J) \geq 2K.$$

At last, pick the threshold increment $\mathbf{m} \in \mathbb{N}$ such that $\mathbf{m} \geq \max\{m_0, m_1, m_2\}$ and that $n - \mathbf{m}$ is less than the level of $\hat{R}$. This concludes the proof of Theorem 3.5.1. □

3.6 Loss of horizontal width

Let us fix $\lambda \gg \tau$ and let $n \in \mathbb{N}$ be such that $2\lambda l_n < 1$. The key players of this section are as follows.

- ▷ L and R are pieces in $\mathbf{H}$ of combinatorial distance $\text{dist}(I, L) \asymp l_n$ and length at least l_n satisfying $|L| \asymp |R| \asymp l_n$;
- ▷ $U^0 \subset \mathbb{C}^*$ is a topological disk containing $L \cup R$ such that $\mathbf{H} \backslash U^0 = (\lambda L)^c$.

Remark 3.6.1. In Case ⊚ (see §3.1.4), we will also impose the additional assumption that U^0 is disjoint from the connected component of $\hat{\mathbb{C}} \backslash \mathbb{H}$ containing 0, so that for all j, every critical value of f^j in U^0 must lie in $U^0 \cap \mathbf{H}$.

Similar to (a)–(e) in §6.2, we define the corresponding lifts under f^j for $j \in \mathbb{N}$:

(a.) $L^j :=$ the connected component of $f^{-j}(L)$ intersecting $\mathbf{H}$;

(b.) $R^j :=$ the connected component of $f^{-j}(R)$ intersecting $\mathbf{H}$;

(c.) $\Upsilon^j := L^j \cup R^j$;

(d.) $U^j :=$ the connected component of $f^{-j}(U^0)$ containing Υ^j.

(e.) $\mathcal{F}^j :=$ the canonical horizontal lamination $\mathcal{F}^h_{\text{can}}(U^j, \Upsilon^j)$ of $U^j \backslash \Upsilon^j$.

We are restricting our map f to a sequence of branched coverings

$$\ldots \xrightarrow{f} \left(U^3, \Upsilon^3\right) \xrightarrow{f} \left(U^2, \Upsilon^2\right) \xrightarrow{f} \left(U^1, \Upsilon^1\right) \xrightarrow{f} \left(U^0, \Upsilon^0\right)$$

and we will study how the width of $\mathcal{F}^j$ behaves as j increases. The goal of this section is to prove the following proposition.

Proposition 3.6.2. *For any $\Delta > 1$, $\delta \in (0,1)$, and sufficiently large λ, there are some constants $m_1 = m_1(\Delta,\delta) \in \mathbb{N}$, $m_2 = m_2(\Delta,\delta) \in \mathbb{N}$ and $\mathbf{K} = \mathbf{K}(\lambda,\Delta,\delta) > 1$ such that if $W(\mathcal{F}^0) = K \geq \mathbf{K}$, then either*

(1) *there is a level $n + m_1$ combinatorial piece J of width $W_\tau(J) \geq \Delta K$, or*

(2) *there is significant loss in horizontal width: $W(\mathcal{F}^{q_{n+m_2}}) < \delta K$.*

Remark 3.6.3. Recall that the final missing ingredient of the proof of Theorem 3.5.1 is Lemma 3.5.7. Indeed, we can apply Proposition 3.6.2 in the context of Lemma 3.5.7 (e.g. setting $\Delta = 4\varepsilon_1^{-1}$ and $\delta = (2\varepsilon_2)^{-1}$), thereby proving the lemma immediately.

3.6.1 Outline

For every $j \in \mathbb{N}$, since $U^j \backslash \Upsilon^j$ is a disk with two connected compact sets removed, the leaves of $\mathcal{F}^j$ belong to only at most two proper homotopy classes in $U^j \backslash \Upsilon^j$. We say that a homotopically non-trivial proper curve in $U^j \backslash \Upsilon^j$ is of type

A if it connects L^j and R^j, and

B if it starts and ends at the same component of Υ^j.

Naturally, we split $\mathcal{F}^j$ into a disjoint union of type A and B sublaminations $\mathcal{A}^j \cup \mathcal{B}^j$. See Figure 3.7.

To illustrate the main idea, let us consider the unbranched covering map $f^r : U^r \backslash f^{-r}(\mathrm{CV}) \to U^0 \backslash \mathrm{CV}$ of large degree (depending on λ, δ, and Δ) where $r = r(\delta,\Delta) \geq 1$ is a fixed integer and CV is the set of critical values of f^r. To compare the widths of $\mathcal{F}^r$ and $\mathcal{F}^0$, we apply the property that, up to an additive constant, curves in $\mathcal{F}^r$ must travel through the canonical rectangles in $\mathcal{T}' = (f^r)^*\mathcal{T}$ (Lemma 3.6.5), where $\mathcal{T}$ is the horizontal thick-thin decomposition of $U^0 \backslash (\Upsilon^0 \cup \mathrm{CV})$. (Refer to §2.2.2.)

However, $\mathcal{T}$ does not just contain type A and B rectangles. It may have some peripheral ones, i.e. those which are homotopically trivial if we forget the punctures CV. We ignore the peripheral rectangles that are combinatorially close to Υ^0 by thickening $\Upsilon^0 = L^0 \cup R^0$ into $\boldsymbol{\Upsilon} = \mathbf{L} \cup \mathbf{R}$. Rectangles that are combinatorially distinct (type D) can be assumed to have width at most $\kappa W(\mathcal{F}^0)$ for some fixed $\kappa = \kappa(\delta,\Delta) > 1$ (Lemma 3.6.7).

There is a unique (up to homotopy) proper curve $\boldsymbol{b}$ in $U^0 \backslash \Upsilon^0$ separating L^0 and R^0, and it is crossed by $\mathcal{A}^0$ once and $\mathcal{B}^0$ twice. See Figure 3.7. There is a lift β of $\boldsymbol{b}$ under f^r which separates L^r and R^r, and automatically the set $\mathcal{T}'[\beta]$ of rectangles in $\mathcal{T}'$ that intersect β has total width about $Z^0 := W(\mathcal{A}^0) + 2W(\mathcal{B}^0)$. Since $\mathcal{F}^r$ overflows $\mathcal{T}'[\beta]$, we arrive at the conclusion that the *asymmetric width*

$$Z^j := W\left(\mathcal{A}^j\right) + 2W\left(\mathcal{B}^j\right)$$

is non-increasing (Corollary 3.6.9).

In order to upgrade monotonicity to a strict loss, we will construct several proper curves like $\boldsymbol{b}$ that are distinct rel CV. Hence, we construct a separating strip made of proper curves β_L and β_R that are as close as possible to $\mathbf{L}'$ and $\mathbf{R}'$ respectively and we can assume that the set $\mathcal{T}'[\beta_L]$ (resp. $\mathcal{T}'[\beta_R]$) of rectangles in $\mathcal{T}'$ intersecting β_L (resp. β_R) has width at most $(2\kappa+1)Z^0$. (See Figure 3.9 for the construction.) This leads us to an estimate for Z^r (Proposition 3.6.11), which implies that either

(i) *persistent* rectangles, i.e. the ones in $\mathcal{T}'_{\text{per}} := \mathcal{T}'[\beta_L] \cap \mathcal{T}'[\beta_R]$, have total width $\asymp Z^0$ (the left side of Figure 3.10), or

(ii) we can apply the series law (the middle and right parts of Figure 3.10) and conclude that Z^r is strictly less than Z^0 with a factor depending on κ.

The treatment for Case (i) hinges on the key property that vertical leaves of persistent rectangles are all homotopic rel critical points of f^r (Lemma 3.6.12). This property forces their image to frequently submerge in and out through $\mathbf{H}$ (Figures 3.11 and 3.12) at a rate controlled by r. For sufficiently large r, this enables us to construct a large τ-degeneration (Lemma 3.6.13) and ultimately achieve (1).

In Case (ii), we run an inductive process to get the desired shrinking factor δ, which yields (2). This completes the proof of the proposition.

3.6.2 Decomposition of canonical laminations

Let us fix

$$K := W\left(\mathcal{F}^0\right) \quad \text{and} \quad r := q_{n+m}$$

for some sufficiently large integer $m \in \mathbb{N}$. From now on until the end of §3.6.5, we consider the unbranched covering map

$$f^r : U^r \backslash f^{-r}(\mathrm{CV}) \to U^0 \backslash \mathrm{CV}$$

where $\mathrm{CV} = \mathrm{CV}(f^r)$ denotes the set of critical values of f^r in U^0.

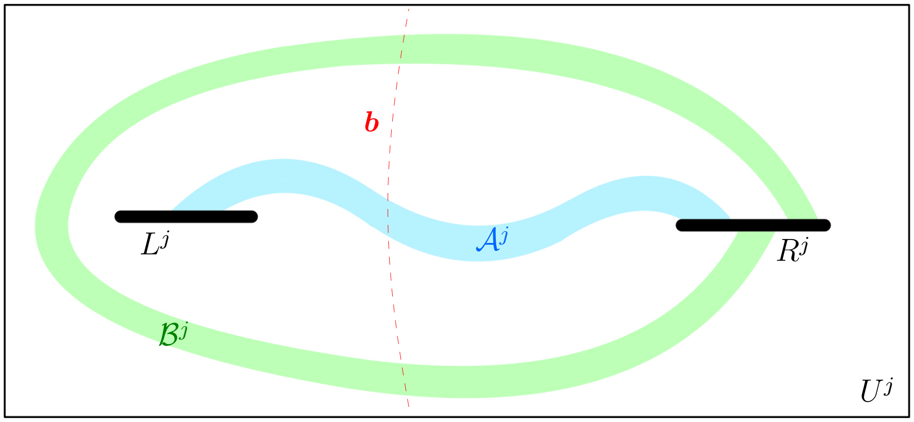

Figure 3.7: The decomposition of $\mathcal{F}^j$. The intersection number between $\boldsymbol{b}$ and $\mathcal{A}^j$ is one, and that between $\boldsymbol{b}$ and $\mathcal{B}^j$ is two.

Lemma 3.6.4. *There is an absolute constant $C > 1$ such that $|\mathrm{CV}| = O(\lambda C^m)$. In particular, the degree of f^r depends only on m and λ.*

Proof. Consider the piece $J = \overline{U^0} \cap \mathbf{H}$. It suffices to fix a critical value $v \in \mathbf{H}$ of f and estimate the size of $\mathcal{O}_v := \{(f|_{\mathbf{H}})^{-i}(v) \in J \ : \ 0 \leq i \leq q_{n+m} - 1\}$. By Proposition 2.1.2, $\mathcal{O}_v$ divides J into $|\mathcal{O}_v| + 1$ pieces $J_1, \ldots, J_{|\mathcal{O}_v|+1}$ of length at least l_{n+m}. Let $C > 1$ be the constant from Proposition 2.1.3. Then,

$$\lambda l_n \asymp |J| = \sum_i |J_i| \geq |\mathcal{O}_v| \cdot l_{n+m} \geq |\mathcal{O}_v| \cdot C^{-m} l_n$$

which implies that $|\mathcal{O}_v| = O(\lambda C^m)$. □

Let $\mathcal{T} := \mathrm{TTD}^h(U^0, \Upsilon^0 \cup \mathrm{CV})$ denote the horizontal thick-thin decomposition of $U^0 \backslash (\Upsilon^0 \cup \mathrm{CV})$. (Refer to §2.2.2 for details.) According to Proposition 2.2.11, the horizontal thick-thin decomposition $\mathcal{T}'$ of $U^r \backslash f^{-r}(\Upsilon^0 \cup \mathrm{CV})$ is the full lift $(f^r)^* \mathcal{T}$ of $\mathcal{T}$.

As we apply Lemma 2.2.12 to the inclusion $U^r \backslash f^{-r}(\Upsilon^0 \cup \mathrm{CV}) \subset U^r \backslash \Upsilon^r$, we obtain the following fundamental property relating the widths of $\mathcal{F}^r$ and $\mathcal{T}'$.

Lemma 3.6.5. *There exist some sublamination $\mathcal{F}^r_{\mathrm{sub}} \subset \mathcal{F}^r$ and some constant $C = C(m, \lambda) > 0$ such that*

$$W(\mathcal{F}^r) - C \leq W(\mathcal{F}^r_{\mathrm{sub}})$$

and for every leaf γ of $\mathcal{F}^r_{\mathrm{sub}}$, every component of $\gamma \backslash f^{-r}(\Upsilon^0)$ is either

(1) *a homotopically trivial proper curve in $U^r \backslash f^{-r}(\Upsilon^0 \cup \mathrm{CV})$, or*

(2) *a proper curve in a canonical rectangle in $\mathcal{T}'$.*

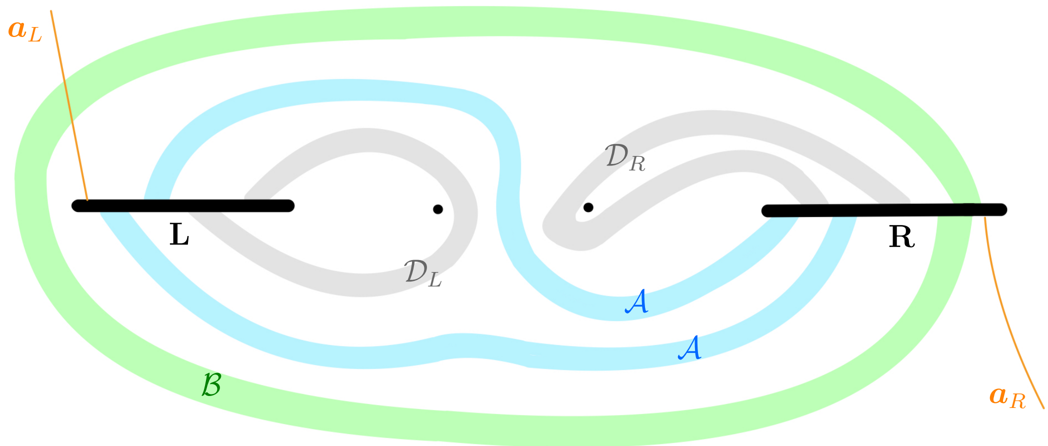

Figure 3.8: An example of the decomposition of $\mathcal{T}$. Note that $\mathcal{E}$ is inside of $\boldsymbol{\Upsilon} = \mathbf{L} \cup \mathbf{R}$.

By taking into account the critical values of f^r, we enrich the canonical lamination with the presence of peripheral arcs. We say that a proper curve in $U^0 \backslash (\Upsilon^0 \cup \mathrm{CV})$ is *peripheral* if it has a trivial proper homotopy class in $U^0 \backslash \Upsilon^0$. We will decompose $\mathcal{T}$ into a disjoint union

$$\mathcal{T} := \mathcal{A} \cup \mathcal{B} \cup \mathcal{P}$$

where $\mathcal{A}$ consists of canonical rectangles of type A (leaves of $\mathcal{F}(\mathcal{A})$ are of type A) in $U^0 \backslash \Upsilon^0$, $\mathcal{B}$ consists of canonical rectangles of type B, and $\mathcal{P}$ consists of peripheral rectangles. Denote by A and B the total widths of $\mathcal{A}$ and $\mathcal{B}$ respectively.

Observe that the width of $\mathcal{F}^0$ should be close to $A + B$. The following lemma is an immediate consequence of Proposition 2.2.10 and Lemma 3.6.4.

Lemma 3.6.6. *There is some $C = C(m, \lambda) > 0$ such that*

$$\left|A - W\left(\mathcal{A}^0\right)\right| \leq C \quad \textit{and} \quad \left|B - W\left(\mathcal{B}^0\right)\right| \leq C$$

Let us pick a definite constant $\eta > 1$ such that the combinatorial distance between ηI and ηL is still $\asymp l_n$. Let us split $\mathcal{P}$ into a disjoint union

$$\mathcal{P} = \mathcal{D} \cup \mathcal{E}, \quad \mathcal{D} = \mathcal{D}_L \cup \mathcal{D}_R, \quad \mathcal{E} = \mathcal{E}_L \cup \mathcal{E}_R,$$

where for $J \in \{L, R\}$,

- ▷ rectangles in $\mathcal{D}_J \cup \mathcal{E}_J$ are attached to J^0,
- ▷ every leaf of $\mathcal{F}(\mathcal{D}_J)$ intersects $(\eta J^0)^c$, and
- ▷ every leaf of $\mathcal{F}(\mathcal{E}_J)$ is disjoint from $(\eta J^0)^c$.

See Figure 3.8.

By design, peripheral rectangles in $\mathcal{E}$ are combinatorially close to Υ^0. We will remove $\mathcal{E}$ from consideration by absorbing it into Υ^0 as follows. Let us define $\mathbf{L}$ to be the hull of the union of L^0 and $\mathcal{E}_L$, i.e. the smallest compact full subset of U^0 containing $L^0 \cup \mathcal{E}_L$. Similarly, we define $\mathbf{R}$ to be the hull of $R^0 \cup \mathcal{E}_R$. Denote by $\mathbf{L}'$ and $\mathbf{R}'$ the connected components of $f^{-r}(\mathbf{L})$ and $f^{-r}(\mathbf{R})$ that contain L^0 and R^0 respectively. Let

$$\mathbf{\Upsilon} := \mathbf{L} \cup \mathbf{R} \quad \text{and} \quad \mathbf{\Upsilon}' := \mathbf{L}' \cup \mathbf{R}'.$$

Meanwhile, peripheral rectangles in $\mathcal{D}$ are a source of τ-degeneration. Let us denote the widths of $\mathcal{D}$, $\mathcal{D}_L$, and $\mathcal{D}_R$ by D, D_L, and D_R respectively.

Lemma 3.6.7. *There is an absolute constant $m_0 \in \mathbb{N}$ such that if*

$$m \geq m_0 \quad \textit{and} \quad D \geq \kappa K$$

for some $\kappa > 1$, then there is a combinatorial piece J of level $n+m$ and width $W_\tau(J) \geq \varepsilon\kappa K$ for some constant $\varepsilon = \varepsilon(m) > 0$.

Proof. Suppose without loss of generality that $D_L \geq \frac{\kappa}{2} K$. There exists a constant $m_0 \in \mathbb{N}$ depending on η such that for any level $n + m_0$ combinatorial subpiece J of L^0, the thickened piece τJ is contained in ηL^0. Let us assume $m \geq m_0$. The piece L^0 can be covered by N level $n+m$ combinatorial pieces $J_1, \ldots, J_N$ for some integer $N = N(m) \in \mathbb{N}$. Let us split $\mathcal{F}(\mathcal{D}_L)$ into sublaminations $\mathcal{L}_1, \ldots, \mathcal{L}_N$ where leaves of $\mathcal{L}_i$ start at points on J_i. For each I, since $\mathcal{D}_L$ crosses $(\eta L^0)^c$ which is contained in $(\tau J_i)^c$, then $\mathcal{L}_i$ overflows the curve family $\mathcal{F}_\tau(J_i)$. Therefore,

$$\frac{\kappa}{2} K \leq D_L = \sum_{i=1}^{N} W(\mathcal{L}_i) \leq \sum_{i=1}^{N} W_\tau(J_i) \leq N \max_i W_\tau(J_i).$$

Consequently, there is some $i \in \{1, \ldots, N\}$ such that $W_\tau(J_i) \geq \frac{\kappa}{2N} K$. □

3.6.3 Separating curves

From now on, let us assume without loss of generality that $\mathcal{B}$ starts from and ends at R^0. Given a proper curve α in $U^r \backslash \Upsilon^r$, we will denote by $\mathcal{T}'[\alpha]$ the union of rectangles in $\mathcal{T}'$ that intersect α.

Let us fix vertical rays $\boldsymbol{a}_L$ and $\boldsymbol{a}_R$ in $U^0 \backslash (\Upsilon^0 \cup \mathrm{CV})$ where

- ▷ $\boldsymbol{a}_L$ connects ∂U^0 to L^0 and $\boldsymbol{a}_R$ connects ∂U^0 to R^0;
- ▷ $\boldsymbol{a}_L$ is crossed by $\mathcal{B}$ exactly once and is disjoint from $\mathcal{T} \backslash \mathcal{B}$;

▷ $\boldsymbol{a}_R$ is disjoint from $\mathcal{T}$.

The first assumption states that $\boldsymbol{a}_L$ and $\boldsymbol{a}_R$ are vertical cuts of $U^0\backslash(\Upsilon^0 \cup \mathrm{CV})$, whereas the other two assumptions state that the minimal intersection number relative to $\mathcal{T}$ is achieved. See Figure 3.8.

For $J \in \{L, R\}$, let $\alpha_{J,+}$ and $\alpha_{J,-}$ be the unique pair of lifts of $\boldsymbol{a}_J$ under f^r that are attached to $\mathbf{J}'$ and are closest to $\boldsymbol{\Upsilon}'\backslash\mathbf{J}'$. Such lifts exist because $f^r : J^r \to J^0$ is a branched covering of degree at least 2.

To estimate the width of $\mathcal{F}^r$, we will identify rectangles in $\mathcal{T}'$ that cross a number of proper curves in U^r separating $\mathbf{L}'$ and $\mathbf{R}'$. These curves are constructed with the aid of $\boldsymbol{a}_L$ and $\boldsymbol{a}_R$ as follows. Refer to Figure 3.9 for a schematic picture.

Lemma 3.6.8 (Middle curve). *There exist a proper curve $\boldsymbol{b}$ in U^0 and a proper curve β in U^r with the following properties.*

(1) *$\boldsymbol{b}$ disjoint from $\boldsymbol{\Upsilon} \cup \mathrm{CV} \cup \boldsymbol{a}_L \cup \boldsymbol{a}_R$ and separates $\mathbf{L}$ from $\mathbf{R}$.*

(2) *$\mathcal{B}$ crosses $\boldsymbol{b}$ twice, $\mathcal{A}$ crosses $\boldsymbol{b}$ once, and $\mathcal{P}$ is disjoint from $\boldsymbol{b}$.*

(3) *β is a lift of $\boldsymbol{b}$ under f^r that separates $\mathbf{L}'$ from $\mathbf{R}'$.*

(4) *Every rectangle in $\mathcal{T}'$ crosses β at most once, and $W(\mathcal{T}'[\beta]) = A + 2B$.*

Proof. The existence of $\boldsymbol{b}$ satisfying (1) and (2) is clear. (See Figures 3.7 and 3.8.) Let Q denote the connected component of $U^0\backslash\boldsymbol{b}$ containing $\mathbf{L}$. The unique lift Q' of Q under f^r which contains $\mathbf{L}'$ must be disjoint from $\mathbf{R}'$. Since f^r is a proper map on U^r, there exists a unique connected component β of $\partial Q'\backslash\partial U^r$ that is a lift of $\boldsymbol{b}$ and separates $\mathbf{L}'$ and $\mathbf{R}'$.

To prove (4), it suffices to show that every rectangle $\mathcal{R}$ in $\mathcal{B}$ admits exactly two distinct lifts in $\mathcal{T}'[\beta]$, and each of them crosses β exactly once. If otherwise, then there would exist a unique lift $\mathcal{R}'$ of $\mathcal{R}$ in $\mathcal{T}'[\beta]$ which crosses β exactly twice. In this case, $\mathcal{R}'$ would be crossing both $\alpha_{L,-}$ and $\alpha_{L,+}$, hence $\mathcal{R}$ would be crossing $\boldsymbol{a}_L$ twice, which is impossible. □

For $j \in \mathbb{N}$, let us consider the asymmetric width

$$Z^j := W(\mathcal{A}^j) + 2W(\mathcal{B}^j)$$

on $U^j\backslash\Upsilon^j$. (One may compare with the notion of *asymmetric modulus* in [Lyu97].)

Corollary 3.6.9 (Monotonicity). *There exists a constant $C = C(m, \lambda) > 0$ such that*

$$Z^r - C \le Z^0.$$

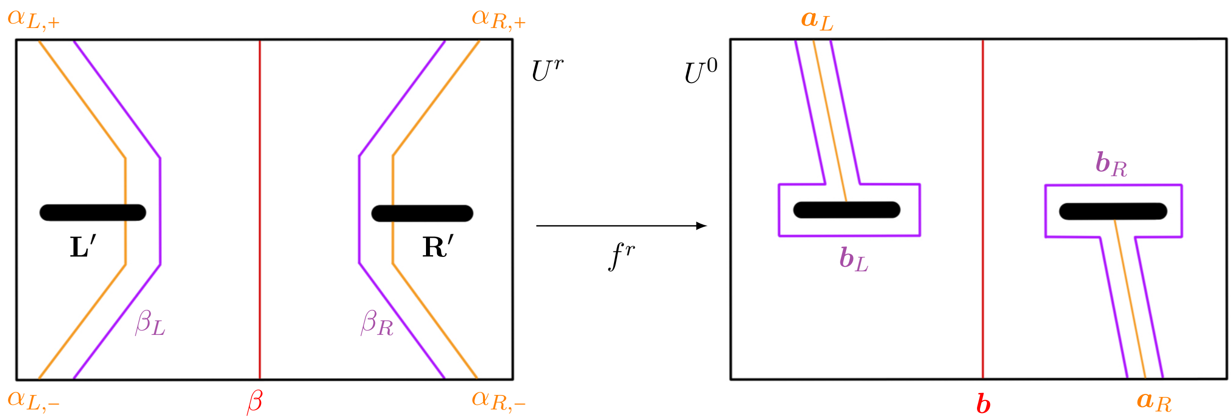

Figure 3.9: A schematic diagram of the construction of separating proper curves β, β_L, and β_R.

Proof. Consider $\mathcal{F}^r_{\text{sub}}$ from Lemma 3.6.5 and split them into $\mathcal{A}^r_{\text{sub}} \cup \mathcal{B}^r_{\text{sub}}$ according to the topological type. Observe that $\mathcal{A}^r_{\text{sub}}$ crosses β once, whereas $\mathcal{B}^r_{\text{sub}}$ crosses β twice. By Lemma 3.6.8 (4), $\mathcal{A}^r_{\text{sub}}$ admits a restriction $\mathcal{A}^r_{\text{res}}$ properly contained in $\mathcal{T}'[\beta]$, whereas $\mathcal{B}^r_{\text{sub}}$ admits two disjoint restrictions $\mathcal{B}^r_{\text{res},1}$ and $\mathcal{B}^r_{\text{res},2}$ that are properly contained in $\mathcal{T}'[\beta]$. Then,

$$\begin{aligned} Z^r - C &\le W(\mathcal{A}^r_{\text{res}}) + 2\left[W(\mathcal{B}^r_{\text{res},1}) \oplus W(\mathcal{B}^r_{\text{res},2})\right] \\ &\le W(\mathcal{A}^r_{\text{res}}) + \frac{1}{2}\left[W(\mathcal{B}^r_{\text{res},1}) + W(\mathcal{B}^r_{\text{res},2})\right] \le W\left(\mathcal{T}'[\beta]\right) \le A + 2B. \end{aligned}$$

At last, apply Lemma 3.6.6 and we are done. □

Our next goal is to upgrade monotonicity to a strict loss. To do so, let us introduce two other separating curves $\boldsymbol{b}_L$ and $\boldsymbol{b}_R$.

Lemma 3.6.10 (Left and right curves)**.** *There exist proper curves $\boldsymbol{b}_L$ and $\boldsymbol{b}_R$ in U^0 and proper curves β_L and β_R in U^r with the following properties. For each $J \in \{L, R\}$,*

(1) *$\boldsymbol{b}_J$ is disjoint from $\boldsymbol{\Upsilon} \cup \mathrm{CV} \cup \boldsymbol{a}_L \cup \boldsymbol{a}_R \cup \boldsymbol{b}$ and separates $\mathbf{J}$ and $\boldsymbol{b}$;*

(2) *$\mathcal{B} \cup \mathcal{D}_J$ crosses $\boldsymbol{b}_J$ twice, $\mathcal{A}$ crosses $\boldsymbol{b}_J$ once, and $\mathcal{P} \backslash \mathcal{D}_J$ is disjoint from $\boldsymbol{b}_J$;*

(3) *β_J is a lift of $\boldsymbol{b}_J$ that separates $\mathbf{L}'$ and β and is close to $\mathbf{J}' \cup \alpha_{J,+} \cup \alpha_{J,-}$;*

(4) *$W\left(\mathcal{T}'[\beta_J]\right) = A + 2B + 2D_J$.*

Moreover, the strip $\Pi \subset U^r$ cut out by β_L and β_R contains a piece I of length $\asymp l_n$.

Proof. For $J \in \{L, R\}$, pick an extremely small $\varepsilon > 0$ such that the ε-neighborhood O_J of $\mathbf{J} \cup \boldsymbol{a}_J$ is disjoint from $\mathrm{CV} \backslash \mathbf{J}$. Let us set $\boldsymbol{b}_J := \partial O_J \cap U^0$, then (1)–(3) is immediate. Item (4) follows in a similar manner as the proof of Lemma 3.6.8. Moreover, the existence of $I \subset \Pi$ follows from the property that $\mathbf{L}$ and $\mathbf{R}$ have combinatorial distance $\asymp l_n$. □

3.6.4 Non-persistence induces width loss

We say that a rectangle in $\mathcal{T}'$ is *persistent* if it crosses both β_L and β_R, i.e. it belongs in

$$\mathcal{T}'_{\text{per}} := \mathcal{T}'[\beta_L] \cap \mathcal{T}'[\beta_R].$$

Denote the total widths of persistent and non-persistent rectangles in $\mathcal{T}'[\beta]$ by

$$Z_{\text{per}} := W(\mathcal{T}'_{\text{per}}) \quad \text{and} \quad Z_{\text{non}} := A + 2B - Z_{\text{per}}$$

respectively. In this subsection, we prove the following non-dynamical result.

Proposition 3.6.11 (Key estimate). *There exists some constant $C = C(m, \lambda) > 0$ such that*

$$Z^r - C \le Z_{\text{per}} + Z_{\text{non}} \oplus 2(Z_{\text{non}} + D).$$

The idea is captured in Figure 3.10. Most leaves of $\mathcal{F}^r$ travel through either $\mathcal{T}'_{\text{per}}$ (the left part of the figure) or $\mathcal{T}'[\beta] \backslash \mathcal{T}'_{\text{per}}$ (the middle and the right parts). The former case gives the term Z_{per}. In the latter case, they must also travel through $(\mathcal{T}'[\beta_L] \cup \mathcal{T}'[\beta_R]) \backslash \mathcal{T}'_{\text{per}}$, which has total width $2(Z_{\text{non}} + D)$, and thus the series law can be applied to generate the harmonic sum. In Section 3.6.6, we will show from this inequality that Z^r shrinks provided that Z_{per} and D are small relative to K.

Proof. Consider the sublaminations $\mathcal{A}^r_{\text{sub}} \subset \mathcal{A}^r$ and $\mathcal{B}^r_{\text{sub}} \subset \mathcal{B}^r$ from Lemma 3.6.5. For some $C = C(m, \lambda) > 0$,

$$W(\mathcal{A}^r) - C \le W\left(\mathcal{A}^r_{\text{sub}}\right) \quad \text{and} \quad W(\mathcal{B}^r) - C \le W\left(\mathcal{B}^r_{\text{sub}}\right),$$

and every leaf of $\mathcal{A}^r_{\text{sub}} \cup \mathcal{B}^r_{\text{sub}}$ travels through rectangles in $\mathcal{T}'$.

Let us assume that $\mathcal{B}^r$ is attached to R^r; if otherwise, $\mathcal{B}^r_{\text{sub}}$ would be empty because no rectangles in $\mathcal{T}'$ can cross $\alpha_{R,+} \cup \alpha_{R,-}$. Let us define two disjoint restrictions $\mathcal{B}^r_1$ and $\mathcal{B}^r_2$ of $\mathcal{B}^r_{\text{sub}}$ as follows. Denote by Q the connected component of $U^r \backslash (\alpha_{L,+} \cup \alpha_{L,-})$ containing L^r. For $\gamma \in \mathcal{B}^r_{\text{sub}}$, let us fix a parametrization $\gamma : (0,1) \to U^r$ travelling around L^r in an anticlockwise manner. Consider the set T_γ of times $t \in (0,1)$ such that $\gamma(t)$ is in $f^{-r}(\Upsilon^0) \cap Q$. Note that T_γ is non-empty because no rectangle in $\mathcal{T}'$ crosses both $\alpha_{L,+}$ and $\alpha_{L,-}$ simultaneously. Let $t_{\gamma,1} := \min T_\gamma$ and $t_{\gamma,2} := \max T_\gamma$. Then, we define restrictions

$$\mathcal{B}^r_1 := \left\{ \gamma|_{(0, t_{\gamma,1})} \,:\, \gamma \in \mathcal{B}^r_{\text{sub}} \right\} \quad \text{and} \quad \mathcal{B}^r_2 := \left\{ \gamma|_{(t_{\gamma,2}, 1)} \,:\, \gamma \in \mathcal{B}^r_{\text{sub}} \right\}.$$

Let us consider the lamination

$$\mathcal{G} := \mathcal{A}^r_{\text{sub}} \cup \mathcal{B}^r_1 \cup \mathcal{B}^r_2$$

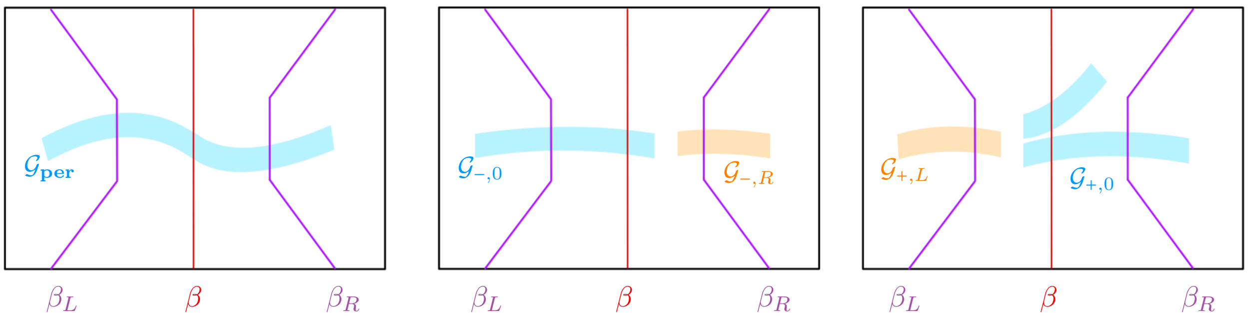

Figure 3.10: The lamination $\mathcal{G} = \mathcal{G}_{\text{per}} \cup \mathcal{G}_- \cup \mathcal{G}_+$ has width at least Z^r. $\mathcal{G}_{\text{per}}$ crosses both β_L and β_R. In contrast, $\mathcal{G}_-$ overflows $\mathcal{G}_{-,0}$ and $\mathcal{G}_{-,R}$, whereas $\mathcal{G}_+$ overflows $\mathcal{G}_{+,0}$ and $\mathcal{G}_{+,L}$.

Every leaf of $\mathcal{G}$ crosses β_L, ends at R^r (thus crosses β_R too), and travels through rectangles in $\mathcal{T}'$. Moreover, there is some $C = C(m, \lambda) > 0$ such that

$$Z^r - C \le W(\mathcal{A}^r_{\text{sub}}) + 2\left[W(\mathcal{B}^r_1) \oplus W(\mathcal{B}^r_2)\right] \le W(\mathcal{G}). \tag{3.6.1}$$

For $\gamma \in \mathcal{G}$, let γ_0, γ_L, and γ_R be the connected components of $\gamma \backslash f^{-r}(\Upsilon^0)$ that are crossing β, β_L, and β_R respectively. Let us split $\mathcal{G}$ into a disjoint union of three sublaminations $\mathcal{G}_{\text{per}} \cup \mathcal{G}_- \cup \mathcal{G}_+$ defined as follows. For $\gamma \in \mathcal{G}$,

- ▷ $\gamma \in \mathcal{G}_{\text{per}}$ if γ_0 crosses both β_L and β_R;
- ▷ $\gamma \in \mathcal{G}_-$ if γ_0 crosses β_L but not β_R;
- ▷ $\gamma \in \mathcal{G}_+$ if γ_0 does not cross β_L.

For $\bullet \in \{+, -\}$ and $\mathrm{x} \in \{0, L, R\}$, let us denote $\mathcal{G}_{\bullet,\mathrm{x}} := \{\gamma_{\mathrm{x}} : \gamma \in \mathcal{G}_\bullet\}$. By design, $\mathcal{G}_{-,0}$ and $\mathcal{G}_{-,R}$ are disjoint, and $\mathcal{G}_{+,0}$ and $\mathcal{G}_{+,L}$ are disjoint. See Figure 3.10. Then,

$$\begin{aligned} W(\mathcal{G}) &\le W(\mathcal{G}_{\text{per}}) + W(\mathcal{G}_{-,0}) \oplus W(\mathcal{G}_{-,R}) + W(\mathcal{G}_{+,0}) \oplus W(\mathcal{G}_{+,L}) \\ &\le W(\mathcal{G}_{\text{per}}) + W(\mathcal{G}_{-,0} \cup \mathcal{G}_{+,0}) \oplus \left[W(\mathcal{G}_{-,R}) + W(\mathcal{G}_{+,L})\right]. \end{aligned} \tag{3.6.2}$$

Since $\mathcal{G}_{\text{per}}$ travels through $\mathcal{T}'_{\text{per}}$ and $\mathcal{G}_{-,0} \cup \mathcal{G}_{+,0}$ travels through $\mathcal{T}'[\beta] \backslash \mathcal{T}'_{\text{per}}$, then

$$W(\mathcal{G}_{\text{per}}) \le Z_{\text{per}} \quad \text{and} \quad W(\mathcal{G}_{-,0} \cup \mathcal{G}_{+,0}) \le Z_{\text{non}}. \tag{3.6.3}$$

Since $\mathcal{G}_{-,R}$ travels through $\mathcal{T}'[\beta_R] \backslash \mathcal{T}'_{\text{per}}$ and $\mathcal{G}_{+,L}$ travels through $\mathcal{T}'[\beta_L] \backslash \mathcal{T}'_{\text{per}}$, then by Lemma 3.6.10,

$$W(\mathcal{G}_{-,R}) \le Z_{\text{non}} + 2D_R \quad \text{and} \quad W(\mathcal{G}_{+,L}) \le Z_{\text{non}} + 2D_L. \tag{3.6.4}$$

Hence, combining (3.6.1), (3.6.2), (3.6.3), and (3.6.4) gives us the desired inequality. □

3.6.5 Persistence amplifies degeneration

Let us consider the strip $\Pi \subset U^r$ from Lemma 3.6.10, and a proper lamination $\mathcal{L}_{\text{per}}$ in Π that is a restriction of the canonical lamination of $\mathcal{T}'_{per}$. Clearly, $\mathcal{L}_{\text{per}}$ connects β_L and β_R, and its width satisfies

$$W(\mathcal{L}_{\text{per}}) \geq Z_{\text{per}}.$$

Let us denote by $\text{CP} = \text{CP}(f^r)$ the set of critical points of f^r.

Lemma 3.6.12. *All leaves of $\mathcal{L}_{\text{per}}$ are properly homotopic to each other in $\Pi \backslash \text{CP}$.*

Proof. Pick any two distinct leaves γ_1 and γ_2 of $\mathcal{L}_{\text{per}}$. Then, $\gamma_1 \cup \gamma_2 \cup \beta_L \cup \beta_R$ must enclose a disk O' contained in Π. Denote by O the disk enclosed by $f^r(\gamma_1) \cup f^r(\gamma_2) \cup \boldsymbol{b}_L \cup \boldsymbol{b}_R$. By the maximum principle, $f^r : O' \to O$ is a proper holomorphic map, and by the argument principle, $f^r|_{O'}$ must be univalent. In particular, O' contains no critical points of f^r. □

Lemma 3.6.13 (Persistence $\longrightarrow$ τ-degeneration)**.** *For any $M > 1$, there exist constants $m_0 = m_0(M) \in \mathbb{N}$ and $\mathbf{K}_2 = \mathbf{K}_2(M, \lambda) > 0$ such that if*

$$m \geq m_0, \quad K \geq \mathbf{K}_2, \quad \textit{and} \quad Z_{\text{per}} \geq 0.1K, \tag{3.6.5}$$

then there is a level $n + m$ combinatorial piece J of width $W_\tau(J) \geq MK$.

Proof. Let us set $t := q_{n+m-2}$ and $s := r - t$. Assume that (3.6.5) holds, and so

$$W(\mathcal{L}_{\text{per}}) \geq 0.1K. \tag{3.6.6}$$

For every critical point c of f, the backward orbit $\{(f|_{\mathbf{H}})^{-i}(c)\}_{i=0,\dots,t-1}$ partitions $\mathbf{H}$ into pieces of length between l_{n+m-2} and l_{n+m-4}. By lifting this tiling by f^s, observe that CP partitions $f^{-s}(\mathbf{H})$ into preimages of pieces of length at most l_{n+m-4}.

Before we proceed, we will first sketch the idea behind our construction. The horizontal lift $\mathcal{L}_{\text{per}}$ of the persistent lamination must cross through a large number of *fences*, which are connected subsets of $f^{-s}(\mathbf{H})$ separating β_L and β_R in Π, as shown in Figure 3.11. These fences can be chosen such that their images under f^s have alternating configuration shown in Figure 3.12. As these fences are tiled by CP, then by Lemma 3.6.12, $\mathcal{L}_{\text{per}}$ must intersect a common tile G_i from each fence $\#_i$. We then apply the series law to obtain a large τ-degeneration.

Now, let us delve into the details. By Lemma 3.6.10, there exists a piece I in Π of length $\asymp l_n$. Recall the two distinct cases ⦿ and ◎ outlined in §3.1.4.

Case ⦿: Assuming m is large enough (depending on N), there is a sequence

$$x_1^\infty,\, x_1^0,\, x_2^\infty,\, x_2^0,\, \dots,\, x_{2N}^\infty,\, x_{2N}^0$$

of critical points of f^s, written in consecutive order, with the following properties.

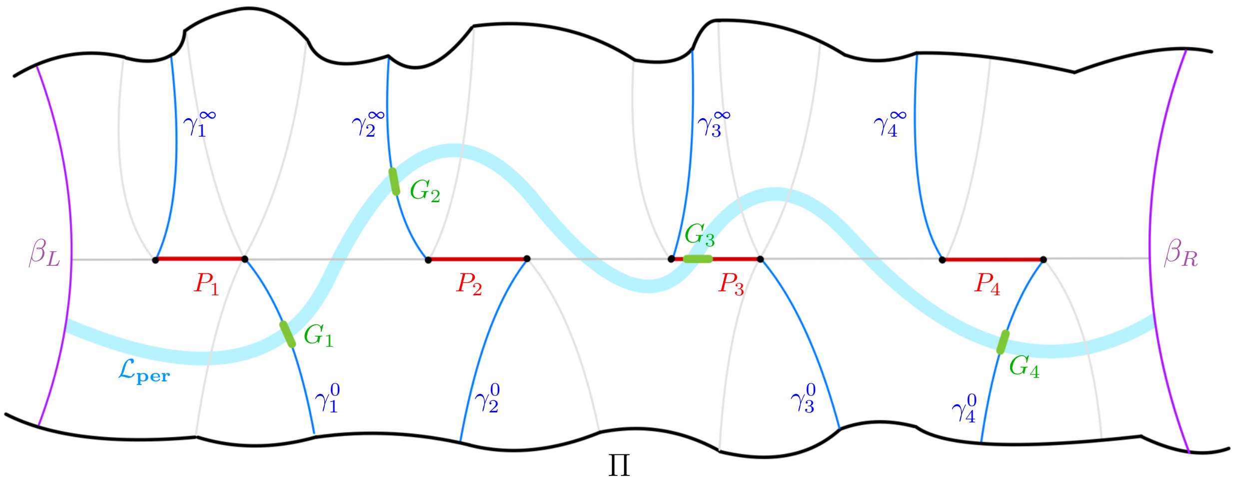

Figure 3.11: The lamination $\mathcal{L}_{\text{per}}$ crosses *fences* $\#_j = \gamma_j^\infty \cup P_j \cup \gamma_j^0$ through *gates* G_j in consecutive order.

(i) All the x_i^∞'s and x_i^0's are located on I, with x_1^∞ being the closest to β_L and x_{2N}^0 being the closest to β_R combinatorially.

(ii) Every x_i^∞ (resp. x_i^0) is the root of an outer (resp. inner) bubble $\mathbf{B}_i^\infty$ (resp. $\mathbf{B}_i^0$) of generation at most s.

(iii) The pieces $P_i := [x_i^\infty, x_i^0]$ have length at least l_{n+m-4} and are of distance at least $\frac{\tau-1}{2} l_{n+m-4}$ away from each other.

For every odd (resp. even) i and $\bullet \in \{0, \infty\}$, the critical value $f^s(x_i^\bullet)$ partitions[4] $\overline{U^t} \cap \mathbf{H}$ into two pieces, one of which, which we will denote by $J_i^\bullet$, intersects L^t (resp. R^t). Denote by $\gamma_i^\bullet$ the lift of $J_i^\bullet$ under f^s that lies within the bubble $\mathbf{B}_i^\bullet$. By (ii), each $\gamma_i^\bullet$ intersects $\mathbf{H}$ precisely at the critical point $x_i^\bullet$. Define our *fences* as

$$\#_i := \gamma_i^\infty \cup P_i \cup \gamma_i^0.$$

By (i) and (iii), they satisfy the following properties. (See Figures 3.11 and 3.12.)

(iv) The $\#_i$'s are pairwise disjoint connected subsets of $\overline{\Pi} \cap f^{-s}(\mathbf{H})$ which separate β_L and β_R.

(v) For each $l \in \{1, \dots, N\}$, the images $f^s(\#_{2l-1})$ and $f^s(\#_{2l})$ are disjoint pieces in $\mathbf{H}$ that are at least $\frac{\tau-1}{2} l_{n+m-4}$ apart from each other.

[4]In Case ⦿, we partition using the radial segment in $\mathbf{H}$ containing $f^s(x_i^\bullet)$.

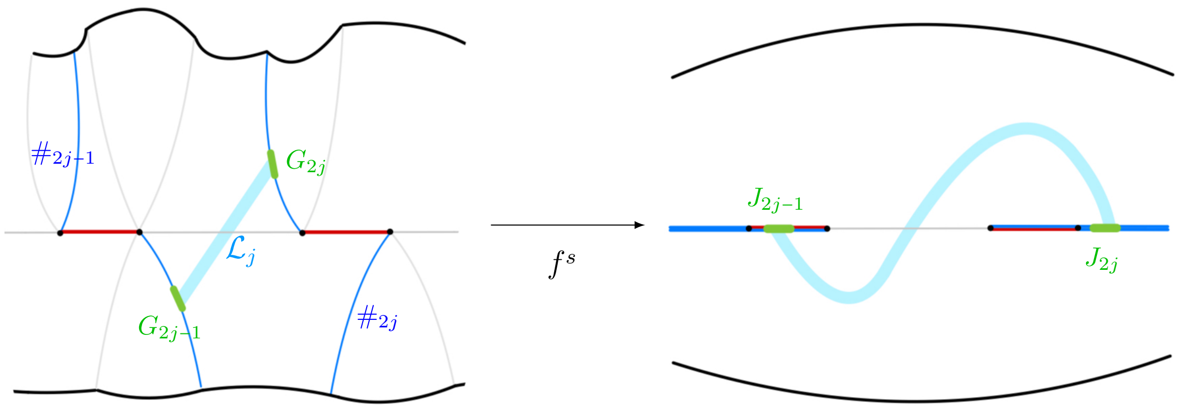

Figure 3.12: The fences $\#_{2j-1}$ and $\#_{2j}$ are constructed such that their images under f^s have τ-separation.

By property (iv), $\mathcal{L}_{\text{per}}$ crosses each fence in consecutive order, namely $\#_i$ first before $\#_{i+1}$. As CP induces tiling on fences, Lemma 3.6.12 implies the existence of connected compact subsets $G_i \subset \#_i$ (the *gates* of the fence) where the images $J_i := f^s(G_i)$ are level $n+m-4$ combinatorial pieces and $\mathcal{L}_{\text{per}}$ crosses the G_i's in consecutive order.

Therefore, there exist pairwise disjoint laminations $\mathcal{L}_1, \ldots, \mathcal{L}_N$ such that each $\mathcal{L}_j$ is a restriction of $\mathcal{L}_{\text{per}}$ that connects G_{2j-1} and G_{2j}. Let k be such that $\mathcal{L}_k$ is the widest among all the $\mathcal{L}_j$'s. By property (v), since each J_i lies within $f^s(\#_i)$, then $f^s(\mathcal{L}_k)$ overflows $\mathcal{F}_\tau(J_{2k})$. By Propositions 2.2.2 and 2.2.3, and by (3.6.6),

$$W_\tau(J_{2k}) \geq W\left(f^s(\mathcal{L}_k)\right) \geq N \cdot W\left(f^s\left(\mathcal{L}_{\text{per}}\right)\right) = N \cdot W\left(\mathcal{L}_{\text{per}}\right) \geq 0.1NK.$$

Consider the constant $C > 1$ from Proposition 2.1.3. There exists a level $n+m$ combinatorial subpiece J of J_{2k} with width $W_\tau(J) \geq 0.1C^{-4}NK$. Finally, set $N = \lceil 10C^4M \rceil$ and we are done.

Case $\odot$: The proof is similar to the previous case, but the construction of fences needs a small adjustment. Following Remark 3.6.1, we assume that the U^j's are disjoint from the connected component of the complement of the Herman ring $\mathbb{H}$ containing 0. In particular, U^r does not contain any inner bubbles. We will instead take the bubbles $\mathbf{B}_i^\infty$ and $\mathbf{B}_i^0$ described in (ii) to both be outer bubbles. Although the corresponding fences $\#_i$ will no longer separate β_L and β_R, we claim that most of $\mathcal{L}_{\text{per}}$ still cross every fence in consecutive order.

Indeed, the set of leaves in $\mathcal{L}_{\text{per}}$ that are disjoint from some fence $\#_i$ overflows the family $\mathcal{L}_i'$ of curves in $\mathbb{H} \cap U^r$ that skip P_i, i.e. they all connect two disjoint intervals in $\mathbf{H}$ are adjacent to P_i and are at most λl_n in length. By uniformizing $\mathbb{H}$ and applying Proposition 2.2.7, the width of $\mathcal{L}_i'$ is at most some constant depending on λ and N. Therefore, for sufficiently large $\mathbf{K}_2 = \mathbf{K}_2(\lambda, N) > 0$, we can assume that the width of the sublamination $\mathcal{L}''$ consisting of leaves in $\mathcal{L}_{\text{per}}$ that cross the fences in consecutive order is at least half of $\mathcal{L}_{\text{per}}$. The same remaining argument holds for $\mathcal{L}''$, and at the last moment we take $N = \lceil 20C^4 M \rceil$ instead. $\square$

3.6.6 Proof of Proposition 3.6.2

The results in the preceding subsections can be summarized as follows.

Lemma 3.6.14 (Degeneration vs. loss of Z^j)**.** *Given any $M \geq 1$, there exist constants $m = m(M) \in \mathbb{N}$, $\nu = \nu(M) \in (0,1)$, and $\mathbf{K}_1 = \mathbf{K}_1(M, \lambda) > 0$ such that if $W(\mathcal{F}^j) \geq \mathbf{K}_1$ for some $j \in \mathbb{N}$, then either*

(1) *there is a level $n + m$ combinatorial piece J of width $W_\tau(J) \geq M \cdot W(\mathcal{F}^j)$, or*

(2) *$Z^{j+q_{n+m}} \leq \nu Z^j$.*

Proof. Let $K = W(\mathcal{F}^0)$ and $r = q_{n+m}$. Following Sections §3.6.2–3.6.5, assume $j = 0$ without loss of generality and denote by $C = C(m, \lambda)$ any positive constant depending only on m and λ. By Lemmas 3.6.7 and 3.6.13, for sufficiently high integers m and κ depending on M, either item (1) holds, or

$$D \leq \kappa K \qquad \text{and} \qquad Z_{\text{per}} \leq 0.1K. \tag{3.6.7}$$

We will show that the latter assertion implies (2). By Proposition 3.6.11,

$$Z^r - C \leq Z_{\text{per}} + (Z^0 - Z_{\text{per}}) \oplus (2 + 2\kappa) Z^0.$$

Set $\nu' := 0.1 + 0.9 \oplus (2 + 2\kappa)$; clearly, $0 < \nu' < 1$. By (3.6.7), the inequality simplifies to

$$Z^r - C \leq \nu' Z^0.$$

Set $\nu = (1 + \nu')/2$ and assume $\mathbf{K}_1 \geq 2C/(1 - \nu')$. Then, $Z^r \leq \nu Z^0$. $\square$

At last, we are ready to prove the main result of this section. We will apply Lemma 3.6.14 many times until the shrinking factor is as low as we want.

Proof of Proposition 3.6.2. Fix $\Delta > 1$ and $\delta \in (0,1)$. We will be applying Lemma 3.6.14 using the constant $M = \delta^{-1}\Delta$. Consider the constants m, ν, and $\mathbf{K_1}$ from the lemma. Set $r := q_{n+m}$ and $\mathbf{K} := \delta^{-1}\mathbf{K}_1$, and let us assume that $W(\mathcal{F}^0) = K \geq \mathbf{K}$. Let us pick $\mathbf{t} \in \mathbb{N}$ such that $\nu^{\mathbf{t}} \leq \delta/2$. Our goal is to prove that either

(a.) there is a level $n + m$ combinatorial piece J of τ-width at least ΔK,

or there is some t between 1 and $\mathbf{t}$ such that (b.t) $W(\mathcal{F}^{rt}) \le \delta K$.

The proof below involves another related assertion, which is

(c.t) $Z^{rt} \le \nu^t Z^0$.

Claim. If (c.t) holds, then either (a.), or (b.t), or (c.$t+1$) holds.

Proof. Suppose (c.t) holds and (b.t) fails. By the lemma, either there is a level $n+m$ combinatorial piece J of τ-width at least $\delta^{-1}\Delta \cdot W(\mathcal{F}^{rt})$, or $Z^{r(t+1)} \le \nu Z^{rt}$. If the former assertion holds, since (b.t) does not hold, then

$$W_\tau(J) \ge \delta^{-1}\Delta \cdot W(\mathcal{F}^{rt}) \ge \Delta K.$$

If the latter assertion holds instead, then by (c.t), $Z^{r(t+1)} \le \nu^{t+1} Z^0$. □

Trivially, (c.0) holds. As we apply the claim above for $t = 0, 1, \dots, \mathbf{t}-1$, we conclude that either (a.) holds, or (b.t) holds for some t between 1 and $\mathbf{t}-1$, or (c.$\mathbf{t}$) holds. The latter case implies (b.$\mathbf{t}$) because

$$W\left(\mathcal{F}^{r\mathbf{t}}\right) \le Z^{r\mathbf{t}} \le \nu^{\mathbf{t}} Z^0 \le \frac{\delta}{2} Z^0 \le \delta K.$$

Therefore, either (a.) holds or (b.t) holds for some $t \le \mathbf{t}$. □

3.7 A priori bounds for Herman rings of the simplest configuration

We are now prepared to prove the first main theorem of the chapter. The results in Sections §3.2–3.6 are compiled together to obtain the following theorem.

Theorem 3.7.1 (Amplification Theorem)**.** *There is an absolute constant $\tau > 1$ and some constants $\mathbf{K} > 1$, $m \in \mathbb{N}$, and $N \in \mathbb{N}$ depending only on d_0, d_∞, and $\beta(\theta)$ such that if*

there is a $[K, \tau]$-wide combinatorial piece $I \subset \mathbf{H}$ of level $n \ge N$

where $K \ge \mathbf{K}$ then

there is a $[2K, \tau]$-wide combinatorial piece $J \subset \mathbf{H}$ of level $n' \ge N$

where $|n' - n| \le m$.

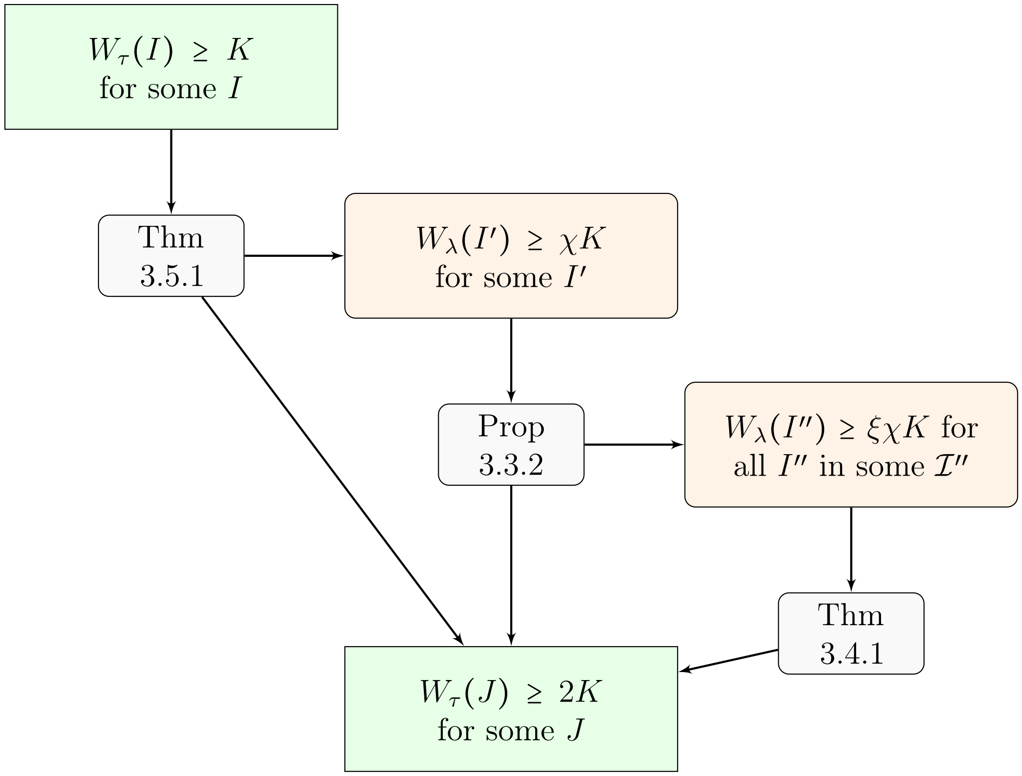

Figure 3.13: Implication diagram illustrating the amplification process.

The motivation behind the Amplification Theorem comes from D. Dudko and Lyubich's motto in [DL22]:

"If life is bad now, it will be worse tomorrow."

This is in the same spirit as Kahn's general strategy in his proof of a priori bounds for infinitely renormalizable quadratic polynomials of bounded primitive combinatorics in [Kah06].

Proof. Set $\tau := 10$. Fix a large constant $\lambda \gg \tau$, and set $N := \mathbf{n}_\lambda$ and $m := \mathbf{m}_\lambda + \mathbf{m}$, where $\mathbf{m}_\lambda$ and $\mathbf{m}$ are constants from Theorems 3.4.1 and 3.5.1 respectively. We will take $\mathbf{K}$ to be sufficiently high so that all the arguments below hold.

Suppose I is a $[K, \tau]$-wide level $\geq N$ combinatorial piece in $\mathbf{H}$, where $K \geq \mathbf{K}$. By Theorem 3.5.1, either there is a $[2K, \tau]$-wide combinatorial piece J or there is a $[\chi K, \lambda]$-wide combinatorial piece L. In the latter case, apply Proposition 3.3.2 for the value $\Xi = 2/\chi$ such that either there is a $[2K, \tau]$-wide combinatorial piece J or there is an almost tiling $\mathcal{I}''$ consisting of $[\xi\chi K, \tau]$-wide pieces. If the latter holds, apply Theorem 3.4.1 to $\mathcal{I}''$ to obtain a $[\Pi_\lambda \xi \chi K, \tau]$-wide combinatorial piece $J \subset \mathbf{H}$. Refer to Figure 3.13 for an illustration. Finally, we choose the constant λ such that $\Pi_\lambda \xi \chi \geq 2$. Then, J is the piece we are looking for. $\square$

Proposition 3.1.11 (1) is not directly applicable to $\mathbf{H}$ in Case ⊙. In particular, τ-degeneration can always be found amongst pieces of level $\gg \log(\operatorname{mod}(\mathbb{H})^{-1})$. To prove a priori bounds for f, we will switch between pieces of $\overline{\mathbb{H}}$ and intervals of a boundary component of $\mathbb{H}$ of sufficiently deep level depending on $\operatorname{mod}(\mathbb{H})$.

Proof of Theorem A. Let $\mathbb{H}$ the Herman ring of f and denote by $\mu > 0$ the conformal modulus of $\mathbb{H}$. By Corollary 3.1.3, it is sufficient to prove the theorem when $\mu < \mu_0$ for some fixed $0 < \mu_0 < 1$.

Let Y^0 and Y^∞ be the connected components of $\hat{\mathbb{C}} \backslash \overline{\mathbb{H}}$ containing 0 and ∞ respectively. Denote the boundary components of $\mathbb{H}$ by

$$H^0 := \partial Y^0, \quad \text{and} \quad H^\infty := \partial Y^\infty.$$

Let τ, m, $\mathbf{K}$, and N be constants from Theorem 3.7.1, and let C be the constant from Proposition 2.1.3. It is sufficient to show that every interval I in $H^0 \cup H^\infty$ of length $\leq l_N$ must have width $W_\tau(I) \leq C\mathbf{K}$.

Let $M \in \mathbb{N}$ be such that

$$l_{M+1} \leq \mu < l_M. \tag{3.7.1}$$

Pick the threshold μ_0 to be small enough such that $M > N + 2m$. All the combinatorial intervals and pieces considered below will be of level $\geq N$, and similar to the shallow-deep treatment in Sections §3.4–3.5, they will be distinguished into two:

Herman scale $N \leq n < M$,

Siegel scale $n \geq M$.

Note that these scales coincide with the ones introduced in §1.3 and §3.1.4.

Lemma 3.7.2. *If there is a $[K,\tau]$-wide combinatorial interval $I^\bullet \subset H^\bullet$ at the Siegel scale for some $\bullet \in \{0,\infty\}$ and $K \geq \mathbf{K}$, then there is a $[2K,\tau]$-wide combinatorial interval $J^\bullet \subset H^\bullet$ of level at least N.*

Proof. By applying Theorem 3.7.1 in Case ⊚, we can obtain from $I^\bullet$ a $[2K,\tau]$-wide combinatorial interval $J^\bullet \subset H^\bullet$ of level $\geq N_1 - m > N$. □

To amplify degeneration about intervals at the Herman scale, we will thicken them to pieces of $\overline{\mathbb{H}}$, amplify via Theorem 3.7.1 in Case ⦿, and convert pieces to intervals to obtain more degenerate intervals.

Lemma 3.7.3. *If there is a $[K,\tau]$-wide combinatorial interval $I^\bullet \subset H^\bullet$ at the Herman scale for some $\bullet \in \{0,\infty\}$ and $K \geq \mathbf{K}$, then there is a $[2K,\tau]$-wide combinatorial interval $J^\dagger \subset H^\dagger$ at the Siegel scale for some $\dagger \in \{0,\infty\}$.*

Proof. Let $I \subset \overline{\mathbb{H}}$ be the combinatorial piece such that $I \cap H^\bullet = I^\bullet$. The piece I is also at the Herman scale and $[K,\tau]$-wide in $\overline{\mathbb{H}}$. By inductively applying the Amplification Theorem, we obtain an infinite sequence of combinatorial pieces $J_1, J_2, \ldots$ where each J_i is $[2^i K, \tau]$-wide.

Let $t \geq 2$ be a fixed integer that is to be determined later. By compactness, it is impossible for every piece in $\{J_{it}\}_{i \geq 1}$ to be at the Herman scale. Let $j \geq 1$ be the smallest integer such that J_{jt} is at the Siegel scale. The piece $J := J_{jt}$ has τ-width $W_\tau(J) \geq 2^{jt}K \geq 2^t K$. Note that the level n_1 of J must satisfy

$$M \leq n_1 < M + tm. \tag{3.7.2}$$

Denote the horizontal sides of J by $P^0 := J \cap H^0$ and $P^\infty := J \cap H^\infty$. For each $\dagger \in \{0, \infty\}$, we denote by $Q^\dagger$ the union of $P^\dagger$ and both of its neighboring combinatorial intervals of level $n_1 + 1$, and by $R^\dagger$ the union of $(\tau P^\dagger)^c$ and both of its neighboring combinatorial intervals of level $n_1 + 1$.

Claim 1. The width of curves in $\mathcal{F}_\tau(J)$ that cross through (intersect both horizontal sides of) any component of $\overline{\tau J \backslash J}$ is at most some absolute constant $C_1 > 0$.

Proof. Indeed, suppose A is one of the two components of $\overline{\tau J \backslash J}$. As a conformal rectangle, the width of A is equal to $|A|/\mu$. Note that

$$\frac{|A|}{\mu} = \frac{\tau - 1}{2} \cdot \frac{l_{n_1}}{\mu} \leq \frac{\tau - 1}{2} \cdot \frac{l_{n_1}}{l_{M+1}} \leq \frac{C}{2}(\tau - 1),$$

where the first inequality follows from (3.7.1) and the second follows from (3.7.2). As there are two possible A's to consider, the claim follows by taking $C_1 = C \cdot (\tau - 1)$. ☐

Claim 2. The width of curves in $\mathcal{F}_\tau(J)$ that do not restrict to curves joining $Q^\sharp$ and $R^\flat$ for some $\sharp, \flat \in \{0, \infty\}$ is at most some constant $C_2(t) > 0$.

Proof. If a curve in $\mathcal{F}_\tau(J)$ does not have a subcurve joining some $Q^\sharp$ and $R^\flat$, then it must have a subcurve that is proper in A and connects the vertical sides of A, where A is one of the four level $n_1 + 1$ combinatorial pieces of $\overline{\mathbb{H}}$ next to J or $(\tau J)^c$. The width w_A of proper curves in A connecting the vertical sides of A satisfies

$$w_A = \frac{\mu}{l_{n_1+1}} \leq \frac{l_M}{l_{n_1+1}} \leq C^{tm+1},$$

where the first inequality follows from (3.7.1) and the second is from (3.7.2). As there are four possible A's to consider, our claim follows from taking $C_2 = 4C^{tm+1}$. ☐

From both claims above, there is some $\dagger \in \{0, \infty\}$ such that the width $W\left(Q^\dagger, R^\dagger\right)$ of curves joining $Q^\dagger$ and $R^\dagger$ satisfies

$$W\left(Q^\dagger, R^\dagger\right) \geq \frac{W_\tau(J) - C_1 - C_2(t)}{2} \geq 2^{t-1}K - \frac{C_1 + C_2(t)}{2}.$$

By replacing $\mathbf{K}$ with a higher constant depending on t if necessary, we have

$$W\left(Q^\dagger, R^\dagger\right) \geq 2^{t-2}K.$$

There is an absolute constant $s \in \mathbb{N}$ such that for any combinatorial subinterval $J^\dagger$ of $Q^\dagger$ of level $n_1 + s$, the piece $(\tau J^\dagger)^c$ contains $R^\dagger$. Therefore, there is a level $n_1 + s$ combinatorial subinterval $J^\dagger \subset Q^\dagger$ such that

$$W_\tau(J^\dagger) \geq \frac{|J^\dagger|}{|Q^\dagger|} \cdot W\left(Q^\dagger, R^\dagger\right) > 2^t K.$$

Finally, we can pick t to be sufficiently high such that $J^\dagger$ is $[2K, \tau]$-wide. □

Suppose for a contradiction that on one of the boundary components, say H^∞, there exists an interval I^∞ of length $\leq l_N$ and τ-width at least CK where $K \geq \mathbf{K}$. Then, I^∞ admits a $[K, \tau]$-wide combinatorial subinterval $I_0^\infty \subset I^\infty$ of level $\geq N$. The two lemmas above imply that there is an increasing sequence of positive integers $\{i_j\}_{j \in \mathbb{N}}$ and $[2^{i_j} K, \tau]$-wide combinatorial intervals $I_{i_j}^\bullet \subset H^\bullet$ for all $j \in \mathbb{N}$ for some common $\bullet \in \{0, \infty\}$. This would contradict Proposition 3.1.11 (1) and thus conclude the proof of Theorem A. □

3.8 Construction of Herman curves

In this section, for every $f \in \mathcal{H}$, we denote by $\mathbb{H}_f$ the Herman ring of f. We will apply Theorem A to study the limit space $\partial\mathcal{H}$ in Rat_d.

Throughout this section, we will denote by $\mathbb{A}(r, R)$ the round annulus $\{r < |z| < R\}$ of inner and outer radii r and R. For brevity, we will also encode the data $(d_0, d_\infty, \beta(\theta))$ with the symbol $\clubsuit$.

3.8.1 Precompactness

Given rational maps f and g, we write $f \sim g$ to denote that f and g are conformally conjugate. Note that a Möbius transformation preserves the space $\mathcal{H}$ by conjugation if and only if it is a linear map $z \mapsto \lambda z$. While our a priori bounds is geometric in flavour, it implies a compactness property that is algebraic in flavour.

Theorem 3.8.1. *For any $\mu > 0$ and $N \in \mathbb{N}$, the quotient space*

$$\{f \in \mathcal{H} \,|\, \mathrm{mod}(\mathbb{H}_f) < \mu\} /_\sim$$

is precompact.

The following lemma will serve as a key ingredient in the proof.

Lemma 3.8.2 (Bounded shape about 0 and ∞)**.** *Let $f \in \mathcal{H}$. The union of the inner (resp. outer) boundary component of $\mathbb{H}_f$ and all the inner (resp. outer) bubbles of generation* 1 *is contained in some round annulus $\mathbb{A}(\varepsilon r, r)$ where $0 < \varepsilon < 1$ depends only on $\clubsuit$.*

Proof. We will prove the lemma for the inner boundary and inner bubbles. The treatment for the outer case is analogous. Denote by Y^0 the connected component of the complement of $\mathbb{H}_f$ containing 0, and by Y_1^0 the component of $f^{-1}(Y^0)$ that is contained in Y^0. Let $H^0 := \partial Y^0$ and $H_1^0 := \partial Y_1^0$. By conjugating with a linear map, we can assume that the maximum Euclidean distance between 0 and a point on H_1^0 is 1. It is sufficient to find a lower bound ε on $\operatorname{dist}(0, H_1^0)$.

Denote by ζ a point on H^0 that is closest to 0, by I_ζ the level 2 combinatorial interval on H^0 centered at ζ, and by $I'_\zeta := (f|_{H^0})^{-1}(I_\zeta)$ the lift of I_ζ inside H^0. Let κ be the harmonic measure of I_ζ in Y^0 about 0. As $f : Y_1^0 \to Y^0$ is a degree d_0 covering map that is branched only at 0, the harmonic measure of I'_ζ on Y_1^0 about 0 is κ/d_0. Since $Y_1^0 \subset Y^0$ and $H^0 \subset H_1^0$, then the harmonic measure of I'_ζ in Y^0 about 0 is at least κ/d_0. Note that since $l_2 < \max\{\theta, 1-\theta\}$, the intervals I'_ζ and I_ζ must be disjoint. As such, κ must be bounded above by

$$\kappa < \left(1 + \frac{1}{d_0}\right)^{-1} < 1. \tag{3.8.1}$$

By assumption, the Euclidean diameter of H^0 is greater than 1. Since the conjugacy $\phi : H^0 \to \mathbb{T}$ between $f|_{H^0}$ and $R_\theta|_{\mathbb{T}}$ is a $K(\clubsuit)$-quasisymmetry, every connected component of $I_\zeta \backslash \{\zeta\}$ has diameter greater than some $L_1 = L_1(\clubsuit) > 0$. As H^0 is a quasicircle, there is also some small $L_2 = L_2(\clubsuit) > 0$ such that $H^0 \backslash I_\zeta$ is disjoint from the disk $\mathbb{D}(\zeta, L_2)$. Together with (3.8.1), this implies that ζ cannot be arbitrarily close to 0, that is, $\operatorname{dist}(0, H^0) > \varepsilon'$ for some $\varepsilon' = \varepsilon'(\clubsuit) > 0$.

The outer boundary of every inner bubble of generation 1 is contained in H_1^0, and its harmonic measure in Y_1^0 about 0 is simply the constant $1/d_0$. Using a similar argument, we conclude that every inner bubble of generation 1 is of distance at least some constant $\varepsilon(\clubsuit) > 0$ away from 0. □

Lemma 3.8.3. *Let $f \in \mathcal{H}$. There is a K-quasiconformal map $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with the following properties.*

(1) *ϕ maps the Herman ring $\mathbb{H} = \mathbb{H}_f$ onto some annulus $\mathbb{A} = \mathbb{A}(r, re^{2\pi \operatorname{mod}(\mathbb{H})})$;*

(2) *ϕ is conformal in $\mathbb{H}$;*

(3) *$\phi|_{\overline{\mathbb{H}}}$ is a conjugacy between $f|_{\overline{\mathbb{H}}}$ and rigid rotation $R_\theta|_{\overline{\mathbb{A}}}$;*

(4) *ϕ fixes* 0 *and* ∞*;*

(5) K *depends only on* $\clubsuit$.

Proof. By Theorem A, it is immediate that there is a map $\phi : \overline{\mathbb{H}} \to \overline{\mathbb{A}}$ satisfying (1)-(3) that restricts to a $K'(\clubsuit)$-quasisymmetric map from $\mathbb{H}$ to ∂A. By Lemma 3.8.2, the control of $\partial\mathbb{H}$ relative to 0 and ∞ allows us to extend ϕ to a global quasiconformal map satisfying (4) and (5). □

Proof of Theorem 3.8.1. Let $f \in \mathcal{H}$ be a rational map such that $\mathrm{mod}(\mathbb{H}_f) < \mu$. Denote by H^0 and H^∞ the inner and outer boundary components of $\mathbb{H}_f$. By conjugating f with a linear map, assume that the maximum Euclidean distance between 0 and a point in H^0 is 1.

The rational map f must be of the form

$$f(z) = \lambda z^{d_0} \frac{(z - z_1) \dots (z - z_{d_\infty - 1})}{(z - p_1) \dots (z - p_{d_0 - 1})},$$

where $\mathcal{Z} := \{z_1, \dots, z_{d_\infty - 1}\}$ and $\mathcal{P} := \{p_1, \dots, p_{d_0 - 1}\}$ are the sets of zeros and poles of f respectively. To prove precompactness, it is sufficient to show that there exists some $\varepsilon = \varepsilon(\clubsuit, \mu) > 0$ such that

(i) $\mathcal{Z} \cup \mathcal{P} \subset \mathbb{A}(\varepsilon, \varepsilon^{-1})$,

(ii) $\mathrm{dist}(\mathcal{Z}, \mathcal{P}) > \varepsilon$, and

(iii) $\varepsilon < |\lambda| < \varepsilon^{-1}$.

From our choice of normalization, the outer boundary H^∞ must contain some point w such that $|w| \leq e^{2\pi\mu}$. Indeed, if otherwise, $\mathbb{H}_f$ would contain the annulus $\overline{\mathbb{A}(1, e^{2\pi\mu})}$ which would contradict the assumption that $\mathrm{mod}(\mathbb{H}_f) < \mu$. As a consequence of Lemma 3.8.2, there is some $\varepsilon_1 = \varepsilon_1(\clubsuit, \mu) > 0$ such that

$$f^{-1}(\overline{\mathbb{H}_f}) \subset \mathbb{A}(\varepsilon_1, \varepsilon_1^{-1}). \tag{3.8.2}$$

Since the zeros and poles are enclosed by bubbles of generation 1, we obtain (i).

Next, (ii) follows directly from the claim below.

Claim. There is some $\varepsilon_2 = \varepsilon_2(\clubsuit, \mu) > 0$ such that $\mathrm{dist}\left(\overline{\mathbb{H}_f}, \mathcal{Z} \cup \mathcal{P}\right) > \varepsilon_2$.

Proof. Let us pick a pole $p \in \mathcal{P}$. The treatment for zeros is analogous. Recall the notation H_1^0, Y^0 and Y_1^0 used in the proof of Lemma 3.8.2. Let $c \in H^0$ be the critical point that is the root of the inner bubble B of generation 1 that encloses p. Let ϕ be the $K(\clubsuit)$-quasiconformal

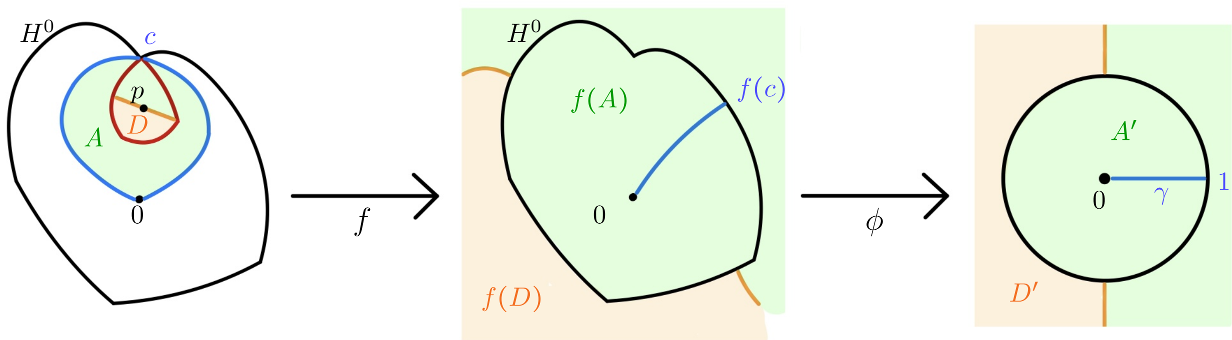

Figure 3.14: Construction of the annulus A surrounding D.

map from Lemma 3.8.3. We can normalize ϕ such that it maps the inner boundary H^0 to the unit circle and the critical value $f(c)$ to 1.

Let γ be the straight segment $[0,1]$ and let D' be the closure of the left half plane minus $\mathbb{D}$. By construction, the annulus $A' := \hat{\mathbb{C}} \backslash (D' \cup \gamma)$ has modulus equal to some universal constant $\kappa > 0$. Let A (resp. D) be the unique lift of A' (resp. D') under $\phi \circ f$ that intersects the bubble B. See Figure 3.14.

Since ϕ maps $(Y^0, 0)$ to $(\mathbb{D}, 0)$, the harmonic measure of $f(D) \cap H^0$ in Y^0 about 0 is at least some $\delta(\clubsuit) > 0$. Therefore, the harmonic measure of $D \cap H_1^0$ in Y_1^0 about 0 is at least δ/d_0. Combined with (3.8.2), the diameter of D must be bounded above by some $\delta'(\clubsuit, \mu) > 0$. Since $\mathrm{mod}(A) \geq \kappa/K$, we can apply Teichmüller estimates (cf. [Ahl06, §3]) and conclude that the distance between the two boundary components of A is at least some constant $\varepsilon_2(\clubsuit, \mu) > 0$. Finally, as A separates the pole p from $\overline{\mathbb{H}_f}$, then $\mathrm{dist}\left(\overline{\mathbb{H}_f}, p\right) > \varepsilon_2$. □

The claim and (3.8.2) imply that every $w \in \mathcal{Z} \cup \mathcal{P}$ satisfies $\varepsilon_2 \leq |1 - w| \leq 1 + \varepsilon_1^{-1}$. Moreover, as $f(1)$ lies on the inner boundary H^0, then $\varepsilon_1 \leq |f(1)| \leq \varepsilon_1^{-1}$. These two observations imply (iii), and we are done. □

3.8.2 Degeneration of Herman rings

Theorem 3.8.1 implies that $\partial\mathcal{H}/_\sim$ is compact. By a priori bounds, we are finally able to establish a formal relation between the two spaces $\mathcal{H}$ and $\mathcal{X}$.

Corollary 3.8.4. *$\partial\mathcal{H}$ is contained in $\mathcal{X}$.*

Proof. Suppose $f_n \to f$ for some sequence of rational maps $f_n \in \mathcal{H}$. We will show that the limit f must lie in $\mathcal{H} \cup \mathcal{X}$.

Due to uniform convergence, both 0 and ∞ remain superattracting fixed points for f of local degrees d_0 and d_∞ respectively. In particular, the Julia sets $J(f_n)$ must all be contained in $\mathbb{A}(\varepsilon, \varepsilon^{-1})$ for some $0 < \varepsilon < 1$ independent of n, and that the moduli μ_n of the Herman rings $\mathbb{H}_n$ of f_n are bounded above by $\frac{1}{\pi}\log\frac{1}{\varepsilon}$. Moreover, for sufficiently high n, f_n has a free critical point c_n of the same local degree independent of n located on the inner boundary of $\mathbb{H}_n$, and $c_n \to c$ where $\varepsilon \le |c| \le \varepsilon^{-1}$.

By Lemma 3.8.3, every f_n admits a $K(\clubsuit)$-quasiconformal map $\phi_n : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that is conformal in $\mathbb{H}_n$, fixes 0 and ∞, maps c_n to 1, and restricts to a conjugacy between $f_n|_{\overline{\mathbb{H}_n}}$ and the rigid rotation R_θ on the closed annulus $A_n := \overline{\mathbb{A}(1, e^{2\pi\mu_n})}$. By the compactness of normalized K-quasiconformal maps, ϕ_n has a subsequence converging to a K-quasiconformal map ϕ which fixes 0 and ∞ and maps c to 1.

By passing to a further subsequence, suppose $\mu_n \to \mu$ for some limit $\mu \ge 0$. As $n \to \infty$, A_n converges in the Hausdorff topology to $A := \overline{\mathbb{A}(1, e^{2\pi\mu})}$ on which we have the conjugacy:

$$R_\theta = \lim_{n\to\infty} \phi_n f_n \phi_n^{-1} = \phi f \phi^{-1}.$$

Moreover, $\overline{\mathbb{H}_n}$ converges to $\mathbf{H} := \phi^{-1}(A)$. Since all free critical points of f_n lie on $\partial\mathbb{H}_n$, then all free critical points of f also lie on $\partial\mathbf{H}$. If $\mu = 0$, then $\mathbf{H}$ must be a Herman quasicircle and f is in $\mathcal{X}$. Else, f is in $\mathcal{H}$ and $\mathbf{H}$ is the closure of a Herman ring of f. □

Corollary B then follows from the corollary above. In the proof, notice that $\mathbf{H}$ is independent of any choice of convergent subsequence taken. In particular, we have simultaneously shown:

Corollary 3.8.5. *For $f \in \overline{\mathcal{H}}$, let $\mathbf{H}_f$ denote either the closure of the Herman ring of f or the Herman quasicircle of f. Then, $f \mapsto \mathbf{H}_f$ is continuous in the Hausdorff topology.*

Recall that the combinatorics of $\mathbf{H}_f$ can be encoded by elements of the space $\mathcal{C} = \mathcal{C}_{d_0,d_\infty}$. (See Definition 3.1.6.)

Corollary 3.8.6. *The map $\partial\mathcal{H} \to \mathcal{C}, f \mapsto comb(f)$ is a continuous surjection.*

Proof. Continuity of comb$(\cdot)$ follows directly from Corollary 3.8.5.

Pick any arbitrary combinatorial data $\sigma \in \mathcal{C}$. By Theorem 3.1.7, there is a rational map $f_1 \in \mathcal{H}$ with a Herman ring $\mathbb{H}_1$ of modulus 1 with comb$(f) = \sigma$. By deforming the complex structure of $\mathbb{H}_1$ (see [BF14, §6.1]), we obtain a real analytic family of rational maps $\{f_t\}_{0<t\le 1}$ in $\mathcal{H}$ where each f_t has a Herman ring $\mathbb{H}_t$ of modulus t with the same combinatorics $\mathcal{C}$.

From Theorem 3.8.1, by appropriately normalizing f_t, there is a sequence $\{t_n\}_{n\in\mathbb{N}}$ in $(0,1)$ such that as $n \to \infty$, the modulus t_n converges to 0 and f_{t_n} converges to a degree d rational map f. Clearly, f cannot lie in $\mathcal{H}$ because otherwise it would contradict the continuity of the moduli of Herman rings guaranteed in Corollary 3.8.5. Therefore, by Corollary 3.8.4, f must lie on $\partial\mathcal{H}$ and it has the same combinatorics σ. □

Chapter 4

Rigidity

This chapter begins with a general discussion of the local dynamics of holomorphic maps near Herman quasicircles. We then give a proof of rigidity of rational maps in $\mathcal{X}$, and discuss some of the immediate applications. In the second half of this chapter, we initiate the study of renormalization theory of critical quasicircle maps.

4.1 Approximate rotation

Let us fix a holomorphic map $f : U \to \hat{\mathbb{C}}$ on an open subset U of $\hat{\mathbb{C}}$ and assume that f admits a Herman quasicircle $\mathbf{H} \subset U$ with bounded type rotation number θ. By Proposition 2.1.6, $\mathbf{H}$ necessarily contains an inner critical point and an outer critical point of f.

By Theorem 2.1.7, there exists a quasiconformal map $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ sending $\mathbf{H}$ to the unit circle $\mathbb{T} \subset \mathbb{C}$ and conjugating $f_{\mathbf{H}}$ and the irrational rotation R_θ. We will fix such a map ϕ.

Consider the function

$$L(z) := \log\left(\operatorname{dist}(\phi(z), \mathbb{T})\right). \tag{4.1.1}$$

Given a point z near $\mathbf{H}$, we will measure the rate of escape of iterates of z using the function L. For any $\kappa > 0$, denote the open annulus

$$A_\kappa := \{-\infty \le L(z) < -\kappa\}.$$

Suppose f extends to a holomorphic map on an annular neighborhood $A = A_\kappa$ of $\mathbf{H}$. We select κ to be high enough such that the only critical points of f inside of A lie on $\mathbf{H}$. Without loss of generality, we will also assume that A avoids 0 and ∞. Let us split A into two annuli

$$A^0 := A \cap Y^0 \quad \text{and} \quad A^\infty = A \cap Y^\infty,$$

where Y^0 and Y^∞ are the connected components of $\hat{\mathbb{C}} \backslash \mathbf{H}$ containing 0 and ∞ respectively. We call A, A^0, and A^∞ a *collar*, *inner collar*, and *outer collar* of $f|_{\mathbf{H}}$.

Let us define the quasi-rotation

$$F(z) \coloneqq \phi^{-1}(e^{2\pi i\theta}\phi(z)).$$

Clearly, F coincides with f on $\mathbf{H}$. Let us equip $Y \coloneqq \hat{\mathbb{C}} \backslash \mathbf{H}$ with the hyperbolic metric. The following definition is inspired by [McM98, §3].

Definition 4.1.1. Suppose an iterate $f^i : U \to V$ is well defined for some $i \geq 0$ and some pair of topological disks $U, V \subset A$. We say that $f^i : U \to V$ is an *approximate rotation* if it is a univalent function of bounded distortion such that for all $x \in U \backslash \mathbf{H}$,

$$d_Y(f^i(x), F^i(x)) = O(1).$$

Given a Jordan domain U and a pair of disjoint arcs I and J on the boundary of U, we denote by $\mathcal{L}_U(I, J)$ the extremal length of the family of proper curves in U connecting I and J. The domain U is conformally equivalent to a Euclidean rectangle where I and J correspond to the vertical sides of unit length and $\mathcal{L}_U(I, J)$ is equal to the length of the horizontal side.

Lemma 4.1.2 (One-sided approximate rotation)**.** *For any $\bullet \in \{0, \infty\}$, any point $z \in \mathbf{H}$, and sufficiently small scale $r > 0$, there exists an approximate rotation*

$$f^i : (U, y) \to (V, c)$$

such that

(1) *both U and V are contained in $A^\bullet$,*

(2) *the point y lies on $\partial U \cap \mathbf{H}$ and the interval $[y, z]$ is contained on $\partial U \cap \mathbf{H}$,*

(3) *c is an outer critical point of f if $\bullet = \infty$, an inner critical point of f if $\bullet = 0$, and*

(4) *we have*

$$\mathcal{L}_U([y, z], \partial U \backslash \mathbf{H}) \succ 1 \qquad \textit{and} \qquad \operatorname{dist}(z, \partial U \backslash \mathbf{H}) \asymp r.$$

For any $0 < a < \pi$ and any small interval $I \subset \mathbb{T}$, we define $H_a(I) \subset \mathbb{D}$ to be the Jordan disk enclosed by the interval I together with the unique circular arc in $\mathbb{D}$ that has the same endpoints as I and meets the circular arc $\mathbb{T} \backslash I$ at an angle of a. To prove Lemma 4.1.2, we will use the following tool.

Lemma 4.1.3 ([McM98, Lemma 3.3])**.** *Consider a K-quasiconformal map $g : H_\alpha(I) \to \mathbb{D}$ which extends continuously to the identity on some interval $I \subset \mathbb{T}$. For any $\beta \in (0, \alpha)$, there is some constant $C = C(\alpha, \beta, K) > 0$ such that*

$$d_{\mathbb{D}}(g(x), x) \leq C \quad \textit{for all } x \in H_\beta(I).$$

Proof of Lemma 4.1.2. Assume without loss of generality that $\bullet = \infty$. Since ϕ is uniformly continuous with respect to the hyperbolic metric of Y, it is sufficient to prove the lemma in the w-coordinate, where $w = \phi(z)$. (Compare with [McM98, Theorem 3.4].) In the w-coordinate, f is an irrational rotation along $\mathbf{H}$, which is the unit circle, and all iterates of f are quasiregular with uniform dilatation. Let us pick a small scale $r > 0$ and a point $w \in \mathbf{H}$.

Denote by $\{p_k/q_k\}_{k\in\mathbb{N}}$ the rational approximations of θ and consider the combinatorial lengths $l_k := |p_k - q_k\theta|$. Let us pick $n \in \mathbb{N}$ such that $r \asymp l_n$. Apply Lemma 2.1.4 by taking S to be the set of critical values of f on $\mathbf{H}$ in order to obtain a pair of intervals $J_{q_{n+2}} \subset I_{q_{n+2}}$ in $\mathbf{H}$ such that

(i) $J_{q_{n+2}}$ contains the level n combinatorial interval centered at $f^{q_{n+2}}(w)$;

(ii) the endpoints of $J_{q_{n+2}}$ split $I_{q_{n+2}}$ into three connected components each having combinatorial length $\asymp l_n$;

(iii) $I_{q_{n+2}} \backslash J_{q_{n+2}}$ does not contain any critical value of $f^{q_{n+2}}$.

For $j = 0, 1, \dots q_{n+2}$, let

$$J_j := (f|_{\mathbf{H}})^{-q_{n+2}+j}(J_{q_{n+2}}) \quad \text{and} \quad I_j := (f|_{\mathbf{H}})^{-q_{n+2}+j}(I_{q_{n+2}}).$$

By Proposition 2.1.2, there is some minimal $i < q_n + q_{n+1}$ such that J_i contains an outer critical point c. Let $y := (f|_{\mathbf{H}})^{-i}(c)$.

Let $\psi_\infty : \mathbb{D} \to Y^\infty$ be a biholomorphism sending 0 to a point outside of A^∞. By Carathéodory's theorem, ψ_∞ extends to a homeomorphism on the boundary $\mathbb{T} \to \mathbf{H}$. For any $a \in (0, \pi)$, we define the domain

$$H_a^\infty(I_i) := \psi_\infty\left(H_a(\psi_\infty^{-1}(I_i))\right) \subset Y^\infty.$$

Let us pick some $\varepsilon \in \left(0, \frac{\pi}{2}\right)$. There is some constant $\kappa' > \kappa$ such that the annular neighborhood $A_{\kappa'}$ of $\mathbf{H}$ is contained within $A^\infty \cap f^{-1}(A^\infty)$ and there is a well-defined inverse branch of f^{-1} mapping any domain of the form $H_\varepsilon^\infty(I)$ that is contained in $A_{\kappa'}$ to a domain touching $\mathbf{H}$ on its boundary. Now let $V := H_{\frac{\pi}{2}}^\infty(I_{q_{n+2}})$.

Claim. For sufficiently small n, there is a well-defined inverse branch of f^{-i} mapping V to a domain U touching $\mathbf{H}$ along the interval I_0. Moreover,

$$d_Y(f^i(x), F^i(x)) = O(1) \quad \text{for all } x \in U. \tag{4.1.2}$$

Proof. Let us pick a positive constant δ. By taking n to be sufficiently small, we assume that V is contained in $A_{\kappa'+\delta}$. Denote by V_{-1} a univalent lift of V under f that is touching

$\mathbf{H}$. Since I_i does not contain any outer critical value of f^i, V_{-1} touches $\mathbf{H}$ precisely along I_{i-1} and so $F \circ f^{-1}$ is the identity map on I_i. By Lemma 4.1.3, there is some constant $C > 0$ depending on the dilatation of ϕ such that

$$d_Y\left((f|_{V_{-1}})^{-1}(x), F^{-1}(x)\right) \leq C \quad \text{ for all } x \in V.$$

Consequently, we take ε to be small enough and δ to be high enough to beat the constant C so that ultimately, $V_{-1} \subset H^{\infty}_{\varepsilon}(I_{-1}) \subset A_{\kappa'}$. The way κ' and i are chosen ensures that V_{-1} can again be lifted to a domain V_{-2} touching $\mathbf{H}$ along I_{i-2}. By the same argument, we have

$$d_Y\left((f|_{V_{-2}})^{-2}(x), F^{-2}(x)\right) \leq C \quad \text{ for all } x \in V$$

and V_{-2} is again contained in $A_{\kappa'}$. Inductively, we can define the domains V_{-2}, V_{-3}, $\ldots$, V_{-i} by pulling back and set $U = V_{-i}$. $\square$

We can ensure that $f^i : U \to V$ has bounded distortion by shrinking I_i (and thus V and U) by a little bit. By (4.1.2), we conclude that $f^i : U \to V$ is an approximate rotation. It remains to prove (4). Properties (i) and (ii) imply that

$$\operatorname{dist}(\partial V \backslash \mathbf{H}, f^i(w)) \asymp r \qquad \text{and} \qquad \mathcal{L}_V(J_n, \partial V \backslash \mathbf{H}) \succ 1$$

in the w-coordinate. Since f^{-i} is uniformly quasiconformal on V and acts as an isometry along $\mathbf{H}$, we obtain the desired estimates in (4). $\square$

Instead of an approximate rotation on one side of $\mathbf{H}$, we can also consider a two-sided approximate rotation, yielding the following lemma.

Lemma 4.1.4 (Two-sided approximate rotation). *Given any point $z \in \mathbf{H}$ and sufficiently small scale $r > 0$, there is an approximate rotation*

$$f^i : (U, y) \to (V, c)$$

such that y lies on $\mathbf{H}$, c is a critical point of f, and (U, y) is a pointed disk that well contains the interval $[y, z] \subset \mathbf{H}$ and has bounded shape and diameter $\asymp r$.

This lemma is a generalization of [McM98, Theorem 3.4] which was originally formulated for bounded type quadratic Siegel disks.

Proof. We can adapt the same proof as the previous lemma by defining the Jordan domain V by gluing $H^{\infty}_{\frac{\pi}{2}}(I_i)$ and $H^{0}_{\frac{\pi}{2}}(I_i)$. The rest of the proof resumes as before. Replace U with a smaller disk (e.g. a hyperbolic ball $\mathbb{D}_U(y, R)$ for some definite radius $R > 1$) so that (U, y) has bounded shape. $\square$

For each $\bullet \in \{0, \infty\}$, the round annulus $\phi^{-1}(A^\bullet)$ admits a canonical radial foliation connecting its two boundary components. For every point z on $\mathbf{H}$, we denote by $\gamma_z^\bullet$ the unique proper curve in $A^\bullet$ such that $\phi(\gamma_z^\bullet)$ is the radial leaf with endpoint $\phi(z)$.

For every inner (resp. outer) critical point $c \in \mathbf{H}$, let us denote by $d_0(c)$ (resp. $d_\infty(c)$) the inner (resp. outer) criticality of c. See Definition 4.3.7 for details.

Lemma 4.1.5 (Local preimages of $\mathbf{H}$)**.** *For $\bullet \in \{0, \infty\}$, there are $2d_\bullet(c) - 2$ pairwise disjoint open quasiarcs $\Gamma_1^\infty, \ldots, \Gamma_{2d_\bullet(c)-2}^\infty$ in $A^\bullet$ which are all mapped into $\mathbf{H}$ by f and attached to c at one of its endpoints. Every point z on $\Gamma_1^\bullet \cup \ldots \cup \Gamma_{2d_\infty(c)-2}^\bullet$ satisfies*

$$\mathrm{dist}_Y(z, \gamma_c^\bullet) = O(1).$$

Proof. Let $d(c) := d_0(c) + d_\infty(c) - 1$ be the local degree of f at c. There exists an open disk neighborhood $Q \subset A$ of c on which f is a degree $d(c)$ covering map branched only at c. When Q is sufficiently small, the map f on Q is of the form $h(z)^{d(c)} + f(c)$ for some univalent map $h : Q \to \hat{\mathbb{C}}$ with bounded distortion sending c to 0.

The disk Q can be picked such that $\phi(f(Q))$ is a round disk orthogonal to $\mathbb{T}$. Then, the preimage of the interval $\mathbf{H} \cap f(Q)$ will consist of $\mathbf{H} \cap Q$ as well as pairwise disjoint open quasiarcs

$$\Gamma_1^0, \ldots, \Gamma_{2d_0-2}^0, \quad \Gamma_1^\infty, \ldots, \Gamma_{2d_\infty-2}^\infty$$

where each $\Gamma_i^\bullet$ is contained in $A^\bullet \cap Q$ and connects the critical point c to a point in $A^\bullet \cap \partial Q$.

Let us pick a point z on Γ_i^∞ for some i. Let w be a point on $\mathbf{H}$ closest to z, i.e. $|z - w| = \mathrm{dist}(z, \mathbf{H})$. Note that the hyperbolic metric of Y at z is comparable to $\mathrm{dist}(z, \mathbf{H})^{-1}$. As such, in order to prove the lemma, it is sufficient to show that

$$|z - w| \succ |z - c|. \tag{4.1.3}$$

Before we do so, we will introduce another disk neighborhood $\tilde{Q}$ of c in a similar way as Q, except that $\tilde{Q}$ is larger than Q and $\mathrm{mod}\left(\tilde{Q} \backslash \overline{Q}\right) \asymp 1$. If $\tilde{Q}$ does not contain w, then by Teichmüller estimates [Ahl06, §3],

$$|z - c| \le \mathrm{diam}\,(Q) \prec \mathrm{dist}\left(\partial\tilde{Q}, \partial Q\right) \le |z - w|$$

and we are done.

Suppose instead that w lies inside of $\tilde{Q}$. Since $\mathbf{H}$ is a quasicircle, the ratio of the distance between $f(z)$ and $f(w)$ to the diameter of the interval $[f(z), f(w)] \subset \mathbf{H}$ must be bounded above by some definite constant. In particular, since the critical value $f(c)$ lies on $[f(z), f(w)]$, then

$$|f(z) - f(w)| \succ |f(z) - f(c)|.$$

Consider the univalent map $h : \tilde{Q} \to \hat{\mathbb{C}}$ described previously. The estimate above can be rewritten as

$$|h(z)^d - h(w)^d| \succ |h(z)|^d,$$

which implies that

$$|h(z) - h(w)| \succ |h(z)|.$$

Since h has bounded distortion on $\tilde{Q}$, this estimate implies (4.1.3). □

Corollary 4.1.6. *Given any point z in $A^\bullet$ for some $\bullet \in \{0, \infty\}$, if z is sufficiently close to $\mathbf{H}$, there exists an approximate rotation $f^i : U \to V$ such that U contains z, V contains a hyperbolic ball $D \subset Y$ of radius $\asymp 1$ centered at some point on $f^{-1}(\mathbf{H})\backslash\mathbf{H}$, and*

$$\mathrm{dist}_Y(f^i(z), D) = O(1).$$

Proof. Again, assume $\bullet = \infty$. Let $w \in \mathbf{H}$ be the unique point such that z lies on the radial segment γ_w^∞. By Lemma 4.1.2, there is an approximate rotation $f^i : U \to V$ such that

(i) U and V are disks in A^∞ and $z \in U$,

(ii) there is some interval $[w, y] \subset \partial U \cap \mathbf{H}$ such that $c := f^i(y)$ is an outer critical point and $\mathcal{L}_U([w, y], \partial U \backslash \mathbf{H}) \succ 1$,

(iii) $|w - y| = O(|w - z|)$.

Refer to Figure 4.1. Clearly, (iii) implies that

$$\mathrm{dist}_Y(z, \gamma_y^\infty) = O(1).$$

Since f^i is an approximate isometry on U, then

$$\mathrm{dist}_Y(f^i(z), \gamma_c^\infty) = O(1).$$

Because of Lemma 4.1.5 and the fact that c is an outer critical point, we have

$$\mathrm{dist}_Y(f^i(z), f^{-1}(\mathbf{H})) = O(1). \tag{4.1.4}$$

From (ii), we have $\mathcal{L}_V([f^i(w), c], \partial V \backslash \mathbf{H}) \succ 1$. Together with (4.1.4), we conclude that V contains a hyperbolic ball D with the desired properties. □

4.2 No invariant line fields on the Julia set

Consider a rational map f in $\mathcal{X}$. In this section, we will prove the following theorem.

Theorem 4.2.1. *The Julia set $J(f)$ of f does not support any invariant line field.*

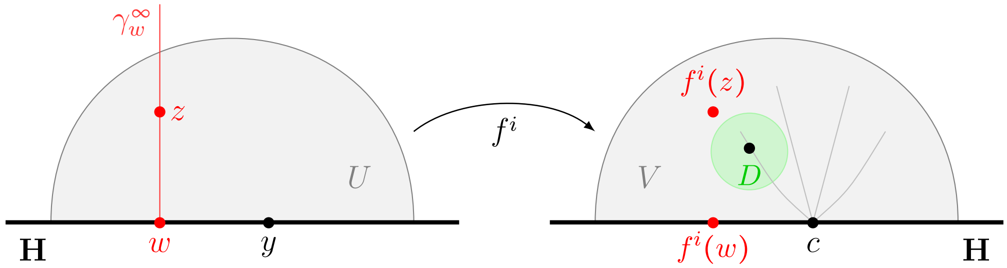

Figure 4.1: The construction in the proof of Corollary 4.1.6.

4.2.1 Hyperbolic geometry off the postcritical set

The postcritical set $P(f)$ is the union of the Herman curve $\mathbf{H}$ of f and the superattractors $\{0, \infty\}$. Denote the complement by

$$\Omega := \hat{\mathbb{C}} \backslash P(f).$$

We shall equip Ω with the hyperbolic metric $\rho_\Omega(z)|dz|$. The set Ω is the maximal open subset such that $f^n : f^{-n}(\Omega) \to \Omega$ is an unbranched covering map for all $n \geq 1$.

For any point z in Ω, denote by $\|f'(z)\|$ the norm of the derivative of f at z with respect to the hyperbolic metric of Ω.

Lemma 4.2.2 ([McM94, Theorems 3.5–3.6])**.** *Consider a point z in Ω.*

(1) *If $f(z)$ is also contained in Ω, then $\|f'(z)\| \geq 1$.*

(2) *If $z \in J(f)$ and $f^n(z) \notin \mathbf{H}$ for all $n \geq 0$, then*

$$\|(f^n)'(z)\| \to \infty \qquad \textit{as } n \to \infty.$$

By Theorem 2.1.7, there exists a quasiconformal map $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that fixes 0 and ∞ and conjugates $f|_{\mathbf{H}}$ and the rigid rotation R_θ on the unit circle $\mathbb{T}$. Recall the function $L(z) := \log(\text{dist}(\phi(z), \mathbb{T}))$ from (4.1.1) as well as the annular neighborhood $A_\kappa := \{-\infty \leq L(z) < -\kappa\}$ of $\mathbf{H}$ for any real number κ.

Let us select a large negative number κ_0 such that the Julia set $J(f)$ of f is compactly contained in the annulus A_{κ_0}. Let us define the *thick part of* Ω by

$$\Omega_{\text{thick}} := \Omega \cap A_{\kappa_0}. \tag{4.2.1}$$

In other words, Ω_{thick} is the sphere $\hat{\mathbb{C}}$ with $\mathbf{H}$ and some small neighborhoods of 0 and ∞ removed.

Lemma 4.2.3. *For every point z in Ω_{thick},*

$$\rho_\Omega(z) \asymp \frac{1}{\operatorname{dist}(z, \mathbf{H})}.$$

Proof. Consider a point z in Ω_{thick} and denote by E the connected component of $\hat{\mathbb{C}} \backslash \mathbf{H}$ containing z. Let us equip E the hyperbolic metric. A standard application of Koebe quarter theorem to any Riemann mapping $(\mathbb{D}, 0) \to (E, z)$ yields the estimate $\rho_E(z) \succ \frac{1}{\operatorname{dist}(z,\mathbf{H})}$. Then, we apply Schwarz Lemma to the inclusion map on the connected component of Ω containing z into E in order to obtain $\rho_\Omega(z) \geq \rho_E(z)$, which gives us the estimate $\rho_\Omega(z) \succ \frac{1}{\operatorname{dist}(z,\mathbf{H})}$.

As we apply Schwarz lemma to the inclusion map $\mathbb{D}(z, \operatorname{dist}(z, P(f))) \hookrightarrow \Omega$, we also obtain the estimate $\rho_\Omega(z) \prec \frac{1}{\operatorname{dist}(z, P(f))}$. We conclude the proof with the observation that $\operatorname{dist}(z, P(f)) \asymp \operatorname{dist}(z, \mathbf{H})$ because z is contained in Ω_{thick}. □

4.2.2 Visiting the critical point from $J(f)$

The key ingredient towards proving Theorem 4.2.1 is the following theorem.

Theorem 4.2.4 (Nearby critical visits)**.** *For every point $z \in J(f)$ and scale $r > 0$, there exist an integer $i \geq 0$, a critical point $c \in \mathbf{H}$ of f, and a pair of pointed disks (U, y) and (V, c) such that*

(1) *$f^i : (U, y) \to (V, c)$ is a univalent map with bounded distortion,*

(2) *(U, y) has bounded shape with diameter $\asymp r$,*

(3) *$|y - z| = O(r)$.*

Theorem 4.2.4 is an analog of [McM98, Theorem 3.2], which was originally stated in the context of bounded type quadratic Siegel disks. The proof relies on the approximate rotation mechanism introduced in §4.1, which is made compatible with the hyperbolic metric of Ω thanks to Lemma 4.2.3.

Lemma 4.2.5. *For every point z in $J(f) \backslash \mathbf{H}$, there is a univalent map*

$$f^i : (B, x) \to (V, c)$$

such that $i \geq 0$, $c \in \mathbf{H}$ is a critical point of f, and B is a hyperbolic ball in Ω of radius $r \asymp 1$ centered at a point x in Ω satisfying $d_\Omega(x, z) = O(1)$.

Proof. For every critical point c on $\mathbf{H}$, consider two nested disk neighborhoods $Q_c \subset \tilde{Q}_c$ of c with the following properties.

(i) Q_c is well contained in $\tilde{Q}_c$, and $\tilde{Q}_c$ is well contained in A_j.

(ii) The map f is a covering map from Q_c and $\tilde{Q}_c$ onto their respective images, branched only at c.

(iii) The map $f|_{\tilde{Q}_c}$ can be written as $f(z) = h(z)^{d(c)} + f(c)$, where h is a univalent map of bounded distortion and $d(c)$ is the local degree of f at c.

By taking $\tilde{Q}_c$ to be sufficiently small, we can further assume that the disks $\tilde{Q}_c$ are pairwise disjoint. For brevity, let us also denote the strict preimage of $\mathbf{H}$ by $\mathbf{H}^{-1} := f^{-1}(\mathbf{H})\backslash\mathbf{H}$.

Pick a point z in $J(f)\backslash\mathbf{H}$. We will split into four cases.

Case 1: $z \in \mathbf{H}^{-1} \cap \mathbf{Q}_c$ for some critical point c.

Let z' be the unique point in the intersection $Q_c \cap \mathbf{H}$ such that $f(z) = f(z')$. By Lemmas 4.1.4 and 4.1.5, there is an approximate rotation

$$f^i : (U', x') \to (V, c)$$

such that $i \geq 1$ and (U', x') is a pointed disk in $\tilde{Q}_c$ that avoids $\mathbf{H}^{-1}$, well contains the interval $[x', z']$, and

$$r_{\text{in}}(U', x') \asymp |x' - c|. \tag{4.2.2}$$

Let (U, x) be the pointed disk containing z such that $f(U) = f(U')$ and $f(x) = f(x')$. See Figure 4.2. Since both U and U' are contained in $\tilde{Q}_c$, there is a univalent map $g : (U, x, z) \to (U', x', z')$ of bounded distortion such that $f \circ g = f$ on U. Therefore, U avoids $\mathbf{H}$, well contains the interval $[x, z]$, and by (4.2.2),

$$r_{\text{in}}(U, x) \asymp |x - c| \geq \text{dist}(x, \mathbf{X}).$$

By Lemma 4.2.3, this implies that U contains a hyperbolic ball $B \subset \Omega$ of definite radius centered at x. Therefore, $f^i : (B, x) \to (V, c)$ is the desired univalent map.

Case 2: $z \in \mathbf{H}^{-1} \backslash \cup_c Q_c$.

Since every component of Ω contains a connected component of $\mathbf{H}^{-1}$, there exist a critical point c and a point z' in $\mathbf{H}^{-1} \cap \mathbf{Q}_c$ such that both z and z' are in the same component of Ω and $\text{dist}_\Omega(z, z') = O(1)$. This reduces us to Case 1.

Case 3: $z \in J(f)\backslash f^{-1}(\mathbf{H})$.

By Corollary 4.1.6, there exists an approximate rotation

$$f^i : (U, w) \to (U', w')$$

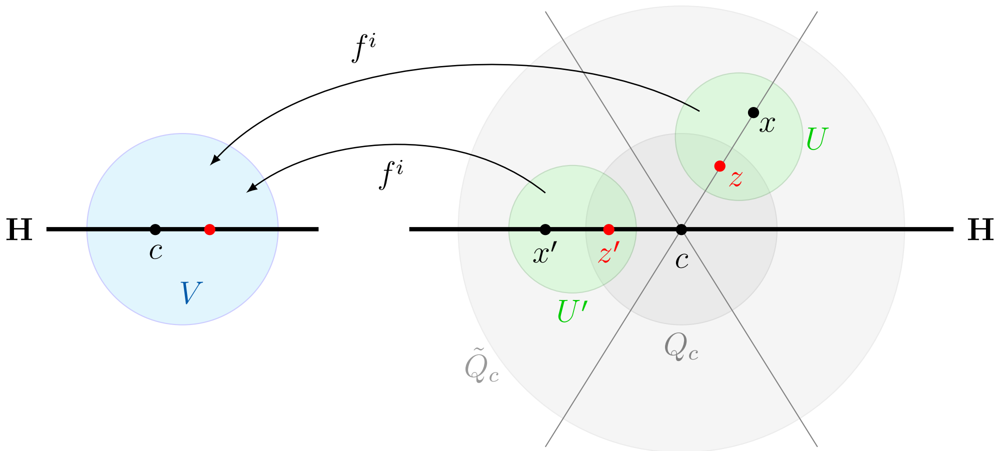

Figure 4.2: Case 1 in the proof of Lemma 4.2.5.

such that w' is in $\mathbf{H}^{-1}$ and that w satisfies $d_\Omega(w,z) = O(1)$ and $\mathrm{dist}_\Omega(w,\partial U) \succ 1$. By Cases 1 and 2, there also exists a univalent map

$$f^j : (B',x') \to (V,c)$$

such that $B' \subset \Omega$ is a hyperbolic ball of radius $\asymp 1$ centered at x', c is a critical point of f in $\mathbf{X}$, and $d_\Omega(w',x') = O(1)$. We can assume that B' is inside of U' by shrinking B' by a little bit. As such, the lift $(f^i|_U)^{-1}(B')$ contains a hyperbolic ball B of radius $\asymp 1$ centered at $x = (f^i|_U)^{-1}(x')$ with distance $d_\Omega(x,z) \coloneqq O(1)$. Therefore, $f^{i+j} : (B,x) \to (V,c)$ is the desired univalent map. □

To prove Theorem 4.2.4, we will apply Lemmas 4.1.4 and 4.2.5 in a similar manner as the Siegel case in [McM98, §3]. (Compare with [McM96, Theorem 8.10].)

Proof of Theorem 4.2.4. For any tangent vector v at a point z, we denote by $|v|$ the Euclidean length of v and $\|v\|$ the hyperbolic length of v with respect to the hyperbolic metric of Ω if $z \in \Omega$. If z is outside of Ω, we set $\|v\| = \infty$. By Lemma 4.2.3,

$$\|v\| \asymp \frac{|v|}{\mathrm{dist}(z,\mathbf{H})} \qquad \text{for all } z \in \Omega_{\mathrm{thick}} \text{ and } v \in T_z\hat{\mathbb{C}}. \tag{4.2.3}$$

Let us fix a point $z_0 \in J_{\mathrm{thick}}$ and a scale $r > 0$. Let v_0 be a tangent vector at z_0 of length $|v_0| = r$. For every $i \in \mathbb{N}$, let $v_i \coloneqq df^i_{z_0}(v_0)$ be the pushforward of v_0 by f^i at $z_i \coloneqq f^i(z_0)$.

Let us fix a small constant $\varepsilon \in (0,1)$ independent of z, which will be determined later. By Lemma 4.2.2, the proof can be split into the following three distinct cases.

Case 1: $1 \le \|v_0\| \le \infty$.

Let w be a point in $\mathbf{X}$ closest to z_0. By (4.2.3), $|z_0 - w| = \mathrm{dist}(z_0,\mathbf{H}) = O(r)$. By Lemma 4.1.4, there is an approximate rotation $f^i : (U,y) \to (V,c)$ such that c is a critical point of f

on $\mathbf{H}$, $|y - w| = O(r)$, and (U, y) has bounded shape with diameter $\asymp r$. This is the univalent map we are looking for.

Case 2: There is some $j \geq 1$ such that $\|v_j\| \geq 1$ but $\|v_{j-1}\| \leq \varepsilon$.

By (4.2.3), the distance between z_{j-1} and $\mathbf{H}$ satisfies $\mathrm{dist}(z_{j-1}, \mathbf{H}) \succ \frac{|v_{j-1}|}{\varepsilon}$. Then, Ω contains a round disk D_{j-1} centered at z_{j-1} of radius $\asymp \frac{|v_{j-1}|}{\varepsilon}$ on which f is univalent. By Koebe quarter theorem, the image $f(D_{j-1})$ contains another round disk $D_j \subset \Omega$ centered at z_j of radius $\asymp \frac{|v_j|}{\varepsilon}$. Denote by D_0 the connected component of $f^{-j}(D_j)$ containing z_0. We have a univalent map $f^j : (D_0, z_0) \to (D_j, z_j)$.

From Case 1, there is a univalent map $f^i : (U', y') \to (V, c)$ of bounded distortion such that c is a critical point of f, $|y' - z_j| = O(|v_j|)$, and (U', y') has bounded shape with diameter $\asymp |v_j|$. Select ε to be just small enough such that U' is well contained in D_j. Let (U, y) be the lift of (U', y') under the map $f^j|_{D_0}$. Since the inverse branch $(f^j|_{D_0})^{-1}$ has bounded distortion on U', then $|y - z_0| = O(r)$ and (U, y) has bounded shape with diameter $\asymp r$. Therefore, $f^{i+j} : (U, y) \to (V, c)$ is the desired univalent map.

Case 3: There is some $j \in \mathbb{N}$ such that $\varepsilon < \|v_j\| < 1$.

By Lemma 4.2.5, there is a univalent map $f^i : (B, x) \to (V, c)$ where c is a critical point of f and $B \subset \Omega$ is a hyperbolic ball of radius $\asymp 1$ centered at a point x which satisfies $d_\Omega(x, z_j) = O(1)$. If z_j is in $\overline{B}$, then set $K' = \overline{B}$; otherwise, set $K' = \overline{B} \cup \gamma$ where $\gamma \subset \Omega$ be the shortest hyperbolic geodesic segment in Ω connecting z_j and a point on ∂B. Let K be the unique lift of K' under f^j containing z_0, and let (U, y) be the lift of (B, x) under $f^j|_K$. The map $f^j : (U, y) \to (B, x)$ is univalent.

By the hyperbolic Koebe distortion theorem [McM94, Theorem 2.29], since K' has bounded hyperbolic diameter in Ω, the map f^j is approximately an expansion by $\|(f^j)'(z_0)\|$ on K. Moreover, since $\|v_j\| \asymp 1$,

$$\|(f^j)'(z_0)\| = \frac{\|v_j\|}{\|v_0\|} \asymp \frac{1}{\|v_0\|}.$$

As the hyperbolic inner radius of B about x satisfies $r_{\mathrm{in},\Omega}(B, x) \asymp 1$, the hyperbolic inner radius of U about y satisfies

$$r_{\mathrm{in},\Omega}(U, y) \asymp \|v_0\|.$$

Also, since $d_\Omega(x, z_j) = O(1)$, we have

$$d_\Omega(y, z_0) = O(\|v_0\|).$$

From (4.2.3) and the two estimates above, we have $r_{\mathrm{in}}(U, y) \asymp r$ and $|y - z_0| = O(r)$. Thus, the map $f^{i+j} : (U, y) \to (V, c)$ is our desired univalent map. $\square$

4.2.3 No invariant line fields

Recall the collar neighborhood A_κ defined in §4.2.1. For every $\kappa > \kappa_0$, define the *local non-escaping set* of f to be

Definition 4.2.6. For every $\kappa > \kappa_0$, define the *local non-escaping set* of f of level κ to be

$$K_\kappa^{\mathrm{loc}} := \{z \in \hat{\mathbb{C}} \, : \, f^n(z) \in \overline{A_\kappa} \text{ for all } n \geq 0\}.$$

Clearly, K_κ^{loc} is contained in $J(f)$, and it is equal to $J(f)$ when κ is sufficiently close to κ_0. Let us prove a slightly stronger version of Theorem 4.2.1. (The local non-escaping set will make an appearance again in §4.5.)

Theorem 4.2.7. *For every $\kappa > \kappa_0$, the local non-escaping set K_κ^{loc} of f does not carry any invariant line field of f.*

In the proof, we will apply the following proposition by Shen.

Proposition 4.2.8 ([She03, Proposition 3.2])**.** *Consider a rational function g of degree ≥ 2 and a forward invariant subset J of $J(g)$. Suppose that for almost every point x in J, there is a constant $C > 1$, a positive integer $N \geq 2$, and a sequence $h_n : U_n \to V_n$ of holomorphic maps such that:*

- (S_1) *$g^i \circ h_n = g^j$ for some $i, j \in \mathbb{N}$;*
- (S_2) *U_n and V_n are topological disks such that as $n \to \infty$,*

$$\mathrm{diam}(U_n) \to 0 \quad \textit{and} \quad \mathrm{diam}(V_n) \to 0;$$

- (S_3) *h_n is a branched covering map of degree between 2 and N;*
- (S_4) *there are some critical point $u_n \in U_n$ of h_n and critical value $w_n = h_n(u_n)$ such that both (U_n, u_n) and (V_n, w_n) have C-bounded shape;*
- (S_5) *U_n and V_n are relatively close to x, i.e.*

$$\mathrm{dist}(x, U_n) \leq C\mathrm{diam}(U_n) \quad \textit{and} \quad \mathrm{dist}(x, V_n) \leq C\mathrm{diam}(V_n).$$

Then, g admits no invariant line field on J.

The idea behind this criterion comes from the fact that at almost every point x in J, any invariant line field on a small neighborhood around x is almost parallel, but the presence of critical points at small scales would carry non-linearity throughout J and contradict such parallel structure. Shen's criterion was inspired by McMullen's treatment of Feigenbaum maps in [McM94; McM96].

Proof of Theorem 4.2.7. It is sufficient to show that the hypothesis of Proposition 4.2.8 holds for every point x in K_κ^{loc}. (In fact, we will show that the constants C and N can be made uniform in x.) There are two cases.

Case 1: x is a critical point of f.

In this case, we will take the sequence $\{h_n\}$ to be the first return maps near x.

Let $\{p_k/q_k\}_{k\in\mathbb{N}}$ denote the best rational approximations of the rotation number θ, and let $l_k := |p_k - q_k\theta|$. Recall the quasiconformal map ϕ from §4.2.1. Pick a sufficiently large $n \in \mathbb{N}$ and let $w_n := f^{q_n}(x)$. By Lemma 2.1.4, there exists a pair of intervals $I' \subset I''$ in $\mathbf{H}$ such that

(i) I' contains the level n combinatorial interval centered at w_n,

(ii) the endpoints of I' split I'' into three components each of combinatorial length $\asymp l_n$, and

(iii) $I''\backslash I'$ contains no critical values of $f^{q_{n+2}}$.

Let V' and V'' be the unique pair of disks such that their closures intersect X_j along intervals I' and I'' respectively, and that both $\phi(\partial V')$ and $\phi(\partial V'')$ are round circles orthogonal to $\mathbb{T}$. Let us pick n to be large enough such that V'' is contained in the collar A_κ.

Let V_n be the Jordan disk that contains V' and is enclosed by the core curve of the annulus $V''\backslash\overline{V'}$. Denote by U', U'', and U_n the connected component containing x of the preimage under f^{q_n} of V', V'' and V_n respectively. By (iii), $f^{q_n} : U''\backslash\overline{U'} \to V''\backslash\overline{V'}$ is an unbranched covering map between two annuli.

We claim that $f^{q_n} : U_n \to V_n$ satisfies (S_1)-(S_5) in Proposition 4.2.8. Indeed, (S_1) is immediate from the construction. (S_3) follows from the fact that, by (ii) and Proposition 2.1.3, the number of critical points of f^{q_n} on U'' is at most some constant N independent of x and n.

Take $u_n = x$. By construction, V' is well contained in V'' and U' is well contained in U''. Since ∂U_n and ∂V_n are core curves of annuli of definite moduli, both U_n and V_n are quasidisks of uniformly bounded dilatation; in particular, they have bounded shape about x and w_n respectively. Thus, (S_4) holds.

By construction, (S_4) ensures that $\mathrm{diam}(U_n) \asymp \mathrm{diam}(V_n)$. As such, ($\mathrm{S}_2$) follows from the fact that $\mathrm{diam}(\phi(V_n)) \asymp l_n \to 0$ as $n \to \infty$, and (S_5) follows from $\mathrm{dist}(x, U_n) = 0$ and $\mathrm{dist}(x, V_n) \leq \mathrm{diam}(U_n)$. This concludes the proof.

Case 2: $x \in K_\kappa^{\mathrm{loc}}$ is not a critical point of f.

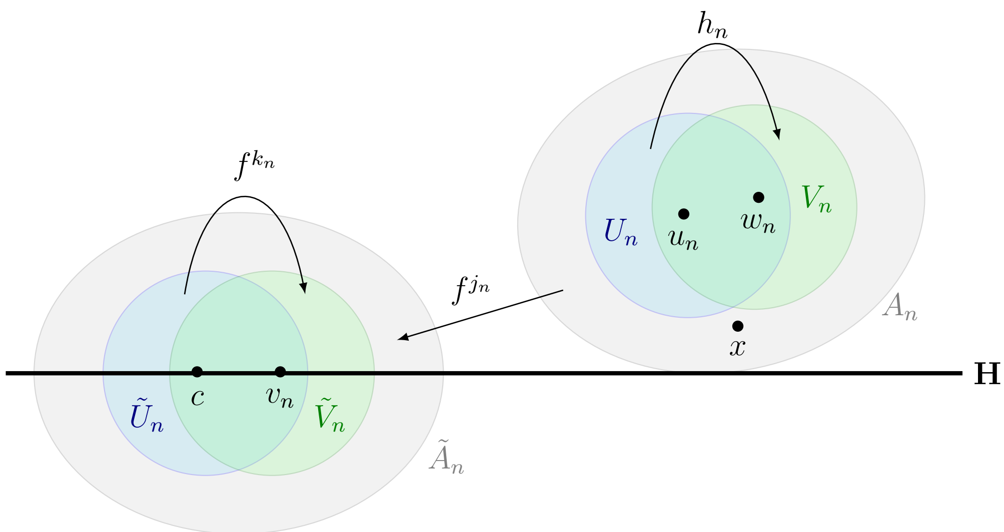

Figure 4.3: The construction of the branched covering map h_n in Case 2.

Fix a sequence $\{r_n\}_{n\in\mathbb{N}}$ of small positive real numbers decreasing to 0. Pick $n \in \mathbb{N}$. By Theorem 4.2.4, there is a univalent map

$$f^{j_n} : (A_n, u_n) \to \left(\tilde{A}_n, c\right)$$

between pointed disks with bounded distortion such that c is a critical point of f, (A_n, u_n) has bounded shape with diameter $\asymp r_n$ and

$$|u_n - x| = O(r_n). \tag{4.2.4}$$

Let $s_n := \operatorname{diam}\left(\tilde{A}_n, c\right)$; this depends on r_n. From Case 1, by appropriately selecting r_n, there is some $k_n \in \mathbb{N}$ and some branched covering map

$$f^{k_n} : \left(\tilde{U}_n, c\right) \to \left(\tilde{V}_n, v_n\right)$$

of degree at most some constant N independent of x and n such that $\left(\tilde{U}_n, c\right)$ and $\left(\tilde{V}_n, v_n\right)$ are pointed disks compactly contained in $\tilde{A}_n$ with bounded shape and diameter $\asymp s_n$, and $\operatorname{dist}\left(c, \tilde{V}_n\right) = O(s_n)$.

Let (U_n, u_n) and (V_n, w_n) be the pointed disks obtained by pulling back $\left(\tilde{U}_n, c\right)$ and $\left(\tilde{V}_n, v_n\right)$ under $f^{j_n}|_{A_n}$ respectively. Then, there is a branched covering map

$$h_n : (U_n, u_n) \to (V_n, w_n)$$

of degree at most N such that $f^{j_n} \circ h_n = f^{k_n + j_n}$ on U_n. See Figure 4.3.

We claim that h_n satisfies (S_1)-(S_5) in Proposition 4.2.8. Indeed, (S_1) and (S_3) are immediate from the construction. Since f^{j_n} has bounded distortion, (U_n, u_n) and (V_n, w_n) have bounded shape with diameter $\asymp r_n$, and $\operatorname{dist}(u_n, V_n) = O(r_n)$. Therefore, ($\mathrm{S}_2$) and ($\mathrm{S}_4$) are satisfied. Together with (4.2.4), we also have (S_5). $\square$

4.3 Combinatorial rigidity of Herman quasicircles

In this section, we will apply Theorem 4.2.1 to complete the proof of Theorem C. We will also provide a few applications on trivial Herman curves and antipode-preserving cubic rational maps.

4.3.1 Combinatorial rigidity

For every rational map f in $\mathcal{X}$, we denote by $\operatorname{comb}(f) \in \mathcal{C} = \mathcal{C}_{d_0,d_\infty}$ the combinatorics of f along its Herman quasicircle, as defined in Definition 3.1.9.

Theorem 4.3.1. *Any two combinatorially equivalent rational maps in $\mathcal{X}$ are conformally conjugate.*

In the proof, we will apply the standard pullback argument to promote combinatorial equivalence to quasiconformal conjugacy. The absence of invariant line fields will allow us to further promote the quasiconformal conjugacy to a conformal one.

Proof. Suppose f_1 and f_2 are two combinatorially equivalent rational maps in $\mathcal{X}$. For each $i \in \{1,2\}$, let $\mathbf{H}_i$ be the Herman quasicircle of f_i and $\phi_i : \mathbf{H}_i \to \mathbb{T}$ be a quasisymmetric conjugacy between f_i and R_θ. By combinatorial equivalence, the conjugacies can be picked such that $\phi_2^{-1} \circ \phi_1$ preserves the critical points of f_1 and f_2 along their Herman curves.

For each $i \in \{1,2\}$ and $\bullet \in \{0,\infty\}$, denote by $b_i^\bullet : (B_i^\bullet, \bullet) \to (\mathbb{D}, 0)$ a Böttcher coordinate for f_i on the immediate basin of attraction $B_i^\bullet$ of $\bullet$, that is, a conformal isomorphism such that $b_i^\bullet \circ f_i(z) = b_i^\bullet(z)^{d_\bullet}$ for all $z \in B_i^\bullet$. Let us also consider the neighborhood $E_i^\bullet \coloneqq \left\{ z \in B_i^\bullet \ : \ |b_i^\bullet(z)| < \frac{1}{2} \right\}$ of $\bullet$ cut out by an equipotential.

Let $h_0 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a quasiconformal map such that

$$h_0(z) = \begin{cases} (b_2^\bullet)^{-1} \circ b_1^\bullet(z), & \text{if } z \in E_1^\bullet, \bullet \in \{0,\infty\}, \\ \phi_2^{-1} \circ \phi_1(z), & \text{if } z \in \mathbf{H}_1, \\ \text{quasiconformal interpolation}, & \text{if otherwise.} \end{cases}$$

Then, h_0 is conformal on $E_1 \coloneqq E_1^0 \cup E_1^\infty$ and provides a conjugacy between f_1 and f_2 on $\mathbf{H}_1 \cup E_1$.

Our choice of ϕ_1 and ϕ_2 ensures that h_0 preserves the covering structure of f_1 and f_2. In particular, we can lift h_0 to a quasiconformal map $h_1 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $f_2 \circ h_1 = h_0 \circ f_1$. This new map h_1 coincides with h_0 on $\mathbf{H}_1 \cup E_1$, restricts to a conformal conjugacy between f_1 and f_2 on $f_1^{-1}(E_1)$, and is homotopic to h_0 rel $P(f_1)$. Moreover, h_1 has the same quasiconformal dilatation as h_0 because both f_1 and f_2 are holomorphic. Repeat this lifting process to obtain an infinite sequence of uniformly quasiconformal homeomorphisms $\{h_n\}_{n\in\mathbb{N}}$ of $\hat{\mathbb{C}}$ such that for all $n \in \mathbb{N}$,

(i) $f_2 \circ h_{n+1} = h_n \circ f_1$;

(ii) $h_{n+1} = h_n$ on $f_1^{-n}(\mathbf{H}_1 \cup E_1)$;

(iii) h_n restricts to a conformal conjugacy between f_1 and f_2 on $f_1^{-n}(E_1)$.

By the compactness of the space of normalised quasiconformal maps, h_n converges in subsequence to a quasiconformal map $h_\infty : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. The limit h_∞ is a conformal conjugacy between f_1 and f_2 on the Fatou sets because $\bigcup_{n\geq 0} f^{-n}(E_i)$ coincides with the Fatou set of f_i for $i \in \{1,2\}$. By continuity, since the Julia sets of f_1 and f_2 are nowhere dense, h_∞ is a global quasiconformal conjugacy between f_1 and f_2.

The absence of invariant line fields on the Julia set implies that $\overline{\partial} h_\infty = 0$ almost everywhere on $J(f_1)$. By Weyl's lemma, h_∞ is indeed a conformal conjugacy between f_1 and f_2 in $\hat{\mathbb{C}}$. □

Let us complete the proof of Theorem C.

Proof of Theorem C. By Corollaries 3.8.4 and 3.8.6 and Theorem 4.3.1, we now know that the map

$$\text{comb} : \mathcal{X}/_\sim \to \mathcal{C}$$

is a continuous bijection and $\partial\mathcal{H}$ is equal to $\mathcal{X}$. By Theorem 3.8.1, the space $\mathcal{X}/\sim$ is Hausdorff and compact, so comb$(\cdot)$ is indeed a homeomorphism. □

4.3.2 Trivial Herman curves

Consider an integer $\mathrm{d} \geq 2$. Let us denote by $\mathcal{B}_{\mathrm{d},\theta}$ the space of rational maps in $\mathcal{X}_{\mathrm{d},\mathrm{d},\theta}$ which are Blaschke products, i.e. those that commute with the reflection $\tau(z) = 1/\bar{z}$ along the unit circle $\mathbb{T}$, or equivalently, those whose Herman quasicircles are $\mathbb{T}$.

For any $T^0 \in \mathrm{SP}^{d_0-1}(\mathbb{T})$ and $T^\infty \in \mathrm{SP}^{d_\infty-1}(\mathbb{T})$, we denote the corresponding element in $\mathcal{C}$ by $\sigma = [(T^0, T^\infty)]$ and say that σ is *symmetric* if $d_0 = d_\infty$ and $T^0 = T^\infty$. If a Herman curve $\mathbf{H}$ has symmetric combinatorics, then every critical point on $\mathbf{H}$ is both an outer and an inner critical point, and its outer and inner criticalities coincide.

Proposition 4.3.2 (Blaschke $\leftrightarrow$ combinatorial symmetry)**.** *Every $f \in \mathcal{B}_{\mathrm{d},\theta}$ has symmetric combinatorics. Conversely, given a symmetric combinatorial data $\sigma \in \mathcal{C}_{\mathrm{d},\mathrm{d}}$, the map $f \in \mathcal{X}_{\mathrm{d},\mathrm{d},\theta}$ realizing σ is conformally conjugate to a Blaschke product, unique up to conjugacy by a rigid rotation.*

Proof. The first statement follows from the observation that for any general rational map $f \in \mathcal{X}_{d_0,d_\infty,\theta}$, if f has combinatorics $[(T^0, T^\infty)]$, then $\tau \circ f \circ \tau$ lies in $\mathcal{X}_{d_\infty,d_0,\theta}$ with combinatorics $[(T^\infty, T^0)]$.

Suppose $f \in \mathcal{X}_{\mathrm{d},\mathrm{d},\theta}$ has a Herman quasicircle $\mathbf{H}$ with symmetric combinatorics $[(T,T)]$. Mark one of the critical points of f and assume it is $z = 1$ after conjugation with a linear map. The rational map $g(z) := \tau \circ f \circ \tau$ has a Herman quasicircle $\tau(\mathbf{H})$ with the same rotation number and the same combinatorics $[(T,T)]$ due to combinatorial symmetry. By Theorem 4.3.1, there is a linear map $L(z) = \lambda z$, $\lambda \in \mathbb{C}^*$ such that $g = L \circ f \circ L^{-1}$. Moreover, L can be chosen to preserve the marked critical points of f and g, which are 1 and $\tau(1) = 1$. Thus, $\lambda = 1$ and $g = f$, which implies that f is a Blaschke product. Uniqueness also follows from rigidity. □

By Theorem C and Proposition 4.3.2, the map comb$(\cdot)$ induces a homeomorphism between the quotient space $\mathcal{B}_{\mathrm{d},\theta}/_\sim$ and the space

$$\left\{[(T,T)] \in \mathcal{C}_{\mathrm{d},\mathrm{d}} \, : \, T \in \mathrm{SP}^{\mathrm{d}-1}(\mathbb{T})\right\}.$$

Observe that the latter is homeomorphic to the quotient $\mathcal{S}_\mathrm{d}$ of $\mathrm{SP}^{\mathrm{d}-1}(\mathbb{T})$ modulo rigid rotations.

Corollary 4.3.3. comb$(\cdot)$ *induces a homeomorphism* $\mathrm{comb}' : \mathcal{B}_{\mathrm{d},\theta}/_\sim \to \mathcal{S}_\mathrm{d}$.

Let $\mathcal{Z}_{\mathrm{d},\theta}$ denote the space of degree d polynomials f that admit a single Siegel disk Z such that Z is centered at 0, has rotation number θ, and contains every free critical point of f on its boundary. We denote by $\mathrm{comb}(f) \in \mathcal{S}_\mathrm{d}$ the combinatorics of $f|_{\partial Z}$, which encodes the combinatorial position of the critical points of f along ∂Z.

The dynamical relation between $\mathcal{B}_{d,\theta}$ and $\mathcal{Z}_{\mathrm{d},\theta}$ can be formulated via the Douady-Ghys surgery. See Theorem 3.1.1.

Corollary 4.3.4. *The Douady-Ghys surgery induces a homeomorphism*

$$\mathrm{DG} : \mathcal{B}_{\mathrm{d},\theta}/_\sim \to \mathcal{Z}_{\mathrm{d},\theta}/_\sim$$

satisfying $\mathrm{comb} \circ \mathrm{DG} = \mathrm{comb}'$.

For $\mathrm{d} = 3$, a variation of this corollary was previously studied in [Zak99].

Proof. Combinatorial rigidity of Siegel polynomials in $\mathcal{Z}_{\mathrm{d},\theta}$ (cf. [Zha08]) ensures that DG is well-defined. The equation $\mathrm{comb} \circ \mathrm{DG} = \mathrm{comb}'$ holds because the surgery preserves the combinatorics. Note that the moduli space $\mathcal{Z}_{\mathrm{d},\theta}/_\sim$ is also compact (for example, one can adapt the proof in §3.8.1) and $\mathrm{comb} : \mathcal{Z}_{\mathrm{d},\theta}/_\sim \to \mathcal{S}_b$ is a homeomorphism. Together with Corollary 4.3.3, we conclude that the map DG is a homeomorphism. □

4.3.3 Antipode-preserving cubic rational maps

We end this section with an application of rigidity to the following family of cubic rational maps

$$f_q(z) = z^2 \frac{q-z}{1+\bar{q}z}, \qquad q \in \mathbb{C}^*.$$

This family was first studied in [BBM18] and is characterized by a simple critical fixed point at 0 and the property that f_q is antipode-preserving, that is, f_q commutes with the antipodal map $z \mapsto -1/\overline{z}$.

Note that f_q and $f_{q'}$ are linearly conjugate if and only if $q' = -q$, so it is natural to consider the q^2-plane as the appropriate parameter space. According to [BBM18], this parameter space has the remarkable property of admitting Herman rings of arbitrary Brjuno rotation number and modulus. Below, we quote a more precise formulation from a sequel [BBM] in progress.

Theorem 4.3.5 (Hair Theorem)**.** *For any Brjuno number $\theta \in (0,1)$, there exists a unique "hair" $\mathscr{H}_\theta$ in the q^2-plane consisting of all maps f_q with a Herman ring of rotation number θ. They satisfy the following properties.*

(1) *For any $m \in (0,\infty)$, there is a unique parameter $q(m)^2$ in $\mathscr{H}_\theta$ such that $f_{q(m)}$ admits a unique invariant Herman ring of modulus m.*

(2) *The map $(0,\infty) \to \mathscr{H}_\theta$, $m \mapsto q(m)^2$ is an analytic and regular parametrization of $\mathscr{H}_\theta$.*

(3) *As $m \to \infty$, $|q(m)| \to \infty$.*

Moreover, the Herman ring locus $\mathscr{H} := \bigcup_\theta \mathscr{H}_\theta$ in the q^2-plane has positive measure. The Hausdorff 1-measure of the intersection $\mathscr{H} \cap \{|q^2| = r\}$ tends to 1 as $r \to \infty$.

Assuming the theorem above, we can apply our rigidity result and deduce that when θ is of bounded type, the corresponding hair $\mathscr{H}_\theta$ lands at a unique point. See Figure 4.4.

Corollary 4.3.6 (Landing of hairs)**.** *When θ is of bounded type, the hair $\mathscr{H}_\theta$ has a unique endpoint $q_\theta^2 := \lim_{m\to 0} q(m)^2$. The map f_{q_θ} lies in $\mathcal{X}_{2,2,\theta}$.*

Proof. Herman rings in $\mathscr{H}_\theta$ automatically lie in the space $\mathcal{H}_{2,2,\theta}$. By Theorem 3.8.1, the set of maps in $\mathscr{H}_\theta$ admitting Herman rings of modulus bounded above by some positive constant is precompact. Therefore, the accumulation set $\partial\mathscr{H}_\theta := \overline{\mathscr{H}_\theta} \backslash \mathscr{H}_\theta$ is non-empty and contained in $\mathcal{X}_{2,2,\theta}$.

Pick any parameter q^2 in $\partial\mathscr{H}_\theta$. Let $\phi : \mathbf{H} \to \mathbb{T}$ be a quasiconformal conjugacy between f_q on its Herman quasicircle and the irrational rotation R_θ. Since f_q commutes with $\tau(z) = -1/\overline{z}$

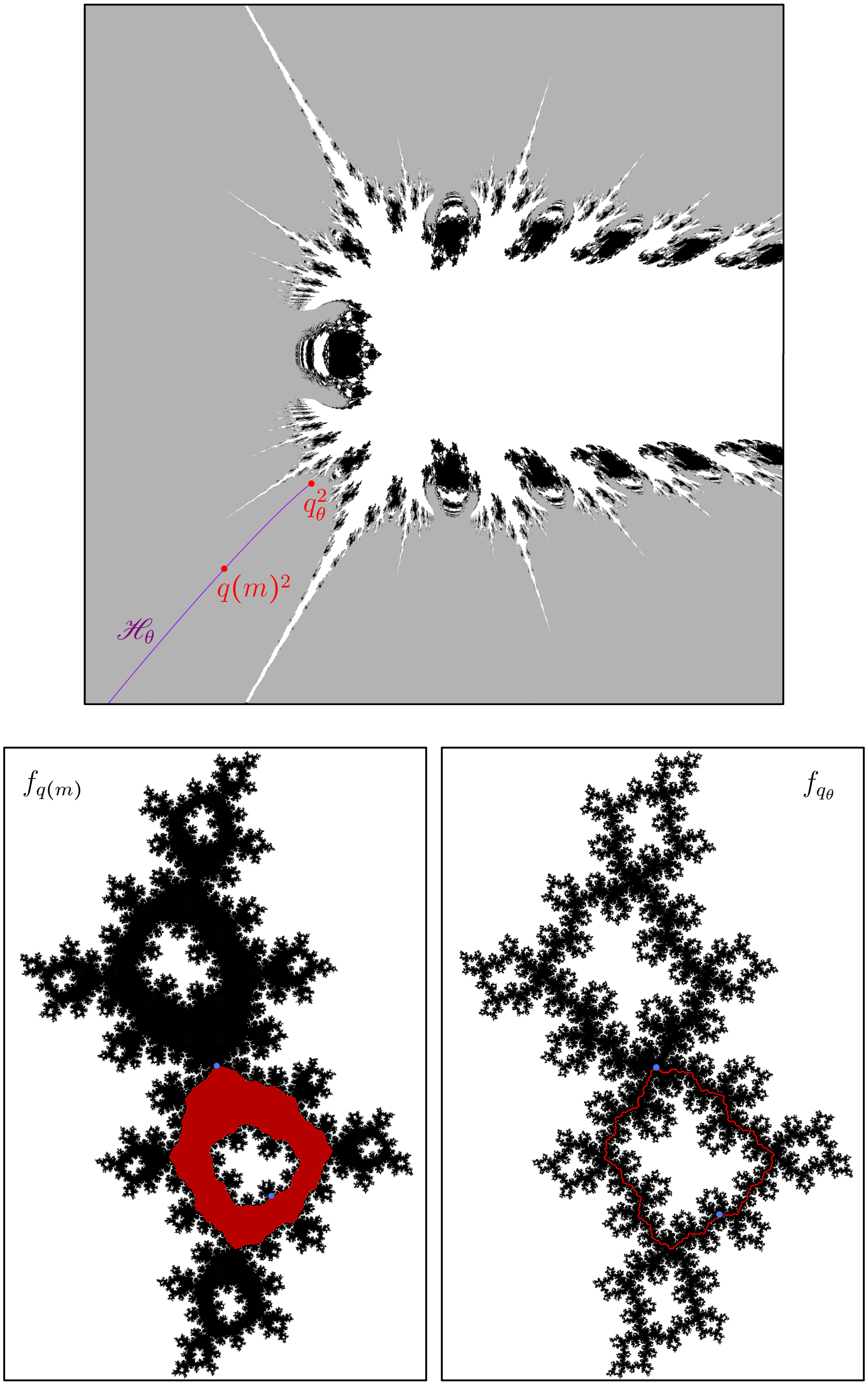

Figure 4.4: Above: The q^2-parameter plane for $\{f_q\}$ containing the golden mean hair $\mathscr{H}_\theta$, colored in purple. Below: The dynamical planes of $f_{q(m)}$ and f_{q_θ} where $q(m)^2 \approx -12.06 - 12.30i$ lies on $\mathscr{H}_\theta$ and $q_\theta^2 \approx -7.05 - 7.41i$ is the endpoint of $\mathscr{H}_\theta$. The Herman ring of $f_{q(m)}$ and the Herman quasicircle of f_{q_θ} are colored in red.

and since ϕ is unique up to post-composition with rigid rotation, then $\phi \circ \tau = -\phi$. In particular, $f_q|_{\mathbf{H}}$ must have combinatorics $[\{1\}, \{-1\}]$. By Theorem 4.3.1, maps in $\partial\mathscr{H}_\theta$ are linearly conjugate to each other, so $\partial\mathscr{H}_\theta$ must be a singleton. □

4.3.4 Realization of critical quasicircle maps

Recall the definition of inner and outer criticality from Definition 2.1.5. In the coming sections, we will go beyond the realm of rational maps and work with critical quasicircle maps.

Definition 4.3.7. We say that a critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$ is (d_0, d_∞)-*critical* if the unique critical point of f on $\mathbf{H}$ has inner criticality d_0 and outer criticality d_∞.

Theorem C implies the existence of a (d_0, d_∞)-critical quasicircle map with bounded type rotation number θ. Indeed, by selecting unicritical combinatorics, the theorem states that there exists a unique rational map $f = f_{d_0, d_\infty, \theta}$ in $\mathcal{X}$ admitting a unique free critical point at $z = 1$ of maximal local degree on its Herman quasicircle. By elementary computation, we know that f is of the form F_c given below for some unique $c \in \mathbb{C}^*$.

Proposition 4.3.8. *Suppose that $F_c \in \mathrm{Rat}_d$ has critical points at 0, ∞, and 1 with local degrees d_0, d_∞, and $d = d_0 + d_\infty - 1$ respectively, and that $F_c(0) = 0$, $F_c(\infty) = \infty$, and $F_c(1) = c \in \mathbb{C}^*$. Then,*

$$F_c(z) := -c \, \frac{\sum\limits_{j=d_0}^{d} \binom{d}{j} \cdot (-z)^j}{\sum\limits_{j=0}^{d_0-1} \binom{d}{j} \cdot (-z)^j}.$$

Proof. The rational map $F_1(z) := c^{-1} F_c(z)$ has superattracting fixed points at 0, ∞, and 1 with local degrees d_0, d_∞, and d respectively. From the behaviour at 0 and ∞, the map F_1 is of the form $z^{d_0} \frac{p(z)}{q(z)}$ where p is a degree $d_\infty - 1$ polynomial and q is a degree $d_0 - 1$ polynomial. Let us present F_1 as

$$F_1(z) = -\frac{(-z)^d + \sum_{j=d_0}^{d-1} a_j (-z)^j}{\sum_{j=0}^{d_0-1} a_j (-z)^j}$$

for some coefficients $a_0, a_1, \ldots, a_{d-1}$. The map $g(z) := 1 - F_1(-z)$ is of the form

$$g(z) = \frac{z^d + \sum_{j=0}^{d-1} a_j z^j}{\sum_{j=0}^{d_0-1} a_j z^j}.$$

From the behaviour of F_1 at 1, $z = -1$ must be a zero of g of order d. Thus, the numerator of g must be divisible by $(z+1)^d = \sum_{j=0}^{d} \binom{d}{j} z^j$. This implies that $a_j = \binom{d}{j}$ for every j, and we are done. □

4.4 Renormalization of critical quasicircle maps

In this section, we begin our study of renormalizations of critical quasicircle maps. Renormalizations are described as commuting pairs, and they admit complex beau bounds, which we will apply to prove quasiconformal rigidity.

Unless otherwise stated, any quasicircle $\mathbf{H} \subset \hat{\mathbb{C}}$ considered will be assumed to separate 0 and ∞. Denote by $Y^0_{\mathbf{H}}$ and $Y^\infty_{\mathbf{H}}$ the connected components of $\hat{\mathbb{C}} \backslash \mathbf{H}$ containing 0 and ∞ respectively.

4.4.1 Commuting pairs

Before we delve into a discussion on renormalization, let us define the abstract notion of commuting pairs relevant in our context.

Let us denote by $\mathbb{H}$ and $-\mathbb{H}$ the standard upper and lower half planes in $\mathbb{C}$ respectively.

Definition 4.4.1. Let $\mathbf{I} \Subset \mathbb{C}$ be a closed quasiarc containing 0 on its interior. A *commuting pair* ζ based on $\mathbf{I}$ is a pair of orientation preserving analytic homeomorphisms

$$\zeta = (f_- : I_- \to f_-(I_-),\ f_+ : I_+ \to f_+(I_+))$$

with the following properties.

(P$_1$) I_- and I_+ are closed subintervals of $\mathbf{I}$ of the form $[f_+(0), 0]$ and $[0, f_-(0)]$ respectively such that $\mathbf{I} = I_- \cup I_+ = f_-(I_-) \cup f_+(I_+)$ and $I_- \cap I_+ = \{0\}$.

(P$_2$) For all $x \in I_\pm \backslash \{0\}$, $f'_\pm(x) \neq 0$.

(P$_3$) Both f_- and f_+ admit holomorphic extensions to a neighborhood B of 0 on which f_- commutes with f_+ and $f_- \circ f_+(\mathbf{I} \cap B) \subset I_-$.

Additionally, a commuting pair ζ is a *critical commuting pair* if

(P$_4$) 0 is a critical point of both f_- and f_+.

The quasiarc $\mathbf{I}$ is called the *base* of ζ. We say that ζ is *normalized* if $f_+(0) = -1$. A critical commuting pair ζ is called a (d_0, d_∞)*-critical commuting pair* if for any quasiconformal map ϕ mapping I_- and I_+ to real intervals $[-1, 0]$ and $[0, 1]$ respectively and for any sufficiently small round disk D centered at $\phi(f_+(f_-(0)))$, the number of connected components of $\phi(f_+ \circ f_-)^{-1}\phi^{-1}(D \cap -\mathbb{H})$ in $-\mathbb{H}$ is d_∞, whereas the number of connected components of $\phi(f_+ \circ f_-)^{-1}\phi^{-1}(D \cap \mathbb{H})$ in $\mathbb{H}$ is d_0. Refer to Figure 4.5 for an illustration when $(d_0, d_\infty) = (3, 2)$.

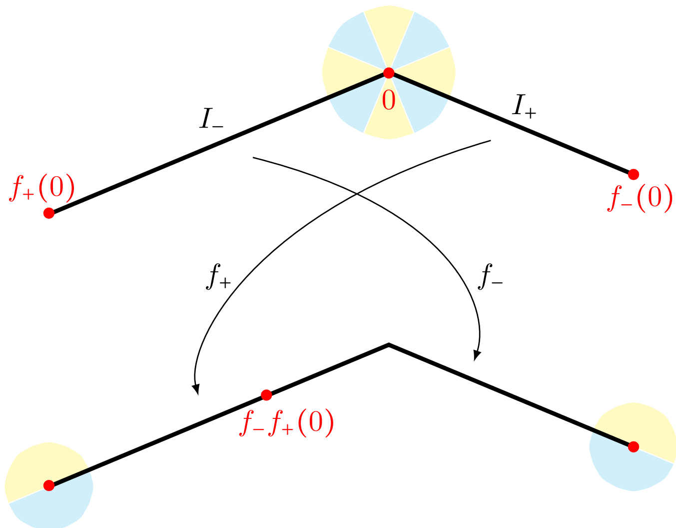

Figure 4.5: A cartoon of a $(3,2)$-critical commuting pair.

Definition 4.4.2. We say that a (d_0, d_∞)-critical commuting pair $\zeta = (f_-, f_+)$ is *renormalizable* if there exists a positive integer $\chi = \chi(\zeta)$ that corresponds to the first time $f_-^{\chi+1} \circ f_+(0)$ lies in the interior of I_+. If renormalizable, we call the (d_∞, d_0)-critical commuting pair

$$p\mathcal{R}\zeta := \left(f_-^{\chi} \circ f_+|_{[0, f_-(0)]},\ f_-|_{[f_-^{\chi} f_+(0), 0]} \right)$$

the *pre-renormalization* of ζ, and we call the normalized (d_0, d_∞)-critical commuting pair obtained by conjugating $p\mathcal{R}\zeta$ with the antilinear map $z \mapsto -f_-(0)\bar{z}$ the *renormalization* $\mathcal{R}\zeta$ of ζ.

If $\mathcal{R}\zeta$ is again renormalizable, we call ζ twice renormalizable, and so on. If ζ is infinitely renormalizable, we define the *rotation number* of ζ to be the irrational number

$$\mathrm{rot}(\zeta) := [0; \chi(\zeta), \chi(\mathcal{R}\zeta), \chi(\mathcal{R}^2\zeta), \ldots].$$

In what follows, we only consider commuting pairs that are infinitely renormalizable. Our renormalization operator transforms the rotation number according to the Gauss map $G(x) := \left\{ \frac{1}{x} \right\}$.

Lemma 4.4.3. *For any critical commuting pair ζ and $n \geq 1$, $\mathrm{rot}(\mathcal{R}^n\zeta) = G^n(\mathrm{rot}(\zeta))$.*

For any $a \in \mathbb{C}$, let us denote by $T_a(z) := z + a$ the translation by a. For any irrational $\theta \in (0,1)$, the (non-critical) commuting pair

$$\mathbf{T}_\theta = \left(T_\theta|_{[-1,0]},\ T_{-1}|_{[0,\theta]} \right) \tag{4.4.1}$$

on intervals along the real line is infinitely renormalizable with rotation number θ. The pair of translations (4.4.1) gives a combinatorial model for normalized critical commuting pairs of the same rotation number.

Gluing the two ends of the real interval $[\theta - 1, \theta]$ by T_1 projects the modified pair of translations $(T_\theta|_{[\theta-1,0]}, T_{\theta-1}|_{[0,\theta]})$ into the standard irrational rotation R_θ on the unit circle $\mathbb{T}$. In general, one can convert a commuting pair to a quasicircle map as follows.

Proposition 4.4.4. *Let $\zeta = (f_-|_{I_-}, f_+|_{I_+})$ be a commuting pair. Let G_ζ be the gluing map which corresponds to identifying z with $f_+(z)$ for every point z in a neighborhood of $f_-(0)$. Then, G_ζ projects the pair $(f_-|_{[f_+f_-(0),0]}, f_+f_-|_{[0,f_-(0)]})$ into a quasicircle map $f_\zeta : \mathbf{H}_\zeta \to \mathbf{H}_\zeta$ having the same rotation number as ζ. If ζ is (d_0, d_∞)-critical, then $f_\zeta : \mathbf{H}_\zeta \to \mathbf{H}_\zeta$ is a (d_0, d_∞)-critical quasicircle map.*

Conversely, we can obtain a commuting pair out of a (d_0, d_∞)-critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$ as follows. By conjugation with a linear map, let us assume that the critical point of f is normalized at 1. Replace $\mathbf{H}$ with its lift under the universal covering $z \mapsto e^{2\pi i z}$. In these logarithmic coordinates, $\mathbf{H}$ is a $\mathbb{Z}$-periodic quasicircle passing through 0 and ∞. Replace f with its corresponding lift F admitting a critical point at $c_0 := 0$ and a critical value $c_1 := F(0)$ located in the interval $[0,1] \subset \mathbf{H}$. Then,

$$\zeta_f := \left(F|_{[-1,0]},\ T_{-1}|_{[0,c_1]}\right)$$

is a commuting pair with the same rotation number as f. Applying the gluing operation from Proposition 4.4.4 to ζ_f results in a (d_0, d_∞)-critical quasicircle map conformally conjugate to $f : \mathbf{H} \to \mathbf{H}$.

We define renormalizations $\mathcal{R}^n f$ of f to be the renormalizations of the commuting pair ζ_f. This can be more explicitly described as follows. Denote by $\{p_n/q_n\}_{n\in\mathbb{N}}$ the best rational approximations of the rotation number θ of f. For every n, let $c_{q_n} := T_{-p_n} F^{q_n}(0)$. The n^{th} *pre-renormalization* of f is the critical commuting pair

$$p\mathcal{R}^n f := p\mathcal{R}^n \zeta_f = \left(T_{-p_n} F^{q_n}\Big|_{[c_{q_{n-1}},0]},\ T_{-p_{n-1}} F^{q_{n-1}}\Big|_{[0,c_{q_n}]} \right),$$

and the n^{th} *renormalization* $\mathcal{R}^n f$ of f is the normalized (d_0, d_∞)-critical commuting pair obtained by conjugating $p\mathcal{R}^n f$ with either the antilinear map $z \mapsto -c_{q_{n-1}} \bar{z}$ if n is odd, or the linear map $z \mapsto -c_{q_{n-1}} z$ if n is even.

4.4.2 Quasicritical circle maps

For any annular neighborhood A of a quasicircle $\mathbf{H}$ and for $\bullet \in \{0, \infty\}$, we denote by $A^\bullet$ the annulus $A \cap Y^\bullet_{\mathbf{H}}$. Given any $\mu > 0$, we say that an open neighborhood A of $\mathbf{H}$ is a *μ-collar neighborhood* of $\mathbf{H}$ if $\mathrm{mod}(A^0) \geq \mu$ and $\mathrm{mod}(A^\infty) \geq \mu$.

Let $f : \mathbf{H} \to \mathbf{H}$ be a (d_0, d_∞)-critical quasicircle map and let $c \in \mathbf{H}$ be its critical point. We call an annular neighborhood A of $\mathbf{H}$ *f-relevant* if it satisfies the following properties.

(R_1) The map f admits a holomorphic extension to an annular neighborhood A of $\mathbf{H}$ on which c is the only critical point.

(R_2) The annulus A can be decomposed into a disjoint union of an open disk neighborhood B of c and a topological rectangle R intersecting $\mathbf{H}$ along $\mathbf{H}\backslash B$.

(R_3) On B, f is a degree $d = d_0 + d_\infty - 1$ covering map branched only at c.

(R_4) The preimage of $f(B) \cap \mathbf{H}$ under f is the union of the interval $B \cap \mathbf{H}$, $2d_\infty - 2$ pairwise disjoint open quasiarcs in $Y^\infty_{\mathbf{H}}$ connecting c and $\partial B \cap Y^\infty_{\mathbf{H}}$, and $2d_0 - 2$ pairwise disjoint open quasiarcs in $Y^0_{\mathbf{H}}$ connecting c and $\partial B \cap Y^0_{\mathbf{H}}$.

(R_5) On the interior of R, f is a conformal isomorphism onto the interior of $f(R)$, and the preimage of $f(R)$ under $f|_A$ is precisely R.

Let us denote by $\mathcal{HQ}(d_0, d_\infty, N, K, \mu)$ the space of (d_0, d_∞)-critical K-quasicircle maps $f : \mathbf{H} \to \mathbf{H}$ with rotation number in Θ_N that admits an f-relevant 2μ-collar neighborhood A of $\mathbf{H}$ whose image $f(A)$ contains a μ-collar neighborhood of $\mathbf{H}$.

Example 4.4.5. The prototypical example of a (d_0, d_∞)-critical quasicircle map comes from the rational map $f = f_{d_0,d_\infty,\theta}$ discussed in §4.3.4. Denote by $\mathbf{H}$ its Herman curve $\mathbf{H}$. Assuming θ is in Θ_N, then $f : \mathbf{H} \to \mathbf{H}$ is in $\mathcal{HQ}(d_0, d_\infty, N, K, \mu)$ where K and μ depend only on d_0, d_∞, and N. By Corollary 4.3.2, when $d_0 = d_\infty = \mathrm{d}$, we know that $\mathbf{H}$ is the unit circle and there exists a unique $\alpha \in [0, 1)$ such that f coincides with the Blaschke product

$$B_{\mathrm{d},\alpha}(z) := e^{2\pi i\alpha} z^{\mathrm{d}} \cdot \frac{\sum_{j=0}^{\mathrm{d}-1} \binom{2\mathrm{d}-1}{j} (-1)^j z^{\mathrm{d}-1-j}}{\sum_{j=0}^{\mathrm{d}-1} \binom{2\mathrm{d}-1}{j} (-1)^j z^j}.$$

To study critical quasicircle maps, we will make use of Avila-Lyubich's theory of quasicritical circle maps in [AL22, §3].

Definition 4.4.6. For any integer $\mathrm{d} \geq 2$, a *d-quasicritical circle map* is an orientation-preserving homeomorphism $g : \mathbb{T} \to \mathbb{T}$ of the circle with the following properties.

(Q_1) The map g admits a $\mathbb{T}$-symmetric quasiregular extension of the form $B_{\mathrm{d},\alpha} \circ h$ on some $\mathbb{T}$-symmetric annular neighborhood A of $\mathbb{T}$ where $\alpha \in [0, 1)$ and h is some $\mathbb{T}$-symmetric quasiconformal map on $\mathbb{C}$.

(Q_2) The annulus A can be decomposed into a disjoint union of a $\mathbb{T}$-symmetric open disk neighborhood B of 1 and a $\mathbb{T}$-symmetric topological rectangle R intersecting $\mathbb{T}$ along $\mathbb{T}\backslash B$.

(Q_3) On B, g is a degree $2d-1$ quasiregular covering map branched only at 1, and it is holomorphic at the set of points z in B such that $\mathbb{T}$ does not separate z and $f(z)$.

(Q_4) The preimage of $g(B) \cap \mathbb{T}$ under g is the union of the interval $B \cap \mathbb{T}$, 2d − 2 pairwise disjoint open quasiarcs in $Y^\infty_\mathbb{T}$ connecting 1 and $\partial B \cap Y^\infty_\mathbb{T}$, and 2d − 2 pairwise disjoint open quasiarcs in $Y^0_\mathbb{T}$ connecting 1 and $\partial B \cap Y^0_\mathbb{T}$.

(Q_5) On the interior of R, g is a conformal isomorphism onto the interior of $g(R)$, and the preimage of $g(R)$ under $f|_A$ is precisely R.

Denote by $\mathrm{Cir}(\mathrm{d}, N, K, \delta)$ the space of all d-quasicritical circle maps $g : \mathbb{T} \to \mathbb{T}$ such that the rotation number θ of g is in Θ_N, the map h in (Q_1) is K-quasiconformal, and that there exists a 2δ-collar annular neighborhood A of $\mathbb{T}$ satisfying (Q_1)–(Q_5) whose image $g(A)$ is also a δ-collar neighborhood of $\mathbb{T}$. Note that our parameters differ slightly from those used by Avila and Lyubich, but it is not difficult to show that they encode equivalent amount of information.

Even though only 2-quasicritical circle maps are discussed in [AL22, §3], the results and proofs still hold for general d-quasicritical circle maps. Such maps admit the usual cross ratio distortion bounds and, as a result, they are quasisymmetrically rigid.

Theorem 4.4.7 ([AL22, Theorem 3.9]). *Every quasicritical circle map $g : \mathbb{T} \to \mathbb{T}$ in $\mathrm{Cir}(d, N, K, \delta)$ is quasisymmetrically conjugate to the irrational rotation with dilatation depending only on (d, N, K, δ).*

Similar to critical circle maps, quasicritical circle maps also admit complex bounds, since the main ingredients of the proof, namely real bounds and Schwarz lemma, are available. We will apply complex bounds later in the proof of Theorem 4.4.16.

The primary motivation behind introducing quasicritical circle maps is that critical quasicircle maps can be identified as a gluing of two quasicritical circle maps.

Proposition 4.4.8. *Every map $f : \mathbf{H} \to \mathbf{H}$ in $\mathcal{HQ}(d_0, d_\infty, N, K, \mu)$ is a welding of two quasicritical circle maps. There is an annular neighborhood A of $\mathbf{H}$ on which f is holomorphic, and a pair of quasicritical circle maps g_0 and g_∞ such that for each $\bullet \in \{0, \infty\}$,*

1. *$g_\bullet$ is in $\mathrm{Cir}(d_\bullet, N, L, \delta)$ for some $L = L(K) > 1$ and $\delta = \delta(d_0, d_\infty, K, \mu) > 0$;*
2. *there is an L-quasiconformal map $\phi_\bullet : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that maps $\mathbf{H}$ to $\mathbb{T}$ and conjugates $f|_{A \cap \overline{Y^\bullet_\mathbf{H}}}$ and $g_\bullet|_{\phi_\bullet(A) \cap \overline{Y^\bullet_\mathbb{T}}}$.*

Proof. Let A be an f-relevant neighborhood of $\mathbf{H}$. For $\bullet \in \{0, \infty\}$, let $\phi_\bullet : Y^\bullet_\mathbf{H} \to Y^\bullet_\mathbb{T}$ denote the Riemann mapping fixing $\bullet$ whose continuous extension to the boundary sends the critical point of f to 1. Since $\mathbf{H}$ is a K-quasicircle, the map $\phi_\bullet$ extends to a global $L(K)$-quasiconformal map sending $\mathbf{H}$ to $\mathbb{T}$. Let $g_\bullet := \phi_\bullet \circ f \circ \phi_\bullet^{-1}$ on $\phi_\bullet(A^\bullet)$ and apply the Schwarz reflection principle to extend $g_\bullet$ to a $\mathbb{T}$-symmetric quasiregular map that restricts to a self homeomorphism of $\mathbb{T}$. Properties (R_1)–(R_5) for f immediately transfer to (Q_1)–(Q_5) for $g_\bullet$, so $g_\bullet$ is the desired quasicritical circle map. □

This proposition is the key towards transferring known results on quasicritical circle maps to critical quasicircle maps. For instance, we have a quantitative version of Theorem 2.1.7.

Lemma 4.4.9. *Given a map $f : \mathbf{H} \to \mathbf{H}$ in $\mathcal{HQ}(d_0, d_\infty, N, K, \mu)$, there is a quasiconformal map h on $\hat{\mathbb{C}}$ that restricts to a conjugacy between $f|_{\mathbf{H}}$ and the rigid rotation $R_\theta|_{\mathbb{T}}$, and has dilatation depending only on $(d_0, d_\infty, N, K, \mu)$.*

Proof. This follows directly from Theorems 4.4.7 and 4.4.8. □

Remark 4.4.10. In general, if f is a multicritical quasicircle map with arbitrary irrational rotation number, then f is still conjugate to irrational rotation. Indeed, similar to Proposition 4.4.8, f is a conformal welding of two *multi-quasicritical circle maps* g_0 and g_∞. In [Pet04, Theorem 1.5], Petersen showed that such maps satisfy cross ratio distortion bounds, which in turn implies that they do not admit any wandering interval. (Compare with [Pet00, §3].)

In the world of commuting pairs, let us denote by $\mathcal{CP}(d_0, d_\infty, N, K, \mu)$ the space of all normalized (d_0, d_∞)-critical commuting pairs $\zeta = (f_-, f_+) : \mathbf{I} \to \mathbf{I}$ with rotation number in Θ_N such that the gluing procedure described in Proposition 4.4.4 produces a critical quasicircle map in $\mathcal{HQ}(d_0, d_\infty, N, K, \mu)$.

The following is a direct consequence of Lemma 4.4.9.

Corollary 4.4.11. *Every critical commuting pair $\zeta : \mathbf{I} \to \mathbf{I}$ in $\mathcal{CP}(d_0, d_\infty, N, K, \mu)$ admits a unique quasisymmetric map $h_\zeta : [-1, \theta] \to \mathbf{I}$ conjugating the pair $\mathbf{T}_\theta$ of translations in* (4.4.1) *with ζ, where θ is the rotation number of ζ. The dilatation of h_ζ depends only on $(d_0, d_\infty, N, K, \mu)$.*

4.4.3 Butterflies

Consider a critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$. From now on, it will be more convenient to use logarithmic coordinates by identifying $\mathbf{H}$ as a $\mathbb{Z}$-periodic quasicircle passing through 0 and ∞ and f as a map on a neighborhood of $\mathbf{H}$ in $\mathbb{C}$ with a critical point at $c_0 := 0$ which commutes with the translation T_1. We will not notationally distinguish the sets $\mathbf{H}$, $Y^0_{\mathbf{H}}$, $Y^\infty_{\mathbf{H}}$ from their respective quotients in $\mathbb{C}/\mathbb{Z}$.

Definition 4.4.12. A *bowtie* is a quadruplet of Jordan domains $(V, U_-, U_+, U_\times)$ in $\mathbb{C}$ together with a quasicircle $\mathbf{H}$ containing 0 and ∞ satisfying the following properties.

(B_1) U_-, U_+, and $U_\times$ are compactly contained in V.

(B_2) $U_- \cap U_+ = \varnothing$ and $\overline{U_-} \cap \overline{U_+} = \{0\} \subset U_\times$.

(B_3) $U_- \backslash U_\times$, $U_\times \backslash U_-$, $U_+ \backslash U_\times$, and $U_\times \backslash U_+$ are all non-empty and connected.

(B_4) $J_- := \mathbf{H} \cap \overline{U_-}$, $J_+ := \mathbf{H} \cap \overline{U_+}$, and $J_\times := \mathbf{H} \cap \overline{U_\times}$ are closed intervals in $\mathbf{H}$, and their interiors are precisely $\mathring{J}_- := \mathbf{H} \cap U_-$, $\mathring{J}_+ := \mathbf{H} \cap U_+$ and $\mathring{J}_0 := \mathbf{H} \cap U_\times$ respectively.

We call $\mathbf{H}$ the *axis* of the bowtie.

Definition 4.4.13. A (d_0, d_∞)-*critical butterfly* $\mathfrak{B}$ is a pair of holomorphic maps (f_-, f_+) together with a bowtie $(V, U_-, U_+, U_\times)$ with some axis $\mathbf{H}$ satisfying the following properties.

(B_5) $f_\pm$ is a univalent map from $U_\pm$ onto $V \backslash \mathbf{H} \cup f_\pm(\mathring{J}_\pm)$.

(B_6) Both f_- and f_+ extend holomorphically to $U_\times$ on which they commute. On $U_\times$, the map $f_- \circ f_+$ is a degree $d = d_0 + d_\infty - 1$ covering map onto $V \backslash \mathbf{H} \cup f_- f_+(\mathring{J}_\times)$ branched only at 0.

(B_7) $\mathbf{H}$ is $f_\pm$-invariant, that is, whenever $f_\pm$ extends holomorphically to a neighborhood E of a point $x \in \mathbf{H}$, then $f_\pm$ sends $E \cap \mathbf{H}$ to a subset of $\mathbf{H}$.

(B_8) $I_- := [f_+(0), 0]$ is a subset of J_-, $I_+ := [0, f_-(0)]$ is a subset of J_+, and $(f_-|_{I_-}, f_+|_{I_+})$ is a (d_0, d_∞)-critical commuting pair.

(B_9) There is some integer $m \geq 1$ such that $f_-(0) = f_+^m(b_+)$ and $f_+(0) = f_-(b_-)$ where $J_- = [b_-, 0]$ and $J_+ = [0, b_+]$.

The *axis* of $\mathfrak{B}$ is the quasicircle $\mathbf{H}$, the *height* of $\mathfrak{B}$ is the integer m, and the *rotation number* $\operatorname{rot}(\mathfrak{B})$ of $\mathfrak{B}$ is the rotation number of the commuting pair $(f_-|_{I_-}, f_+|_{I_+})$. The interval

$$\mathbf{I} := [f_+(0), f_-(0)] = I_- \cup I_+ \subset \mathbf{H}$$

is called the *base* of $\mathfrak{B}$, whereas

$$\mathbf{J} := [b_-, b_+] = J_- \cup J_+ \subset \mathbf{H}$$

is called the *extended base* of $\mathfrak{B}$. We say that $\mathfrak{B}$ is *normalized* if $f_-(0) = -1$. The *domain* of $\mathfrak{B}$ is the Jordan domain

$$U := U_- \cup U_\times \cup U_+.$$

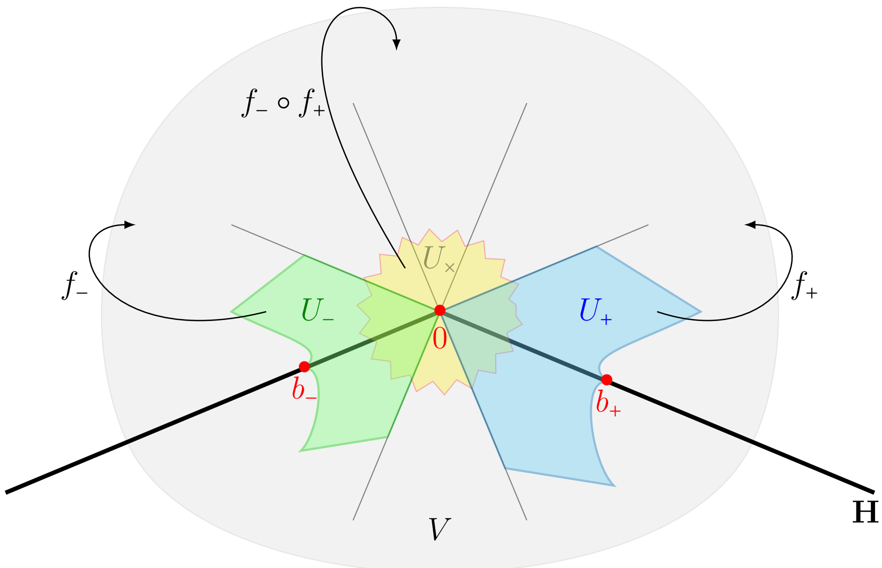

Figure 4.6: A $(3, 2)$-critical butterfly.

The *shadow* of a butterfly $\mathfrak{B}$ is the piecewise holomorphic map $F : U \to V$ where

$$F = \begin{cases} f_- & \text{on } U_-, \\ f_+ & \text{on } U_+, \\ f_- \circ f_+ & \text{on } U_\times \backslash (U_- \cup U_+). \end{cases}$$

The *limit set* of $\mathfrak{B}$ is the non-escaping set of F, namely

$$\Lambda_{\mathfrak{B}} := \bigcap_{n \geq 0} F^{-n}(U).$$

Following de Faria and de Melo [FM99], we will impose geometric assumptions on our butterflies.

Definition 4.4.14. For $K > 1$, a normalized butterfly $\mathfrak{B}$ with axis $\mathbf{H}$ is called a K-*butterfly* if the following conditions are satisfied.

(G_1) $\operatorname{mod}\left(V \backslash \overline{U}\right) \geq K^{-1}$.

(G_2) The components of $\mathbb{C} \backslash \mathbf{H}$, $\mathbb{C} \backslash (\partial V \cup \mathbf{H})$, and $V \backslash \left(\overline{U} \cup \mathbf{H}\right)$ are K-quasidisks.

(G_3) Any two points in the set $\{b_-, f_+(0), 0, f_-(0) = 1, b_+\}$ are at least K^{-1} away from each other.

(G_4) The annulus $\overline{V} \backslash U_\times$ is contained in $\{K^{-1} < |z| < K\}$.

In general, a butterfly is a *K-butterfly* if it is linearly conjugate to a normalized K-butterfly.

We endow the space of butterflies with the topology where a sequence of butterflies $\mathcal{B}_n = \{(f_{-,n}, f_{+,n}), (V_n, U_{-,n}, U_{+,n}, U_{\times,n})\}$ converges to the butterfly $\mathcal{B} = \{(f_-, f_+), (V, U_-, U_+, U_\times)\}$ if

(i) in the Carathéodory topology (refer to [McM94, §5]),

$$(U_{-,n}, f_{+,n}(0)) \to (U_-, f_+(0)), \qquad (U_{\times,n}, 0) \to (U_\times, 0),$$
$$(U_{+,n}, f_{-,n}(0)) \to (U_+, f_-(0)), \qquad (V_n, f_{-,n}f_{+,n}(0)) \to (V, f_-f_+(0));$$

(ii) $\mathcal{B}_n$ has an axis converging to an axis of $\mathcal{B}$ in Hausdorff metric;

(iii) $f_{-,n}$ converges uniformly to f_- on compact subsets of $U_- \cup U_\times$, and $f_{+,n}$ converges uniformly to f_+ on compact subsets of $U_+ \cup U_\times$.

Proposition 4.4.15. *The space of normalized (d_0, d_∞)-critical K-butterflies with rotation number in Θ_N is compact.*

Similar to critical circle maps, critical quasicircle maps also admit complex a priori bounds.

Theorem 4.4.16 (Complex bounds)**.** *Given $f : \mathbf{H} \to \mathbf{H}$ in $\mathcal{HQ}(d_0, d_\infty, N, K, \mu)$, there exist constants $n_0 \in \mathbb{N}$ and $K' > 1$ depending only on $(d_0, d_\infty, N, K, \mu)$ such that for all $n \geq n_0$, the n^{th} pre-renormalization of f extends to a K'-butterfly $\mathcal{B}_n : U_n \to V_n$.*

Recall from Proposition 4.4.8 that f is a welding of two quasicritical circle maps g_0 and g_∞. By [AL22, §3.5], the pre-renormalizations of g_0 and g_∞ have holomorphic extensions admitting a butterfly structure with complex a priori bounds. In the proof below, we will glue the half-butterflies of the two maps in order to obtain a butterfly for the pre-renormalization of f.

Proof. Let U, ϕ_0, ϕ_∞, $g_0 \in \mathrm{Cir}(d_0, N, L, \delta)$, and $g_\infty \in \mathrm{Cir}(d_\infty, N, L, \delta)$ be from Proposition 4.4.8. We will outline the construction of butterflies for g_0 and g_∞ and then glue them to the desired butterfly for f. Let us work in logarithmic coordinates, in which $\mathbf{H} \subset \mathbb{C}/\mathbb{Z}$ is a quasicircle passing through 0 and the critical point of f is at 0. For all $j \in \mathbb{Z}$, we write $c_j := f^j(0)$. For $\bullet \in \{0, \infty\}$ and $j \in \mathbb{Z}$, let $c_0^\bullet = 0$ denote the critical point of $g_\bullet$ and let $c_j^\bullet := g_\bullet^j(0)$.

Let $\{p_n/q_n\}_{n \in \mathbb{N}}$ denote the best rational approximations of the rotation number θ of f. For $\bullet \in \{0, \infty\}$ and $n \geq 2$, observe that the two critical points of $g_\bullet^{q_n}$ that are next to $c_{q_{n-1}}^\bullet$ are $c_0^\bullet$ and $c_{q_{n-1}-q_n}^\bullet$. From now on, we will fix $n \in \mathbb{N}$ larger than some constant $m \in \mathbb{N}$ that is to be determined. Let us recall the construction of butterflies extending the n^{th} pre-renormalization of $g_\bullet$.

For any $k \geq 1$, let $D_k^\bullet$ be the open round disk such that $\partial D_k^\bullet$ intersects $\mathbb{T}$ orthogonally and $D_k^\bullet \cap \mathbb{T}$ is the open interval $\left(c_{q_{k+1}}^\bullet, c_{q_k - q_{k+1}}^\bullet\right) \subset \mathbb{T}$. For $n \gg m$, there exists a $\mathbb{T}$-symmetric univalent lift $A_{n,m}^\bullet$ of $(D_{n-m}^\bullet \backslash \mathbb{T}) \cup \left(c_{q_{n-1}}^\bullet, c_{q_n}^\bullet\right)$ under $g_\bullet^{q_n}$ intersecting $\mathbb{T}$ on the interval $\left(c_{q_{n-1}-q_n}^\bullet, c_0^\bullet\right)$. Similarly, there also exists a $\mathbb{T}$-symmetric univalent lift $B_{n,m}^\bullet$ of $(D_{n-m}^\bullet \backslash \mathbb{T}) \cup \left(c_{q_{n-1}}^\bullet, c_{q_{n-2}}^\bullet\right)$ under $g_\bullet^{q_{n-1}}$ intersecting $\mathbb{T}$ on the interval $\left(c_0^\bullet, c_{q_{n-2}-q_{n-1}}^\bullet\right)$.

Claim. For any $\varepsilon > 0$, there are some constants $n_0, m \in \mathbb{N}$ depending only on ε and $(d_\bullet, N, L, \delta)$ such that $n_0 \geq m$ and for all $n \geq n_0$,

$$\max\{\operatorname{diam}(A_{n,m}^\bullet), \operatorname{diam}(B_{n,m}^\bullet)\} \leq \varepsilon \cdot \operatorname{diam}(D_{n-m}^\bullet). \tag{4.4.2}$$

Proof. Let $I_n^\bullet$ denote the closed interval in $\mathbb{T}$ between $c_0^\bullet$ and $c_{q_n}^\bullet$. Based on the key estimates in [AL22, (3.9)] and [FM99, Proposition 3.2], there are constants $b_1, b_2 > 0$ such that for all $z \in A_{n,m}^\bullet$,

$$\frac{\operatorname{dist}\left(g_\bullet(z), g_\bullet(I_{n-1}^\bullet)\right)}{|g_\bullet(I_{n-1}^\bullet)|} \leq b_1 \cdot \frac{\operatorname{dist}\left(g_\bullet^{q_n}(z), I_{n-1}^\bullet\right)}{|I_{n-1}^\bullet|} + b_2. \tag{4.4.3}$$

[AL22, Lemma 3.6] guarantees that near the critical point, the inverse branch of $g_\bullet$ is highly contracting in big scales relative to $I_{n-1}^\bullet$, which yields

$$C \cdot \left(\frac{\operatorname{diam}(A_{n,m}^\bullet)}{|I_{n-1}^\bullet|}\right)^\sigma \leq b_1 \cdot \frac{\operatorname{diam}(D_{n-m}^\bullet)}{|I_{n-1}^\bullet|} + b_2 \tag{4.4.4}$$

for some constants $C > 0$ and $\sigma > 1$. Applying Theorem 4.4.7 to $g_\bullet$, there are also constants $K_2 > K_1 > 1$ such that for all sufficiently large n,

$$K_1^m \leq \frac{\operatorname{diam}(D_{n-m}^\bullet)}{|I_{n-1}^\bullet|} \leq K_2^m. \tag{4.4.5}$$

Note that all the intermediate constants above depend only on $(d_\bullet, N, L, \delta)$. Let us pick $\varepsilon > 0$. By combining (4.4.4) and (4.4.5), for sufficiently large m, we have

$$\operatorname{diam}(A_{n,m}^\bullet) \leq \varepsilon \cdot \operatorname{diam}(D_{n-m}^\bullet).$$

We can repeat the same analysis for $B_{n,m}^\bullet$. $\square$

Let D_{n-m}, $A_{n,m}$ and $B_{n,m}$ be the interior of the closure of $\bigcup_\bullet \phi_\bullet^{-1}(D_{n-m}^\bullet \cap Y_{\mathbb{T}}^\bullet)$, $\bigcup_\bullet \phi_\bullet^{-1}(A_{n,m}^\bullet \cap Y_{\mathbb{T}}^\bullet)$, and $\bigcup_\bullet \phi_\bullet^{-1}(B_{n,m}^\bullet \cap Y_{\mathbb{T}}^\bullet)$ respectively. Since ϕ_0 and ϕ_∞ are L-quasiconformal, the claim implies that there are some constants $n_0, m \in \mathbb{N}$ depending only on $(\varepsilon, d_0, d_\infty, N, K, \mu)$ such that $n_0 \geq m$ and for $n \geq n_0$,

$$\max\{\operatorname{diam}(A_{n,m}), \operatorname{diam}(B_{n,m})\} \leq \varepsilon \cdot \operatorname{diam}(D_{n-m}). \tag{4.4.6}$$

Let $C_{n,m}$ denote the connected component of $f^{-q_{n-1}}(A_{n,m})$ containing the critical point 0. Since $C_{n,m} \cap \mathbf{H} = (c_{-q_n}, c_{-q_{n-1}})$, the map $f^{q_{n-1}} : C_{n,m} \to A_{n,m}$ is a degree d covering map branched exactly at 0, and $f^{q_{n-1}}$ maps $A_{n,m} \cap C_{n,m}$ univalently onto $(A_{n,m} \backslash \mathbf{H}) \cup (c_{q_{n-1}-q_n}, 0)$. By making n_0 higher if necessary, $A_{n,m} \cup B_{n,m}$ is contained in a neighborhood of the critical point 0 in which $f \equiv \psi(z^d)$ for some univalent map ψ with universally bounded distortion, which implies that $\operatorname{diam}(C_{n,m}) \asymp \operatorname{diam}(A_{n,m} \cap C_{n,m})$. Therefore, (4.4.6) can be upgraded to

$$\max\{\operatorname{diam}(A_{n,m}), \operatorname{diam}(B_{n,m}), \operatorname{diam}(C_{n,m})\} \leq \varepsilon \cdot \operatorname{diam}(D_{n-m}). \tag{4.4.7}$$

By construction, the pointed disk $(D_{n-m}, 0)$ has bounded shape. We can select an appropriate ε such that (4.4.7) implies that the union $U_n := A_{n,m} \cup B_{n,m} \cup C_{n,m}$ is compactly contained in $V_n := D_{n-m}$ and $V_n \backslash \overline{U_n}$ is an annulus with modulus greater than some universal constant. Therefore, the pair $(f^{q_n}, f^{q_{n-1}})$ and the bowtie $(V_n, A_{n,m}, B_{n,m}, C_{n,m})$ form a butterfly $\mathfrak{B}_n$ with axis $\mathbf{H}$ that extends the n^{th} pre-renormalization of f and clearly satisfies (G_1).

It is also clear from the construction that the components of $\hat{\mathbb{C}} \backslash \mathbf{H}$ and $\hat{\mathbb{C}} \backslash (V_n \cup \mathbf{H})$ are K'-quasidisks. Every component of $V_n \backslash \left(\overline{U_n} \cup \mathbf{H}\right)$ is also a K'-quasidisk since its boundary is a union of quasiarcs meeting at definite angles. Hence, (G_2) holds. Condition (G_3) follows from Lemmas 2.1.8 (2) and 4.4.9, and (G_4) follows from the construction of V_n and Koebe distortion theorem. □

In the proof above, the butterfly extending the n^{th} pre-renormalization has height equal to $a_n + 1$, where a_n is the n^{th} term of the continued fraction expansion of the rotation number. If we apply the construction in the proof to the prototypical Example 4.4.5, the corresponding limit set is contained in the Julia set of the rational map, which is nowhere dense.

Corollary 4.4.17. *For any $m \geq 2$ and any $\theta \in \Theta_N$, there exists a (d_0, d_∞)-critical K-butterfly having rotation number θ, height m, and a nowhere dense limit set, where K depends only on (d_0, d_∞, m, N).*

By Proposition 4.4.4, commuting pairs also admit complex bounds.

Theorem 4.4.18. *Given ζ in $\mathcal{CP}(d_0, d_\infty, N, K, \mu)$, there exist some $n_0 \in \mathbb{N}$ and $K' > 1$ depending only on $(d_0, d_\infty, N, K, \mu)$ such that for all $n \geq n_0$, the n^{th} pre-renormalization of ζ extends to a K'-butterfly $\mathfrak{B}_n : U_n \to V_n$.*

4.4.4 Quasiconformal rigidity

Let us fix a bounded type irrational number $\theta \in \Theta_N$ for some $N \geq 1$. Recall from Corollary 4.4.11 that two butterflies of the same criticality and bounded type rotation number must be quasisymmetrically conjugate on their bases.

Lemma 4.4.19. *Suppose two (d_0, d_∞)-critical K-butterflies $\mathbf{B}_1 : U_1 \to V_1$ and $\mathbf{B}_2 : U_2 \to V_2$ have the same height and rotation number θ. The unique quasisymmetric conjugacy between $\mathbf{B}_1$ and $\mathbf{B}_2$ on their bases extend to a quasiconformal conjugacy $h : V_1 \to V_2$ between $\mathbf{B}_1$ and $\mathbf{B}_2$ with dilatation depending only on (d_0, d_∞, N, K).*

The proof is an application of the pullback argument similar to [Far99, Theorem 3.1].

Proof. By Corollary 4.4.11, there exists a unique quasisymmetric conjugacy $h : \mathbf{I}_1 \to \mathbf{I}_2$ between $\mathbf{B}_1$ and $\mathbf{B}_2$ on their bases. Since $\mathbf{B}_1$ and $\mathbf{B}_2$ have the same height, we can extend h to a quasisymmetric conjugacy on the extended bases $\mathbf{J}_1$ and $\mathbf{J}_2$ by setting $h(z) := f_{-,2}^{-1} \circ h \circ f_{-,1}(z)$ for $z \in J_{-,1} \backslash \mathbf{I}_1$ and $h(z) := f_{+,2}^{-k} \circ h \circ f_{+,1}^{k}(z)$ for $z \in J_{+,1} \backslash \mathbf{I}_1$ where $k \in \mathbb{N}$ is the first time $f_{+,1}^{k}(z)$ lies on $\mathbf{I}_1$.

By (G_1) and (G_2), we can perform quasiconformal interpolation and extend h to a global L-quasiconformal map h_0 that is equivariant on the boundaries of the butterflies of $\mathbf{B}_1$ and $\mathbf{B}_2$. Note that the dilatation L depends only on (d_0, d_∞, N, K).

Next, we apply the pullback argument to obtain a sequence of L-quasiconformal maps h_n as follows. Outside of U_1, we take $h_n \equiv h_{n-1}$; within U_1, we set h_n to be the lift of $h_{n-1} : V_1 \to V_2$ via $\mathbf{B}_1$ and $\mathbf{B}_2$. By equivariance, h_n is well-defined, and it gives a conjugacy between $\mathbf{B}_1|_{V_1 \backslash F_1^{-n}(U_1)}$ and $\mathbf{B}_2|_{V_2 \backslash F_2^{-n}(U_2)}$.

By the compactness of the space of normalized L-quasiconformal maps, h_n converges to a subsequential limit $h_\infty : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Note that h_n stabilizes pointwise outside of $\Lambda_{\mathbf{B}_1}$. We will claim that $\Lambda_{\mathbf{B}_1}$ is nowhere dense, which ultimately implies that the limit h_∞ is unique in the sense that $h_n \to h_\infty$ as $n \to \infty$, and h_∞ conjugates $\mathbf{B}_1$ and $\mathbf{B}_2$.

Corollary 4.4.17 guarantees the existence of a (d_0, d_∞)-critical butterfly $\mathbf{B} : U \to V$ that has the same rotation number θ and height m and that its limit set $\Lambda_{\mathbf{B}}$ is nowhere dense. By applying the same pullback argument above, we obtain a quasiconformal map $g : V \to V_1$ that restricts to a conjugacy between $\mathbf{B}$ and $\mathbf{B}_1$ on $V \backslash \Lambda_{\mathbf{B}}$. Since $\Lambda_{\mathbf{B}}$ is nowhere dense, then g extends to a full conjugacy between $\mathbf{B}$ and $\mathbf{B}_1$, which then implies that $\Lambda_{\mathbf{B}_1}$ is indeed nowhere dense. $\square$

Next, we can spread around the quasiconformal conjugacy between butterflies of sufficiently deep renormalizations throughout the entire Herman curves. Compare with [AL22, Theorem 3.19].

Theorem 4.4.20 (Quasiconformal rigidity). *Given any two critical quasicircle maps f_1 and f_2 in $\mathcal{HQ}(d_0, d_\infty, N, K, \mu)$ of the same rotation number, there is an L-quasiconformal map h on $\hat{\mathbb{C}}$ that restricts to a conjugacy between f_1 and f_2 in some δ-collar neighborhoods of their Herman quasicircles. The constants L and δ depend only on $(d_0, d_\infty, N, K, \mu)$.*

Proof. Let $f_1 : \mathbf{H}_1 \to \mathbf{H}_1$ and $f_2 : \mathbf{H}_2 \to \mathbf{H}_2$ be two (d_0, d_∞)-critical quasicircle maps of rotation number θ. Without loss of generality, assume that f_1 is the prototypical Example 4.4.5 $f_1 = f_{d_0,d_\infty,\theta}$.

By Lemma 4.4.9, there is a global quasiconformal map h that sends $\mathbf{H}_1$ to $\mathbf{H}_2$ and restricts to a conjugacy between $f_1|_{\mathbf{H}_1}$ and $f_2|_{\mathbf{H}_2}$. For $i \in \{1, 2\}$ and sufficiently large $m \in \mathbb{N}$, Theorem 4.4.16 states that the m^{th} pre-renormalization of f_i extends to a K'-butterfly $\mathbf{\mathcal{B}}_i = (f_{-,i}, f_{+,i}) : U_i \to V_i$. Clearly, h induces a quasisymmetric conjugacy between $\mathbf{\mathcal{B}}_1$ and $\mathbf{\mathcal{B}}_2$ on their bases. By Lemma 4.4.19, h can be modified to a quasiconformal map $h_\infty : V_1 \to V_2$ that conjugates $\mathbf{\mathcal{B}}_1 : U_1 \to V_1$ and $\mathbf{\mathcal{B}}_2 : U_1 \to V_1$.

It remains to spread the conjugacy around the Herman curve. By [Wan+21] and Douady-Ghys surgery, the immediate basins of attraction of both 0 and ∞ for f_1 have locally connected boundaries. As such, given any point $x \in \mathbf{H}$ that is not an iterated preimage of the critical point c of f_1, there is a unique pair of external rays from 0 and ∞ landing at x; we label the union of these rays as γ_x. The rays $\gamma_{f_{+,1}(c)}$ and $\gamma_{f_{-,1}(c)}$ are mapped to $\gamma_{f_{-,1}f_{+,1}(c)}$ under $f_{-,1}$ and $f_{+,1}$ respectively. As these rays are disjoint (away from 0 and ∞), the union of $\gamma_{f_{-,1}f_{+,1}(c)}$, $\gamma_{f_{\mp,1}(c_0)}$, and ∂V_1 bounds a unique topological rectangle $\Pi_\pm$ containing $I_{\pm,1}$, the base of $\mathbf{\mathcal{B}}_1$. We pull back both rectangles Π_+ and Π_- by iterates of f_1 until the first return to V_1 and obtain a tiling of a neighborhood of $\mathbf{H}_1$ by topological rectangles.

Next, consider the rectangles $h_\infty(\Pi_\pm)$ and construct a similar dynamical tiling for f_2 around $\mathbf{H}_2$. Then, the quasiconformal conjugacy $h_\infty : V_1 \to V_2$ can be lifted via these dynamical tilings to get the desired quasiconformal conjugacy between f_1 and f_2 on the neighborhood of their Herman quasicircles. $\square$

4.5 $C^{1+\alpha}$ rigidity

In this section, we will show that the quasiconformal conjugacy between two (d_0, d_∞)-critical quasicircle maps with the same rotation number θ must be $C^{1+\alpha}$-conformal along the Herman curves. The proof below is an application of McMullen's deep point argument.

4.5.1 Deep points

Definition 4.5.1. We say that a subset S of a compact set $J \subset \mathbb{C}$ is *uniformly deep* if there are positive constants $C, \delta, r > 0$ such that for every point z inside the r-neighborhood of S,

$$\operatorname{dist}(z, J) \leq C \operatorname{dist}(z, S)^{1+\delta}.$$

In other words, if S is uniformly deep, the magnification of J at any point in S converges exponentially fast to the whole plane at a uniform rate.

Let us again consider any rational map f in $\mathcal{X}$, and denote by $\mathbf{H}$ its Herman curve. Following §4.2.1, let us consider a quasiconformal map $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that fixes 0 and ∞ and conjugates $f|_{\mathbf{H}}$ and the rigid rotation $R_\theta|_{\mathbb{T}}$. Recall the function $L(z) := \log\left(\operatorname{dist}(\phi(z), \mathbb{T})\right)$ as well as the annular neighborhood $A_\kappa := \{-\infty \leq L(z) < -\kappa\}$ of $\mathbf{H}$ for any κ. Recall from Definition 4.2.6 the local non-escaping set K_κ^{loc}.

Theorem 4.5.2. *For any $\kappa \in \mathbb{R}$, the Herman curve $\mathbf{H}$ of f is uniformly deep in the local non-escaping set K_κ^{loc}.*

Proof. It suffices to prove the theorem for $A = A_\kappa$ where κ is sufficiently high. By the Hölder continuity of ϕ, for every $z \in A$,

$$L(z) \asymp \log(\operatorname{dist}(z, \mathbf{H})). \tag{4.5.1}$$

Given an approximate rotation $f^i : U \to V$ (rel A), we have $L(f^i(z)) \leq L(z) + O(1)$ for every $z \in U$. In general, we have the following property.

Claim. For every $z \in A$, $L(f(z)) \leq L(z) + O(1)$.

Proof. If z is sufficiently far from any critical point c, i.e. $|z - c| \succ \operatorname{dist}(z, \mathbf{H})$, then f is an approximate rotation on a neighborhood of z. Else, suppose z is close to a critical point c. Let $z' := \phi(z)$, $c' := \phi(c)$, and $F := \phi \circ f \circ \phi^{-1}$. Since F is a quasiregular map that restricts to an isometry of the unit circle $\mathbb{T}$, we have $|F(z') - F(c')| \asymp |z' - c'|$. Therefore,

$$\operatorname{dist}(F(z'), \mathbb{T}) \leq |F(z') - F(c')| \asymp |z' - c'| \asymp \operatorname{dist}(z', \mathbb{T}).$$

By taking the logarithm, this inequality implies the claim. □

Let us equip $\Omega := \hat{\mathbb{C}} \backslash P(f)$ with the hyperbolic metric. For $z \in A$, let $r_z := \operatorname{dist}_\Omega\left(z, f^{-1}(\mathbf{H})\right)$. The norm of $f'(z)$ with respect to the hyperbolic metric of Ω satisfies

$$\|f'(z)\| \geq C(r_z) > 1 \tag{4.5.2}$$

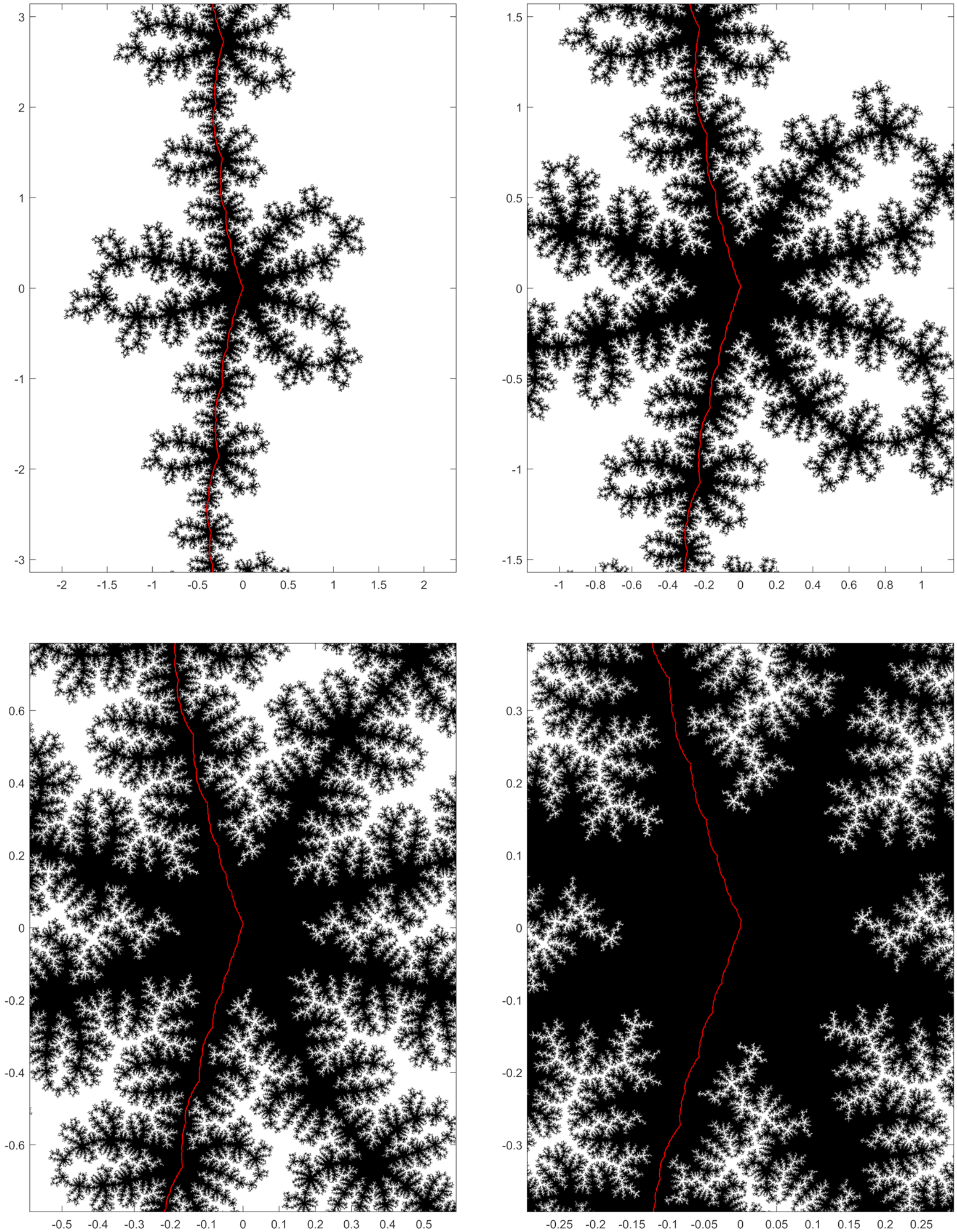

Figure 4.7: The plot of the Julia set of $f = f_{2,3,\theta}$ (Example 4.4.5) in logarithmic coordinates, magnified about its free critical point. The Herman quasicircle is shown in red and its rotation number θ is picked to be the golden mean.

for some function $C(r)$ where $C(r) \to \infty$ as $r \to 0$.

Let us pick any point z in A with $L(z) \ll -\kappa$, and denote $z_i := f^i(z)$ for every $i \in \mathbb{N}$. By Corollary 4.1.6, there is an approximate rotation $f^{n_1} : (U, z) \to (V, z_{n_1})$ such that $\text{dist}_\Omega (z_{n_1}, f^{-1}(\mathbf{H})) = O(1)$. By (4.5.2), we have $\|f'(z_{n_1})\| \geq M > 1$ for some M independent of z. On the other hand, $L(z_{n_1+1}) \leq L(z) + O(1)$ due to the claim above. We repeat this argument inductively to obtain an increasing sequence of positive integers $\{n_j\}_j$ such that

$$\text{dist}_\Omega \left(z_{n_j}, f^{-1}(\mathbf{H})\right) = O(1), \tag{4.5.3}$$

$$\|f'(z_{n_j})\| \geq M, \text{ and} \tag{4.5.4}$$

$$L(z_{n_j}) \leq L(z) + O(j). \tag{4.5.5}$$

By (4.5.1) and (4.5.5), there exists some $m \in \mathbb{N}$ such that $z_{n_i} \in \overline{A}$ for all $i \in \{1, 2, \ldots m\}$ and that

$$m \asymp -L(z) \asymp -\log\left(\text{dist}(z, \mathbf{H})\right). \tag{4.5.6}$$

Then, by (4.5.4) and (4.5.6),

$$\|(f^{n_m})'(z)\| \geq M^{m-1} \succ \text{dist}(z, \mathbf{H})^{-\alpha}, \tag{4.5.7}$$

where $\alpha \asymp \log M$.

By (4.5.3), we can pick an arc $\gamma_m \subset \Omega$ of hyperbolic length $O(1)$ joining z_{n_m} and some point y' in $A \cap f^{-1}(\mathbf{H})$. By (4.5.7), γ_m lifts under f^{n_m} to an arc $\gamma \subset \Omega$ joining z and some point $y \in f^{-n_m-1}(\mathbf{H})$ of hyperbolic length $O(\text{dist}(z, \mathbf{H})^\alpha)$. Therefore, by Lemma 4.2.3,

$$|z - y| = O\left(\text{dist}(z, \mathbf{H})^{\alpha+1}\right).$$

We can make sure that the forward orbit of y stays within $\overline{A}$ and is thus contained in K^{loc}_κ. As such, the estimate above implies that $\mathbf{H}$ is uniformly deep in K^{loc}_κ. □

4.5.2 McMullen's Dynamic Inflexibility Theorem

Consider a pair of (d_0, d_∞)-critical quasicircle maps $f_1 : \mathbf{H}_1 \to \mathbf{H}_1$ and $f_2 : \mathbf{H}_2 \to \mathbf{H}_2$ with rotation number $\theta \in \Theta_N$. By Theorem 4.4.20, there exist collars A_1 and A_2 for $f_1|_{\mathbf{H}_1}$ and $f_2|_{\mathbf{H}_2}$ respectively, and a global quasiconformal map $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ conjugating $f_1|_{A_1}$ and $f_2|_{A_2}$. Our goal is to improve the regularity of the quasiconformal conjugacy h and prove Theorem D. We will do so via McMullen's Dynamic Inflexibility Theorem.

Let *Hol* denote the set of all holomorphic maps $g : U \to \hat{\mathbb{C}}$ where U is any open subset of $\hat{\mathbb{C}}$. Endow *Hol* with the topology where $g_n : W_n \to \hat{\mathbb{C}}$ converges to $g : W \to \hat{\mathbb{C}}$ if for every compact subset $X \subset W$, W_n contains X for all sufficiently large n and $g_n \to g$ uniformly in X.

For each $i \in \{1,2\}$, define

$$\mathcal{F}(f_i) := \left\{[g : U \to \hat{\mathbb{C}}] \in Hol \,:\, f_i^i = g \circ f_i^j \text{ for some } i,j \in \mathbb{N} \text{ on some open } U \subset A_i\right\}.$$

Let us pick a skinnier collar $A_1' \Subset A_1$ for $\mathbf{H}_1$ and let $A_2' = \phi(A_1')$. For $i \in \{1,2\}$, let us define the *local non-escaping set* of f_i rel A_i' to be

$$K^{\text{loc}}(f_i) = \overline{\{z \in A_i \,:\, f_i^n(z) \in A_i' \text{ for all } n \geq 0\}};$$

this is a forward invariant compact subset of A_i. The pair $(\mathcal{F}(f_i), K^{\text{loc}}(f_i))$ forms a holomorphic dynamical system in the sense of McMullen [McM96, §9].

Theorem 4.5.3 ([McM96, Theorem 9.15]). *Suppose there is a K-quasiconformal conjugacy ϕ between two holomorphic dynamical systems $(\mathcal{F}_1, J_1)$ and $(\mathcal{F}_2, J_2)$. If $(\mathcal{F}_1, J_1)$ is uniformly twisting and J_1 has a uniformly deep subset S, then ϕ is uniformly $C^{1+\alpha}$-conformal on S.*

Roughly speaking, a holomorphic dynamical system $(\mathcal{F}, J)$ is *uniformly twisting* if $\mathcal{F}$ has robust nonlinearity at every point in J at every scale. A more precise definition of nonlinearity and uniform twisting can be found in [McM96, §9.3]. In our discussion, we will only require the following criterion for uniform twisting.

Proposition 4.5.4 ([McM98, Proposition 4.7]). *Consider a subset $\mathcal{F}$ of Hol and a compact subset J of $\hat{\mathbb{C}}$. Suppose for any sequence of affine maps $\mathrm{af}_n(z) = \alpha_n(z - \beta_n)$ with $\alpha_n \to \infty$ and $\beta_n \in J$, there is a sequence of maps g_n in $\mathcal{F}$ such that the rescaling $\mathrm{af}_n \circ g_n \circ \mathrm{af}_n^{-1}$ converges in subsequence to a non-constant holomorphic map in Hol with a critical point. Then, the dynamical system $(\mathcal{F}, J)$ is uniformly twisting.*

In our context, we will consider the pair of dynamical systems

$$\left(\mathcal{F}(f_1), K^{\text{loc}}(f_1)\right) \quad \text{and} \quad \left(\mathcal{F}(f_2), K^{\text{loc}}(f_2)\right)$$

which are quasiconformally equivalent via h. Let us assume without loss of generality that f_1 is a rational map in $\mathcal{X}$, e.g. the prototypical Example 4.4.5. In the proof of Theorem 4.2.7, we have shown that the map f_1 and the set $K^{\text{loc}}(f_1)$ satisfy properties (S_1)–(S_5) in the hypothesis of Proposition 4.2.8. These properties immediately imply the hypothesis of Proposition 4.5.4, and so $(\mathcal{F}(f_1), K^{\text{loc}}(f_1))$ is indeed uniformly twisting. Let us choose A_1' to be of the form A_κ. Recall from Theorem 4.5.2 that $\mathbf{H}_1$ is a a uniformly deep subset of $K^{\text{loc}}(f_1)$ with constants depending only on (d_0, d_∞, N). By Theorem 4.5.3, the quasiconformal conjugacy h between f_1 and f_2 must be uniformly $C^{1+\alpha}$-conformal on $\mathbf{H}_1$. This completes the proof of Theorem D.

4.6 Consequences of $C^{1+\alpha}$ rigidity

We end this chapter with a discussion on a number of applications of $C^{1+\alpha}$ rigidity of critical quasicircle maps. Throughout, we will fix $N \geq 1$ and $\theta \in \Theta_N$.

4.6.1 Smoothness

Consider a (d_0, d_∞)-critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$ of rotation number θ.

Corollary 4.6.1. *The quasicircle* $\mathbf{H}$ *is* $C^{1+\alpha}$ *smooth if and only if* $d_0 = d_\infty$.

See Figure 4.8 for an example of a C^1 smooth Herman curve that is not a Euclidean circle.

Proof. Suppose $d_0 \neq d_\infty$. If $\mathbf{H}$ were C^1 smooth near the critical value, then the angle of $\mathbf{H}$ at the critical point is equal to $\frac{\pi(2d_0-1)}{d}$, which is not equal to π, and so it cannot have a tangent.

Now, suppose $d_0 = d_\infty$. Example 4.4.5 gives us a (d_0, d_∞)-critical circle map $g : \mathbb{T} \to \mathbb{T}$ with rotation number θ. By $C^{1+\alpha}$ rigidity, there exists a uniformly $C^{1+\alpha}$-conformal conjugacy $\phi : \mathbb{T} \to \mathbf{H}$ between g and f. Lemma 4.6.2 below implies that ϕ has a complex derivative along $\mathbf{H}$ with α-Hölder regularity. □

Lemma 4.6.2. *The complex derivative of a uniformly* $C^{1+\alpha}$*-conformal map is Hölder continuous with exponent* α.

Proof. Suppose $\phi : U \to V$ is uniformly $C^{1+\alpha}$-conformal, that is, for all $z \in U$, the complex derivative $\phi'(z)$ exists and the function

$$\eta_z(t) := \frac{\phi(z+t) - \phi(z)}{t} - \phi'(z),$$

satisfies $|\eta_z(t)| \leq C\,|t|^\alpha$ for some uniform constant $C > 0$ when $|t|$ is sufficiently small. Whenever two points z, w on U are sufficiently close,

$$|\phi'(z) - \phi'(w)| = |\eta_w(z-w) - \eta_z(w-z)| \leq 2C\,|z-w|^\alpha.$$

This proves the lemma. □

Let us complete the proof of Corollary E with the following proposition.

Proposition 4.6.3. *If* $d_0 \neq d_\infty$, $\mathbf{H}$ *is not* C^1 *smooth at every point, and the Hausdorff dimension of* $\mathbf{H}$ *is greater than some constant* $D > 1$ *which depends only on* d_0, d_∞, N, *and the dilatation of* $\mathbf{H}$.

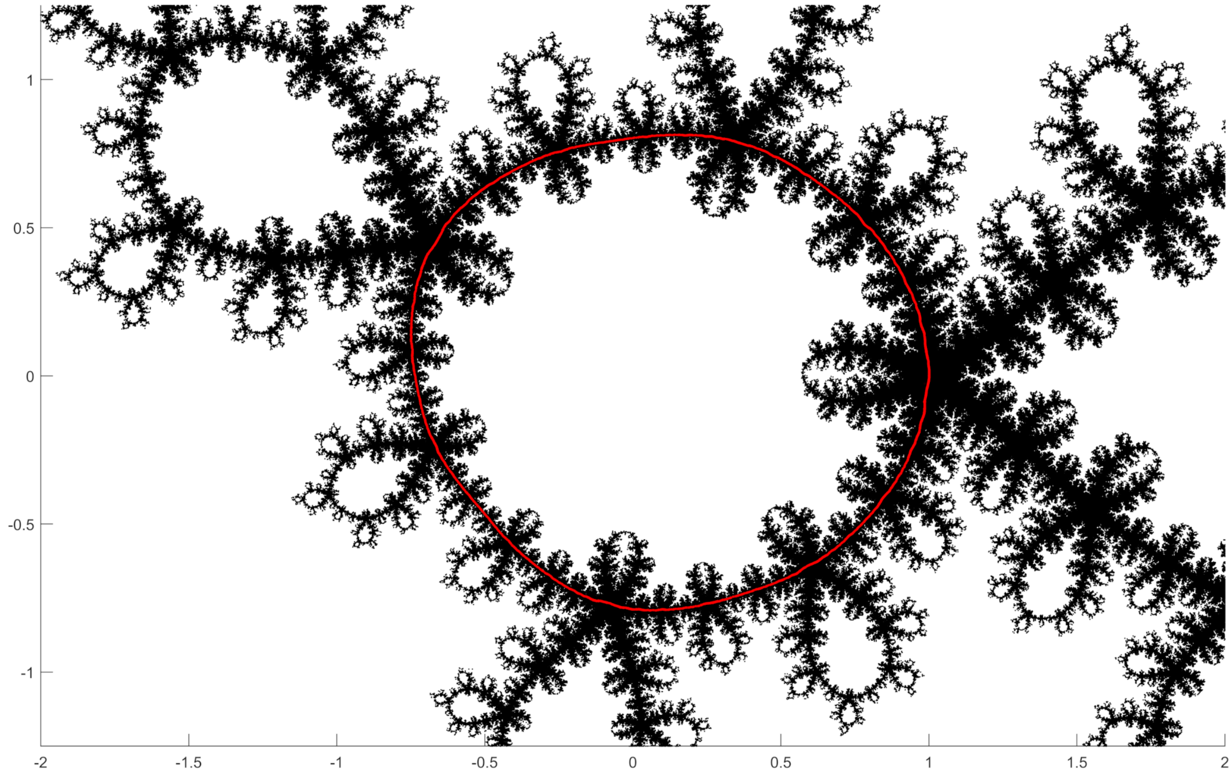

Figure 4.8: A C^1 smooth $(2,2)$-critical Herman curve $\mathbf{H}$ within the Julia set of the cubic rational map

$$f(z) = cz\frac{z^2 - 3z + \frac{\lambda}{c}}{1 + \left(\frac{\lambda}{c} - 3\right)z},$$

which is a perturbation of a Blaschke product of the form $cz^2\frac{z-3}{1-3z}$. It is characterized by fixed points at 0 and ∞ with multipliers $\lambda = 0.9e^{2\pi i\theta}$ and 0 respectively, as well as a critical point at 1 with local degree 3. The critical value $c = f(1) \approx -0.507844 - 0.457336i$ is picked such that $\mathbf{H}$ exists with golden mean rotation number θ.

Proof. Suppose $d_0 \neq d_\infty$. Given a point $x \in \mathbf{H}$ and a small scale $r > 0$, we define

$$\beta_{\mathbf{H}}(x,r) := \frac{1}{r} \inf_{L \in \mathcal{L}(x,r)} \sup_{z \in \mathbb{D}(x,r)} \operatorname{dist}(z, L),$$

where $\mathcal{L}(x,r)$ denotes the set of lines in $\mathbb{C}$ intersecting the disk $\mathbb{D}(x,r)$, as well as

$$\beta_{\mathbf{H}}(x) := \liminf_{r \to 0} \beta_{\mathbf{H}}(x,r).$$

The quantity $\beta_{\mathbf{H}}(x,r)$ measures how far $\mathbf{H}$ is from being a line segment near x at scale r. Due to a result by Bishop and Jones [BJ97], it is sufficient for us to show that $\beta_{\mathbf{H}}(x)$ is uniformly bounded above by some positive constant depending on d_0, d_∞, N, and the dilatation K of $\mathbf{H}$.

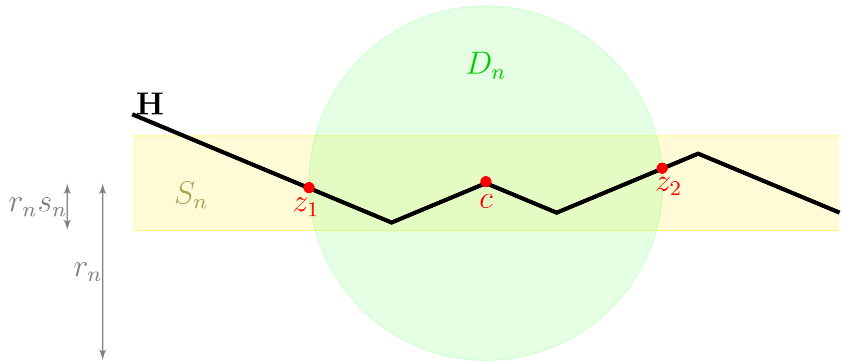

Figure 4.9: $\mathbf{H} \cap D_n$ is contained in the strip S_n.

In the proof below, we will first show that the beta number at the critical point c is positive. We then transfer this property around c to every non-critical point via Koebe distortion. To do this, we will use the *bounded turning* characterization of quasicircles, that is, for any two distinct points a and b on $\mathbf{H}$, the ratio of $|a-b|$ to the diameter of the interval $[a,b] \subset \mathbf{H}$ connecting a and b of the smallest diameter is bounded below by some positive constant depending on K.

Claim. There is some $\beta_0 = \beta_0(d_0, d_\infty, K) > 0$ such that $\beta_{\mathbf{H}}(c) \geq \beta_0$.

Proof. Suppose instead that $\beta_{\mathbf{H}}(c) < \beta_0$ where β_0 is a small constant that is to be determined. Then, there exist sequences of positive real numbers $\{r_n\}$ and $\{s_n\}$ such that $r_n \to 0$ and $s_n \to \beta_0$ as $n \to \infty$, and the intersection of $\mathbf{H}$ and the disk $D_n := \mathbb{D}(c, r_n)$ is contained in the $r_n s_n$-neighborhood S_n of a straight line passing through c. See Figure 4.9.

Let us label the two connected components of $\partial D_n \cap S_n$ by X_1 and X_2. Let $\Gamma = [z_1, z_2]$ denote the closed interval that is the connected component of $\overline{D_n} \cap \mathbf{H}$ containing c. We claim that $z_1 \in X_1$ and $z_2 \in X_2$. Indeed, if otherwise,

$$\frac{|z_1 - z_2|}{\operatorname{diam}(\Gamma)} \leq \frac{\max_i \operatorname{diam}(X_i)}{\operatorname{diam}([c, z_1])} = O(s_n),$$

and as $n \to \infty$, the right hand side becomes small depending on β_0. When β_0 is sufficiently small depending on K, this estimate would contradict the bounded turning property.

Suppose for a contradiction that the local degree $d = d_0 + d_\infty - 1$ is even. Near c, the map f is close to the power map

$$g(z) := f(c) + \frac{f^{(d)}(c)}{d!}(z-c)^d,$$

that is, $|f(z) - g(z)| = O(|z-c|^{d+1})$. Since d is even, g will send both z_1 and z_2 to points that are very close to one another and

$$\begin{aligned}|f(z_1) - f(z_2)| &\le |g(z_1) - g(z_2)| + O(|z_1 - c|^{d+1}) + O(|z_2 - c|^{d+1}) \\ &= O\left(r_n^d(s_n^d + r_n)\right).\end{aligned}$$

However, since the interval $f(\Gamma) = [f(z_1), f(z_2)] \subset \mathbf{H}$ has diameter at least $\asymp r_n^d$,

$$\frac{|f(z_1) - f(z_2)|}{\operatorname{diam}(f(\Gamma))} = O(s_n^d + r_n).$$

Again, as $n \to \infty$, this estimate would contradict the bounded turning property provided that β_0 is sufficiently small depending on d_0, d_∞, and K. Therefore, d must be odd.

Since d is odd, the image $S_n' := f(S_n \cap D_n)$ is close to being a straight strip of width $\asymp r_n^d s_n$ inside of $f(D_n)$, which is close to a round disk of radius $\asymp r_n^d$. Denote by Y^0 and Y^∞ the inner and outer components of $\hat{\mathbb{C}} \backslash \mathbf{H}$.

We claim that the two connected components of $f(D_n) \backslash S_n'$ belong to different components of $\hat{\mathbb{C}} \backslash \mathbf{H}$, which we will denote by $B^0 \subset Y^0$ and $B^\infty \subset Y^\infty$. Indeed, suppose instead that both are contained in Y^∞ without loss of generality. There is some $i \in \{1,2\}$ such that $f(X_i) \cup Y^\infty$ contains a Jordan curve enclosing $f(X_j)$ where $j \in \{1,2\} \backslash \{i\}$. However, this would imply the existence of a closed interval in $\mathbf{H}$ having endpoints in $f(X_i)$ and diameter $\succ r_n^d$, which would again contradict the bounded turning property.

For $\bullet \in \{0, \infty\}$, the number of components of $f^{-1}(B^\bullet)$ contained in $Y^\bullet$ is $d_\bullet$. Since f is close to the d^{th} power map g and S_n' is close to being a thin straight strip, then $d_0 = d_\infty$. This yields a contradiction. ☐

Let us pick any point x on $\mathbf{H}$ and any sufficiently small scale $r > 0$. By Lemma 4.1.4, there is an approximate rotation $f^i : (U, y) \to (V, c)$ such that y lies on $\mathbf{H}$, c is a critical point of f, and (U, y) is a pointed disk that well contains the interval $[x, y] \subset \mathbf{H}$ and has bounded shape and inner radius $r_{\text{in}}(U, y) \asymp r$. (Note that, from this moment on, implicit constants may depend on N.)

Let us denote by δ the inner radius of (V, c). Consider a small disk $B_\varepsilon = \mathbb{D}(y, \varepsilon r)$ where $0 < \varepsilon < 1$. Since f^i has bounded distortion on U, the image $f^i(B_\varepsilon)$ will have bounded shape and diameter $\asymp \varepsilon\delta$.

The claim implies that there is some constant $C = C(d_0, d_\infty, K) > 0$ such that for sufficiently small r (and thus δ), we can find an interval $[a', b'] \subset \mathbf{H}$ contained in $f^i(B_\varepsilon)$ and a point w' on $\mathbf{H}$ such that the distance between w' and the unique straight line $L_{a',b'}$ passing through a' and b' is at least $C\varepsilon\delta$. Denote by a, b, and w the lift of a', b', and w' under $f^i|_U$.

Since f^i has bounded distortion on U, the Euclidean triangle with vertices a', b', w' should be almost similar to that with vertices a, b, w. More precisely, there is some constant $M_\varepsilon > 0$ which shrinks to 0 as $\varepsilon \to 0$ such that

$$\left| \frac{a-w}{b-w} \Big/ \frac{a'-w'}{b'-w'} - 1 \right| \leq M_\varepsilon.$$

Therefore, we can pick ε depending on C such that the distance between w and the unique straight line $L_{a,b}$ passing through a and b satisfies $\operatorname{dist}(w, L_{a,b}) \succ C\varepsilon r$. Together with the bounded turning property, this implies that $\beta_{\mathbf{H}}(x,r) \geq \beta$ for some $\beta = \beta(d_0, d_\infty, K, N) > 0$. □

4.6.2 Universality

Let $f_1 : \mathbf{H}_1 \to \mathbf{H}_1$ and $f_2 : \mathbf{H}_2 \to \mathbf{H}_2$ be two (d_0, d_∞)-critical quasicircle maps of the same rotation number θ. By $C^{1+\alpha}$ rigidity, there exists a uniformly $C^{1+\alpha}$-conformal conjugacy $\phi : \mathbf{H}_1 \to \mathbf{H}_2$ between f_1 and f_2. Corollaries F and G will follow from below.

Corollary 4.6.4. *Quasicircles $\mathbf{H}_1$ and $\mathbf{H}_2$ have the same Hausdorff dimension, lower box dimension, and upper box dimension.*

Proof. By Lemma 4.6.2, the complex derivative ϕ' is continuous. Since quasicircles are compact, the map ϕ must be bi-Lipschitz. Since dimension is bi-Lipschitz invariant, the claim follows. □

We say that a sequence $\{a_n\}_{n\in\mathbb{N}}$ of complex numbers *converges exponentially fast* to a if there are constants $C > 0$, $n_0 \in \mathbb{N}$, and $\lambda \in (0,1)$ such that for all $n \geq n_0$, $|a_n - a| \leq C\lambda^n$. Given some data $\heartsuit$, we also say that $\{a_n\}$ converges $\heartsuit$-exponentially fast to a if the constants C, n_0, λ depend only on $\heartsuit$.

Recall the notion of scaling ratios in (1.1.2).

Corollary 4.6.5. *Asymptotically, f_1 and f_2 have the same scaling ratios:*

$$\frac{s_n(f_2)}{s_n(f_1)} \to 1 \quad \textit{exponentially fast as } n \to \infty.$$

Proof. Assume without loss of generality that 0 is the critical point of both f_1 and f_2, and let $c_n := f_1^n(0)$. By Lemma 2.1.8 (2), the bounded type assumption implies that $|c_{q_n}| \asymp \delta^n$ for some $\delta \in (0,1)$.

Near 0, we can write $\phi(z) = z\,(\lambda + \eta(z))$ where $\lambda \in \mathbb{C}^*$ and $\eta(z) = O(|z|^\alpha)$. Then,

$$\frac{s_n(f_2)}{s_n(f_1)} - 1 = \frac{\phi(c_{q_{n+1}})}{c_{q_{n+1}}} \cdot \frac{c_{q_n}}{\phi(c_{q_n})} - 1 = \frac{\eta(c_{q_{n+1}}) - \eta(c_{q_n})}{\lambda + \eta(c_{q_n})} = O(\delta^{\alpha n}).$$

Therefore, the ratio $s_n(f_2)/s_n(f_1)$ tends to 1 exponentially fast. □

4.6.3 Exponential convergence of renormalizations

Let us fix $K > 1$ and $\mu > 0$. For brevity, we will denote by $\clubsuit$ the data $(d_0, d_\infty, N, K, \mu)$.

Theorem 4.6.6 (Exponential convergence)**.** *Given any two commuting pairs ζ and $\hat{\zeta}$ in $\mathcal{CP}(\clubsuit)$ with the same rotation number, their renormalizations converge together exponentially fast in the following sense. Let us denote the n^{th} renormalization of ζ and $\hat{\zeta}$ by $\mathcal{R}^n\zeta = (f_{n,-}|_{I_{n,-}}, f_{n,+}|_{I_{n,+}})$ and $\mathcal{R}^n\hat{\zeta} = \left(\hat{f}_{n,-}|_{\hat{I}_{n,-}}, \hat{f}_{n,+}|_{\hat{I}_{n,+}}\right)$ respectively. Then,*

(1) *the Hausdorff distance between $I_{n,\pm}$ and $\hat{I}_{n,\pm}$ tends to 0 $\clubsuit$-exponentially fast;*

(2) *for sufficiently large n depending on $\clubsuit$, both $f_{n,\pm}$ and $\hat{f}_{n,\pm}$ extend holomorphically to the $\varepsilon(\clubsuit)$-neighborhood of $I_{n,\pm} \cup \hat{I}_{n,\pm}$ on which the sup norm of $f_{n,\pm} - \hat{f}_{n,\pm}$ converges $\clubsuit$-exponentially fast to 0.*

Proof. Let ψ be a quasiconformal conjugacy between ζ and $\hat{\zeta}$. Recall that the renormalization $\mathcal{R}^n\zeta$ is obtained by conjugating $p\mathcal{R}^n\zeta$ with the map

$$\tau_n(z) := \begin{cases} -c_n z, & \text{if } n \text{ is odd,} \\ -c_n \bar{z}, & \text{if } n \text{ is even,} \end{cases}$$

sending -1 to an endpoint c_n of the base of $p\mathcal{R}^n\zeta$. Similarly, denote by $\hat{\tau}_n(z)$ the corresponding rescaling map for $p\mathcal{R}^n\hat{\zeta}$ with scaling factor $-\hat{c}_n$ where $\hat{c}_n := \psi(c_n)$.

By Theorem 4.4.18, there are constants $n_1 = n_1(\clubsuit) \in \mathbb{N}$ and $L = L(\clubsuit) > 1$ such that for all $n \geq n_1$, the pre-renormalization $p\mathcal{R}^n\zeta$ extends to an L-butterfly with range V_n. Denote the range of the corresponding butterfly for $\mathcal{R}^n\zeta$ by $\mathcal{V}_n = \tau_n^{-1}(V_n)$. By (G$_4$),

$$\operatorname{diam}(\mathcal{V}_n) \leq L \quad \text{and} \quad \operatorname{diam}(V_n) \leq |c_n| L. \tag{4.6.1}$$

Since ψ is $C^{1+\alpha}$-conformal at the critical point 0, there exist positive constants α, C, λ, r depending only on $\clubsuit$ such that for $|z| < r$,

$$\psi(z) = \lambda z + \eta(z), \qquad \text{where} \qquad |\eta(z)| \leq C|z|^{1+\alpha}. \tag{4.6.2}$$

The sequence $\{c_n\}_{n\in\mathbb{N}}$ converges $\clubsuit$-exponentially fast to 0 due to Lemma 2.1.8 (1). By (4.6.1), there is some $n_2 = n_2(\clubsuit) \in \mathbb{N}$ such that for $n \geq n_2$, the disk V_n has diameter at most r. By (4.6.2), for $n \geq n_2$,

$$\sup_{z\in\mathcal{V}_n} \left| \frac{\eta(-c_n z)}{\hat{c}_n} \right| \leq C \frac{|c_n|^{1+\alpha}}{|\hat{c}_n|} \sup_{z\in\mathcal{V}_n} |z|^{1+\alpha} \leq C L^{1+\alpha} \left| \frac{c_n}{\hat{c}_n} \right| |c_n|^\alpha. \tag{4.6.3}$$

Since $\psi(c_n) = \hat{c}_n$, then by (4.6.2) again, the sequence of ratios $\{\hat{c}_n/c_n\}_{n\in\mathbb{N}}$ converges ♣-exponentially fast to the derivative λ of ψ at 0. Therefore, the sequence

$$\sup_{z\in\mathcal{V}_n}\left|\frac{\eta(-c_n z)}{\hat{c}_n}\right|$$

also converges ♣-exponentially fast to 0.

The map $\psi_n := \hat{\tau}_n^{-1}\circ\psi\circ\tau_n$ conjugates $\mathcal{R}^n\zeta_1$ and $\mathcal{R}^n\zeta_2$. For all even $n \geq n_2$ and $z \in \mathcal{V}_n$,

$$|\psi_n(z) - z| \leq \left|\lambda\frac{c_n}{\hat{c}_n} - 1\right||z| + \left|\frac{\eta(-c_n z)}{\hat{c}_n}\right|,$$

and a similar estimate holds for odd n. This implies that the sup norm of $\psi_n - \mathrm{Id}$ on $\mathcal{V}_n$ converges ♣-exponentially fast to 0, and items (1) and (2) follow immediately. □

4.6.4 A horseshoe attractor

Let $\mathcal{CP}(d_0, d_\infty, N)$ be the space of all normalized (d_0, d_∞)-critical commuting pairs ζ with rotation number in Θ_N. Denote by σ the shift map acting on the bi-infinite shift space $\Sigma_N = \{1,\dots,N\}^{\mathbb{Z}}$ of N symbols, equipped with the infinite product topology. Consider the continuous surjection

$$\xi : \Sigma_N \to \Theta_N, \quad (\dots, a_{-2}, a_{-1}, a_0; a_1, a_2, \dots) \mapsto [0; a_1, a_2, \dots].$$

We now prove a more precise formulation of Theorem I.

Theorem 4.6.7 (Renormalization horseshoe)**.** *There is a unique renormalization-invariant compact subset $\mathcal{A} = \mathcal{A}(d_0, d_\infty, N)$ of $\mathcal{CP}(d_0, d_\infty, N)$ satisfying the following properties.*

(1) *There is a topological conjugacy $\Phi : \Sigma_N \to \mathcal{A}$ between the renormalization operator $\mathcal{R} : \mathcal{A} \to \mathcal{A}$ and the shift map $\sigma : \Sigma_N \to \Sigma_N$ such that* $\mathrm{rot}\circ\Phi = \xi$.

(2) *For any $\zeta \in \mathcal{A}$ and $\zeta' \in \mathcal{CP}(d_0, d_\infty, N)$, renormalizations $\mathcal{R}^n\zeta$ and $\mathcal{R}^n\zeta'$ converge together exponentially fast if and only if $\mathcal{R}^m\zeta$ has the same rotation number as $\mathcal{R}^m\zeta'$ for some $m \in \mathbb{N}$.*

In the proof below, we obtain the horseshoe by constructing limits of renormalization towers, and we deduce the rigidity of towers by applying the exponential convergence of renormalizations.

Proof. Consider a bi-infinite sequence $\mathbf{a} = (\dots, a_{-2}, a_{-1}, a_0; a_1, a_2, \dots)$ in Σ_N. For any $k \in \mathbb{Z}$, set $\theta_k := [0; a_{k+1}, a_{k+2}, a_{k+3}, \dots]$ and let f_k be the rational map $f_{d_0,d_\infty,\theta_k}$. By Lemma 4.4.3, whenever $k + l \geq 1$, the $(k+l)^{\text{th}}$ renormalization $\zeta_{k,l} := \mathcal{R}^{k+l} f_{-k}$ has rotation number θ_l.

According to Theorem 4.4.18, if $k+l$ is sufficiently high, then $\zeta_{k,l}$ always extends to a K-butterfly for some $K = K(d_0, d_\infty, N) > 1$. By Proposition 4.4.15, for any $l \in \mathbb{Z}$, there is a subsequence $\{\zeta_{k(i),l}\}_{i\in\mathbb{N}}$ such that as $i \to \infty$, then $k(i) \to \infty$ and $\zeta_{k(i),l}$ converges to some (d_0, d_∞)-critical commuting pair ζ_l of rotation number θ_l. By a diagonal procedure, we can ensure that $\mathcal{R}\zeta_l = \zeta_{l+1}$, giving us a bi-infinite renormalization tower

$$\mathcal{T}_{\mathbf{a}} := (\ldots, \zeta_{-2}, \zeta_{-1}, \zeta_0;\ \zeta_1, \zeta_2, \ldots)$$

of commuting pairs in $\mathcal{CP}(d_0, d_\infty, N)$ where each entry ζ_n extends to a K-butterfly.

Suppose this procedure yields another renormalization tower

$$\mathcal{T}'_{\mathbf{a}} := (\ldots, \zeta'_{-2}, \zeta'_{-1}, \zeta'_0;\ \zeta'_1, \zeta'_2, \ldots).$$

By Theorem 4.6.6, as $n \to \infty$, renormalizations $\mathcal{R}^k\zeta_n$ and $\mathcal{R}^k\zeta'_n$ converge together exponentially fast at a uniform rate independent of k. Since $\mathcal{R}^k\zeta_n = \zeta_{n+k}$ and $\mathcal{R}^k\zeta'_n = \zeta'_{n+k}$, this implies that $\zeta'_n = \zeta_n$ for all $n \in \mathbb{Z}$. Therefore, the tower $\mathcal{T}_{\mathbf{a}}$ is uniquely defined.

Set $\Phi(\mathbf{a})$ to be the zeroth entry ζ_0 of $\mathcal{T}_{\mathbf{a}}$ and set

$$\mathcal{A} := \{\Phi(\mathbf{a}) \in \mathcal{CP}(d_0, d_\infty, N) \,:\, \mathbf{a} \in \Sigma_N\}.$$

We have a surjective mapping $\Phi : \Sigma_N \to \mathcal{A}$ satisfying $\mathrm{rot} \circ \Phi = \xi$. Since Σ_N is compact and Hausdorff, in order to prove (1), it remains to show that Φ is injective and continuous.

Suppose $\Phi(\mathbf{a}_1) = \Phi(\mathbf{a}_2)$ for some $\mathbf{a}_1, \mathbf{a}_2 \in \Sigma_N$. By the identity theorem, the associated towers are equal, namely $\mathcal{T}_{\mathbf{a}_1} = \mathcal{T}_{\mathbf{a}_2}$. For each $i \in \{1, 2\}$, we can recover back the bi-infinite sequence $\mathbf{a}_i$ by evaluating the rotation number of each entry of $\mathcal{T}_{\mathbf{a}_i}$, so then $\mathbf{a}_1 = \mathbf{a}_2$. Therefore, Φ is injective.

Let us show that Φ is continuous. Suppose a sequence of elements $\mathbf{a}^{(n)}$ in Σ_N converges to $\mathbf{a}$. For each $n \in \mathbb{N}$, denote the associated tower by

$$\mathcal{T}_{\mathbf{a}^{(n)}} = (\ldots, \zeta_{-2}^{(n)}, \zeta_{-1}^{(n)}, \zeta_0^{(n)};\ \zeta_1^{(n)}, \zeta_2^{(n)}, \ldots).$$

By passing to a subsequence, each $\zeta_k^{(n)}$ converges to some commuting pair ζ_k as $n \to \infty$, forming a limiting renormalization tower

$$\mathcal{T} = (\ldots, \zeta_{-2}, \zeta_{-1}, \zeta_0;\ \zeta_1, \zeta_2, \ldots).$$

For every $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, Corollary 4.4.11 states that there is a unique quasisymmetric conjugacy $h_k^{(n)}$ between the pair of translations $\mathbf{T}_{\theta_k^{(n)}}$ and $\zeta_k^{(n)}$. Clearly, as $\mathbf{a}^{(n)} \to \mathbf{a}$, the pair $\mathbf{T}_{\theta_k^{(n)}}$ converges to $\mathbf{T}_{\theta_k}$. Since the dilatation of $h_k^{(n)}$ is uniform, the sequence $h_k^{(n)}$ subsequently converges to a quasisymmetric map h_k, which conjugates $\mathbf{T}_{\theta_k}$ with ζ_k. In particular, $\operatorname{rot}(\zeta_k) = \theta_k$. By the uniqueness of renormalization towers, $\mathcal{T}$ coincides with $\mathcal{T}_{\mathbf{a}}$. Therefore, $\Phi\left(\mathbf{a}^{(n)}\right) \to \Phi(\mathbf{a})$.

Let us now prove property (2). Pick $\zeta \in \mathcal{A}$ and $\zeta' \in \mathcal{CP}(d_0, d_\infty, N)$. If $\mathcal{R}^m\zeta$ and $\mathcal{R}^m\zeta'$ have the same rotation number for some $m \in \mathbb{N}$, then again by Theorem 4.6.6, $\mathcal{R}^n\zeta$ and $\mathcal{R}^n\zeta'$ converge together exponentially fast. Otherwise, there is an infinite sequence $\{n_k\}_{k\in\mathbb{N}}$ such that the continued fraction expansions of the rotation numbers of $\mathcal{R}^{n_k}\zeta$ and $\mathcal{R}^{n_k}\zeta'$ have different first term, which clearly implies that $\mathcal{R}^{n_k}\zeta$ and $\mathcal{R}^{n_k}\zeta'$ cannot converge together. □

4.6.5 Self-similarity

Consider a bi-infinite sequence

$$\mathbf{a} = (\ldots, a_{-2}, a_{-1}, a_0; a_1, a_2, \ldots\ldots) \in \Sigma_N$$

such that $\mathbf{a}$ is s-periodic for some $s \geq 1$, i.e. $\mathbf{a} = \sigma^s\mathbf{a}$. To lighten our notation, we assume that s is even. (Else, replace s by $2s$.) Let

$$\theta_0 := [0; a_1, a_2, a_3, \ldots].$$

Then, θ_0 is also s-periodic under the Gauss map $G(x) = \left\{\frac{1}{x}\right\}$.

We say that two subsets P and Q of $\mathbb{C}$ are *linearly equivalent* if there is a linear map g such that $g(P) = Q$. Below is a more precise version of Theorem H.

Theorem 4.6.8 (Self-similarity)**.** *There exists a complex number $\mu_{\mathbf{a}} \in \mathbb{D}^*$ such that the following holds. Let $A_{\mathbf{a}}(z) := \mu_{\mathbf{a}} z$.*

(1) *Consider the conjugacy $\Phi : \Sigma_N \to \mathcal{A}$ described in Theorem 4.6.7. The base of $\Phi(\mathbf{a})$ extends to a unique $A_{\mathbf{a}}$-invariant quasicircle $\mathbf{H}_{\mathbf{a}}$.*

(2) *$\mu_{\sigma^2\mathbf{a}} = \mu_{\mathbf{a}}$, and $\mathbf{H}_{\sigma^2\mathbf{a}}$ is linearly equivalent to $\mathbf{H}_{\mathbf{a}}$.*

(3) *Suppose $f : \mathbf{H} \to \mathbf{H}$ is a (d_0, d_∞)-critical quasicircle map of rotation number θ where $G^k(\theta) = \theta_0$ for some even integer $k \geq 0$. Assume 0 is the critical point of f.*

   (a) *$A_{\mathbf{a}}^{-n}(\mathbf{H})$ converges in the Hausdorff metric to an $A_{\mathbf{a}}$-invariant quasicircle linearly equivalent to $\mathbf{H}_{\mathbf{a}}$.*

   (b) *Let $c_l := f^l(0)$ for all $l \in \mathbb{N}$. Then,*

$$\frac{c_{q_{n+s}}}{c_{q_n}} = \prod_{i=1}^{s} s_{n+i}(f) \to \mu_{\mathbf{a}} \qquad \textit{exponentially fast as } n \to \infty.$$

Proof. Since $\zeta := \Phi(\mathbf{a})$ satisfies the equation $\mathcal{R}^s\zeta = \zeta$, there is a linear map $A_{\mathbf{a}}(z) := \mu_{\mathbf{a}} z$ such that $0 < |\mu_{\mathbf{a}}| < 1$ and $p\mathcal{R}^s\zeta = A_{\mathbf{a}} \circ \zeta \circ A_{\mathbf{a}}^{-1}$. This immediately implies (1). Also, we have $p\mathcal{R}^{s+2}\zeta = A_{\mathbf{a}} \circ p\mathcal{R}^2\zeta \circ A_{\mathbf{a}}^{-1}$, which after normalization yields the equation $p\mathcal{R}^s(\mathcal{R}^2\zeta) = A_{\mathbf{a}} \circ \mathcal{R}^2\zeta \circ A_{\mathbf{a}}^{-1}$. This implies (2).

Suppose $f : \mathbf{H} \to \mathbf{H}$ satisfies the hypothesis in (3). By Theorem D, the quasisymmetric conjugacy ψ between $\mathcal{R}^k f$ and ζ extends to a quasiconformal conjugacy that is $C^{1+\alpha}$ near the critical point. Let $\phi(z) := \psi(z)/\psi'(0)$, which conjugates $\mathcal{R}^k f$ with a rescaling of ζ, and satisfies

$$\phi(z) = z\,(1 + \eta(z)) \quad \text{where} \quad \eta(z) = O(|z|^\alpha).$$

Denote by $\hat{\mathbf{H}}_{\mathbf{a}}$ the rescaling of $\mathbf{H}_{\mathbf{a}}$ by $\psi'(0)^{-1}$.

For any sufficiently large $n \in \mathbb{N}$, there is some $l \geq k$ such that the base $\mathbf{I} \subset \mathbf{H}$ of $p\mathcal{R}^l f$ has diameter $\asymp |\mu_{\mathbf{a}}|^n$. Denote by $d_H(\cdot,\cdot)$ the Hausdorff distance. Then,

$$d_H\left(A_{\mathbf{a}}^{-n}(\mathbf{I}), A_{\mathbf{a}}^{-n}\phi(\mathbf{I})\right) \leq |\mu_{\mathbf{a}}|^{-n} d_H\left(\mathbf{I}, \phi(\mathbf{I})\right) \prec |\mu_{\mathbf{a}}|^{-n}\mathrm{diam}(\mathbf{I})^{1+\alpha} \prec |\mu_{\mathbf{a}}|^{\alpha n}.$$

As $n \to \infty$, the Hausdorff distance between $A_{\mathbf{a}}^{-n}(\mathbf{I})$ and $A_{\mathbf{a}}^{-n}\phi(\mathbf{I})$, which is a subinterval of $\hat{\mathbf{H}}_{\mathbf{a}}$, shrinks exponentially fast.

The proof of (b) is similar to Corollary 4.6.5. Let us write $\phi(z) = z\,(1 + \eta(z))$ where $\eta(z) = O(|z|^\alpha)$. By Lemma 2.1.8 (2), there is some $\delta \in (0,1)$ such that $|c_{q_n}| \asymp \delta^n$. For $n \geq k$, we have the equation $\phi(c_{q_{n+s}}) = A_{\mathbf{a}}\phi(c_{q_n})$, which implies that as $n \to \infty$,

$$\mu_{\mathbf{a}} - \frac{c_{q_{n+s}}}{c_{q_n}} = \frac{\phi(c_{q_{n+s}})}{\phi(c_{q_n})} - \frac{c_{q_{n+s}}}{c_{q_n}} = \frac{c_{q_{n+s}}}{c_{q_n}} \cdot \frac{\eta(c_{q_{n+s}}) - \eta(c_{q_n})}{1 + \eta(c_{q_n})} = O(\delta^{\alpha n}).$$

Therefore, the ratio $c_{q_{n+s}}/c_{q_n}$ tends to $\mu_{\mathbf{a}}$ exponentially fast. $\square$

Chapter 5

Hyperbolicity of Renormalization

5.1 Corona renormalization operator

5.1.1 (d_0, d_∞)-critical coronas

For any open annulus A compactly contained in $\mathbb{C}$, we label the boundary components of A by $\partial^0 A$ and $\partial^\infty A$, and make the convention that $\partial^\infty A$ is the outer boundary, i.e. the one that is closer to ∞. We also say that another annulus A' is *essentially* contained in A if A' is a deformation retract of A.

Definition 5.1.1. A (d_0, d_∞)-*critical corona*[1] is a map $f : U \to V$ between two bounded open annuli in $\mathbb{C}$ with the following properties.

1. The boundary components of both U and V are Jordan curves, and U is compactly and essentially contained in V.

2. There is a proper arc $\gamma_1 \subset V$ connecting $\partial^0 V$ and $\partial^\infty V$ such that the preimage $f^{-1}(\gamma_1)$ is disjoint from γ_1 and is a union of $2d-1$ pairwise disjoint arcs

$$\gamma_0 \subset U, \quad \gamma_1^0, \ldots, \gamma_{2(d_0-1)}^0 \subset \partial^0 U, \quad \gamma_1^\infty, \ldots, \gamma_{2(d_\infty-1)}^\infty \subset \partial^\infty U.$$

3. $f : U \to V$ is holomorphic and $f : U \backslash \gamma_0 \to V \backslash \gamma_1$ is a degree d covering map branched at a unique critical point c_0.

The arc γ_1 is called the *critical arc* of f. See Figure 5.1 for an illustration.

[1] The shape of the domain U in Figure 5.1 resembles a crown or a wreath, which is what *corona* means in Latin.

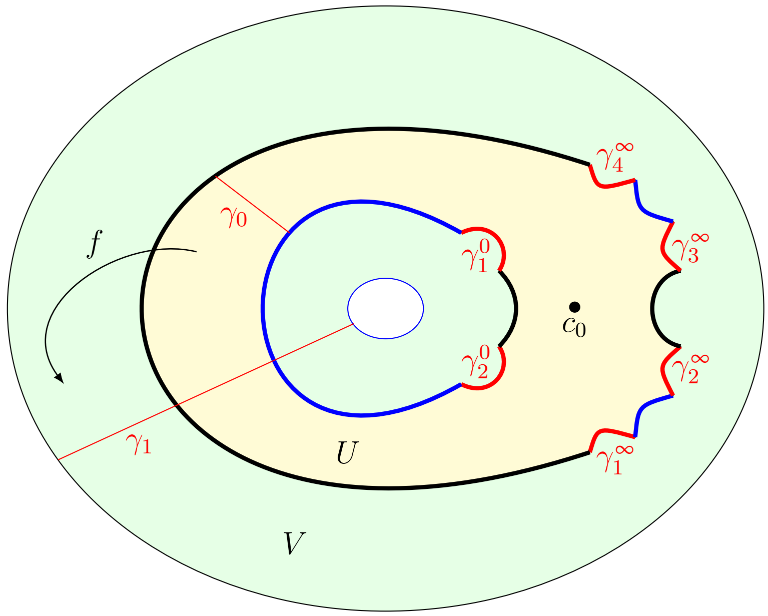

Figure 5.1: A (2,3)-critical corona.

Let $f : U \to V$ be a (d_0, d_∞)-critical corona. For any $\bullet \in \{0, \infty\}$, we divide the boundary component $\partial^\bullet U$ into

$$\partial^\bullet_{\mathrm{L}} U := \partial^\bullet U \cap f^{-1}(\partial^\bullet V) \quad \text{and} \quad \partial^\bullet_{\mathrm{F}} U := \partial^\bullet U \backslash f^{-1}(\partial^\bullet V)$$

according to whether or not it is mapped to the same side the annulus. Each of the above consists of $d_\bullet - 1$ components. Set

$$\partial_{\mathrm{L}} U := \partial^0_{\mathrm{L}} U \cup \partial^\infty_{\mathrm{L}} U \quad \text{and} \quad \partial_{\mathrm{F}} U := \partial^0_{\mathrm{F}} U \cup \partial^\infty_{\mathrm{F}} U.$$

We call $\partial_{\mathrm{L}} U$ the *legitimate boundary* of U and $\partial_{\mathrm{F}} U$ the *forbidden boundary* of U.

For each $\bullet \in \{0, \infty\}$, we properly embed a collection $\mathscr{R}^\bullet$ of $d_\bullet - 1$ pairwise disjoint rectangles within $V \backslash \overline{U}$ such that the union $B^\bullet$ of their bottom horizontal sides is precisely the legitimate boundary $\partial^\bullet_{\mathrm{L}} U$ and the union $T^\bullet$ of their top horizontal sides is a subset of $\partial^\bullet V$. Let us lift $\mathscr{R}^\bullet$ under f such that their top sides are within the legitimate boundary of U. As we repeat this lifting procedure, we obtain a lamination out of the iterated lifts, and its leaves will be called *external ray segments*.

An infinite chain of external ray segments is called an *external ray* of the corona f. We say that γ is an *inner* external ray if γ intersects B^0, and an *outer* external ray if instead γ intersects B^∞.

For each $\bullet \in \{0, \infty\}$, define the map $\pi_\bullet : B^\bullet \to T^\bullet$ sending the bottom endpoint of each leaf of $\mathscr{R}^\bullet$ to the corresponding top endpoint. Consider the partially defined $d_\bullet$ to one self map

$\phi_\bullet := \pi_\bullet^{-1} \circ f$ on $B^\bullet$. Denote by $\mathcal{A}^\bullet$ the set of points of $B^\bullet$ which are invariant under $\phi_\bullet$. Let us identify $\mathbb{T}$ with the quotient $\mathbb{R}/\mathbb{Z}$. There is a semiconjugacy $\theta_\bullet : \mathcal{A}^\bullet \to \mathbb{T}$ between $\phi_\bullet : \mathcal{A}^\bullet \to \mathcal{A}^\bullet$ and the multiplication map $\mathbb{T} \to \mathbb{T}, x \mapsto d_\bullet x \pmod 1$, which is unique up to conjugation with addition by multiples of $\frac{1}{d_\bullet - 1}$.

Given an external ray γ of f, we denote the image by

$$f(\gamma) := f(\gamma \cap U)$$

which is also an external ray of f by definition. The *external angle* of γ is the angle $\theta_\bullet(x)$ where x is the unique point of intersection of γ and $B^\bullet$ for some $\bullet \in \{0, \infty\}$.

5.1.2 Corona renormalization

Definition 5.1.2. A (d_0, d_∞)*-critical pre-corona* is a pair of holomorphic maps

$$F = (f_- : U_- \to S,\ f_+ : U_+ \to S)$$

satisfying the following properties.

1. S is a topological rectangle with vertical sides β_- and β_+.

2. β_0 is a vertical arc in S dividing S into subrectangles T_- and T_+, where $\beta_\pm \subset \partial T_\pm$ and $U_\pm$ is a subrectangle of $T_\pm$ with vertical sides contained in $\beta_\pm$ and β_0.

3. There is a gluing map $\psi : \overline{S} \to \overline{V}$ such that $\psi(\beta_-) = \psi(\beta_+)$, ψ is conformal on a neighborhood of S and injective on $S \backslash (\beta_- \cup \beta_+)$, and ψ projects F into a (d_0, d_∞)-critical corona with critical arc $\psi(\beta_\pm)$.

The gluing map ψ will also be called the *renormalization change of variables* of F. It glues together $f_+(x) \in \beta_-$ and $f_-(x) \in \beta_+$ for every x in $\beta_0 \cap \partial U_\pm$. See Figure 5.2.

Definition 5.1.3. A corona $f : U \to V$ is *renormalizable* if there exists a pre-corona

$$F = (f^{k_-} : U_- \to S,\ f^{k_+} : U_+ \to S)$$

on a rectangle $S \subset V$ such that f^{k_-} and f^{k_+} are the first return maps back to S and

$$\mathbf{\Delta}_F = \bigcup_{i=0}^{k_- - 1} \overline{f^i(U_-)} \cup \bigcup_{j=0}^{k_+ - 1} \overline{f^j(U_+)}$$

is a closed annulus essentially contained in U. We call F the *pre-renormalization* of f, k_- and k_+ the *return times* of F, and $\mathbf{\Delta}_F$ the *renormalization tiling* of F. The corona obtained by projecting F under its gluing map is called the *renormalization* of f.

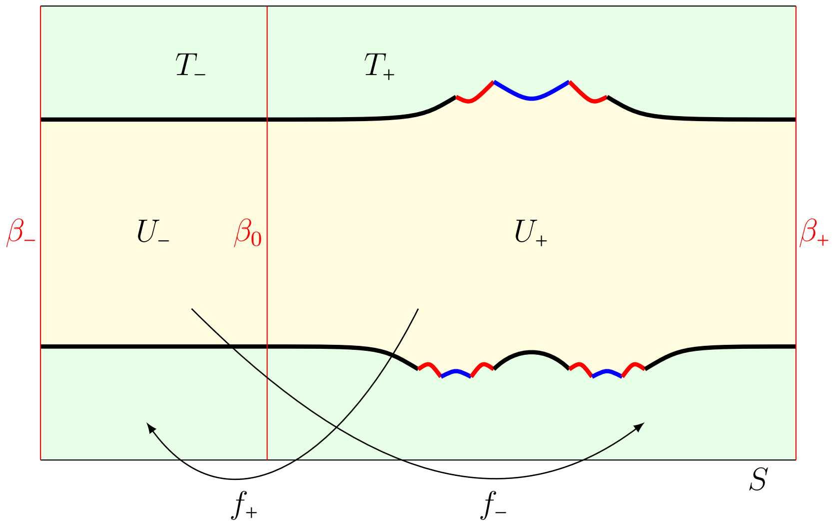

Figure 5.2: A (2,3)-critical pre-corona. It projects to the corona in Figure 5.1 after gluing β_+ and β_-

Example 5.1.4 (Prime renormalization)**.** We say that the renormalization of a corona $f : U \to V$ is *prime* if $k_- + k_+ = 3$. Below is an example of a prime corona renormalization.

Assume that the arcs γ_0, γ_1, and $\gamma_2 := f(\gamma_1)$ are pairwise disjoint. Denote by S_1 the open quadrilateral obtained by cutting V along $\gamma_1 \cup \gamma_2$ which does not contain γ_0. Let us assume further that S_1 does not contain the critical value nor the forbidden boundary of U.

Let us remove S_1 from the dynamical plane. We define $\hat{V}$ to be the Riemann surface with boundary obtained from $\overline{V} \backslash S_1$ by gluing $\gamma_1' := f^{-1}(\gamma_2) \cap \gamma_1$ and its image γ_2 along f. In other words, there is a quotient map $\psi : \overline{V} \backslash S_1 \to \hat{V}$ that is conformal on the interior and $\psi(z) = \psi(f(z))$ for all $z \in \gamma_1'$. We embed the abstract Riemann surface $\hat{V}$ into the plane.

The prime renormalization of f is defined by the induced first return map of f on $\hat{V}$. More precisely, consider the lift S_0 of S_1 under f attached to γ_1. The piecewise holomorphic map

$$\begin{cases} f(z), & \text{if } z \in U \backslash (S_1 \cup f^{-1}(S_1)), \\ f^2(z), & \text{if } z \in S_0 \cap f^{-1}(U). \end{cases}$$

descends via ψ into a corona $\hat{f} : \hat{U} \to \hat{V}$ with critical ray $\hat{\gamma}_1 = \psi(\gamma_1')$.

5.1.3 Banach neighborhood

In what follows, every unicritical holomorphic map $f : U \to V$ under consideration will be assumed to admit a slightly larger domain $\tilde{U}$ with piecewise smooth boundary such that $\tilde{U}$ compactly contains U and f extends to a unicritical holomorphic map on $\tilde{U}$ extending continuously to $\partial \tilde{U}$. We define a *Banach neighborhood* of f to be a neighborhood of f of the form $N_{\tilde{U}}(f,\varepsilon)$, which we define to be the space of holomorphic maps $g:\tilde{U}\to\mathbb{C}$ that extend continuously to $\partial\tilde{U}$, admit a single critical point in $c_0(g)$, and

$$\sup_{z\in\tilde{U}}|f(z)-g(z)|<\varepsilon.$$

We equip $N_{\tilde{U}}(f,\varepsilon)$ with the sup norm over $\tilde{U}$.

Lemma 5.1.5. *Let $f:U\to V$ be a (d_0,d_∞)-critical corona. For sufficiently small $\varepsilon>0$, there is a holomorphic motion ∂U_g of ∂U over $g\in N_{\tilde{U}}(f,\varepsilon)$ such that $g:U_g\to V$ is a (d_0,d_∞)-critical corona with the same codomain V and critical arc γ_1.*

Proof. Let A_δ be the δ-neighborhood of ∂U, where $\delta>0$ is picked small enough such that A_δ contains no critical points of f. For sufficiently small ε, the derivative of $g\in N_{\tilde{U}}(f,\varepsilon)$ is uniformly bounded and non-vanishing on A_δ, and so g has no critical points in A_δ. Thus, we have a well-defined map $\tau_g:\partial U\to A_\delta$ such that $\tau_f=\mathrm{Id}$ and $f=g\circ\tau_g$ on ∂U. Since f has no critical value along ∂U, $\tau_g(z)$ is injective in z and holomorphic in g. Therefore, we have a holomorphic motion of ∂U, and $\tau_g(\partial U)$ bounds an open annulus U_g on which $g:U_g\to V$ is a well-defined (d_0,d_∞)-critical corona with the same critical arc. □

The following theorem is inspired by Yampolsky's holomorphic motions argument [Yam03, §7]. See also [Yam08, Proposition 2.11] and [DL23, §2].

Theorem 5.1.6. *Suppose a unicritical holomorphic map $f:U\to V$ admits a pre-corona which projects to a corona $\hat{f}:\hat{U}\to\hat{V}$ via a quotient map $\psi_f:S_f\to\hat{V}$. For sufficiently small $\varepsilon>0$, there is a compact analytic renormalization operator $\mathcal{R}$ on a Banach neighborhood $N_{\tilde{U}}(f,\varepsilon)$ such that $\mathcal{R}f=\hat{f}$ and for each $g\in N_{\tilde{U}}(f,\varepsilon)$,*

1. *g admits a pre-corona which projects to the corona $\mathcal{R}g:\hat{U}_g\to\hat{V}$, and*
2. *the domain $\partial\hat{U}_g$ and the associated gluing map ψ_g depend holomorphically on g.*

Proof. There exists a pre-corona $F=(f^{k_\pm}:U_\pm\to S)$ and a quotient map ψ_f projecting F to $\hat{f}$. Recall the arcs $\beta_\pm$ and β_0 corresponding to F. For $g\in N_{\tilde{U}}(f,\varepsilon)$, consider the map $\tau_g:\beta_0\cup\beta_\pm\to\mathbb{C}$ defined by setting τ_g to be the identity map on β_0 and the composition $g^{k_\mp}\circ f^{-k_\mp}$ on $\beta_\pm$; this is an equivariant holomorphic motion of $\beta_0\cup\beta_\pm$ for sufficiently small $\varepsilon>0$. By λ-lemma [BR86; ST86], τ_g extends to a holomorphic motion of S over a neighborhood of f.

Let μ_g be the Beltrami differential of τ_g. Define a global Beltrami differential ν_g by setting $\nu_g=(\psi_f)_*\mu_g$ on $\hat{V}$ and $\nu_g\equiv 0$ outside of $\hat{V}$. Integrate ν_g to obtain a unique quasiconformal map ϕ_g fixing ∞, the critical point of f, and the critical value of f. Then, $\psi_g := \phi_g \circ \psi_f \circ \tau_g^{-1}$ is a conformal map on $S_g := \tau_g(S_f)$ depending holomorphically on g.

The gluing map ψ_g projects the pair (g^{k_-}, g^{k_+}) on S_g to a map $\hat{g}$ close to $\hat{f}$. By Lemma 5.1.5, $\hat{g}$ restricts to a corona that has the same range as $\hat{f}$ and depends analytically on g. This yields an analytic operator $g \mapsto \hat{g}$. To make this operator compact, we modify it as follows. Pick another annulus U' where $U \Subset U' \Subset \tilde{U}$. We define $\mathcal{R}$ on $N_{\tilde{U}}(f, \varepsilon)$ to be the renormalization of the restriction of g to U'. $\square$

5.2 Rotational coronas

Throughout this section, we fix a bounded type irrational $\theta \in \Theta_{bdd}$.

Definition 5.2.1 (Inner and outer criticalities)**.** Consider a quasicircle $\mathbf{H} \subset \mathbb{C}$ and denote the bounded and unbounded components of $\hat{\mathbb{C}} \backslash \mathbf{H}$ by Y^0 and Y^∞ respectively. We say that $f : \mathbf{H} \to \mathbf{H}$ is a (d_0, d_∞)-*critical quasicircle map* if it is a critical quasicircle map where for any $\bullet \in \{0, \infty\}$ and any point $z \in Y^\bullet$ close to the critical value of f, there are exactly $d_\bullet$ preimages of z in $Y^\bullet$ that are close to the critical point of f.

When a holomorphic map f is given, we also say that an invariant quasicircle $\mathbf{H} \subset \mathbb{C}$ is a (d_0, d_∞)-*critical Herman quasicircle* if $f : \mathbf{H} \to \mathbf{H}$ is a (d_0, d_∞)-critical quasicircle map.

Definition 5.2.2. A corona $f : U \to V$ is a *rotational corona* if

1. U essentially contains a Herman quasicircle $\mathbf{H}$ that passes through the unique critical point of f;

2. the critical arc γ_1 intersects $\mathbf{H}$ precisely at one point $m(f)$, which we call the *marked point* of f, which splits γ_1 into an inner external ray R^0 and an outer external ray R^∞.

A pre-corona is called *rotational* if it projects to a rotational corona under its renormalization change of variables.

By design, if a (d_0, d_∞)-critical corona is rotational, then it admits a (d_0, d_∞)-critical Herman quasicircle. In this section, we will construct rotational coronas out of critical quasicircle maps and discuss a rigidity property for rotational coronas.

5.2.1 Realization of rotational coronas

Consider our favorite rational map $f = f_{d_0, d_\infty, \theta} \in \mathcal{X}$ from Example 4.4.5. Denote its Herman quasicircle by $\mathbf{H}$. Using external rays, we will show in this subsection that f is corona renormalizable.

For any $n \geq 1$, we refer to the closure of a component of $f^{-n}(\mathbf{H}) \backslash f^{-(n-1)}(\mathbf{H})$ as a *bubble* of *generation* n. Every bubble B of generation n is a quasicircle admitting a unique point, which we will call the *root* of B, that lies on the pre-critical set $f^{-(n-1)}(1)$. We call a bubble B of generation n an *outer bubble* (resp. *inner bubble*) if the bubbles B, $f(B), \ldots, f^{n-1}(B)$ all lie in the connected component of $\hat{\mathbb{C}} \backslash \mathbf{H}$ containing ∞ (resp. 0).

A *limb* of generation one is the closure of a connected component of $J(f) \backslash \{1\}$ that is disjoint from $\mathbf{H}$. In general, a limb L of generation $n \geq 1$ is the connected component of the preimage under f^{n-1} of a limb of generation one. A *filled limb* $\hat{L}$ of generation n is the hull of a limb L of generation n, that is, $\hat{\mathbb{C}} \backslash \hat{L}$ is the unbounded connected component of $\hat{\mathbb{C}} \backslash L$.

Every limb L of generation n contains a unique bubble B_L of generation n. The *root* of L is the root of B_L. We call L an *outer/inner limb* if B_L is an outer/inner bubble.

Let us denote by A_0 and A_∞ the immediate attracting basins of 0 and ∞.

Lemma 5.2.3. *The boundary of A_0 is the closure of the union of $\mathbf{H}$ and all outer bubbles of f, whereas the boundary of A_0 is the closure of the union of $\mathbf{H}$ and all inner bubbles of f. Both ∂A_0 and ∂A_∞ are locally connected. For any $\varepsilon > 0$, all but finitely many inner and outer limbs of f have diameter at most ε.*

Proof. Denote by Y^0 and Y^∞ the connected components of $\hat{\mathbb{C}} \backslash \mathbf{H}$ containing 0 and ∞ respectively. Perform Douady-Ghys surgery [Ghy84; Dou87] (see also [BF14, §7.2]) along $\mathbf{H}$ to replace the dynamics of f in Y^0 with a rotation disk and obtain a degree d_∞ unicritical polynomial P_∞ whose critical point lies in the boundary of an invariant Siegel disk Z_∞ of P_∞. The maps $f|_{\overline{Y^\infty}}$ and $P_\infty|_{\hat{\mathbb{C}} \backslash Z_\infty}$ are quasiconformally conjugate, and this conjugacy sends A_∞ onto the immediate basin of ∞ of P_∞. In particular, the external boundary of the filled outer limbs of f are quasiconformally equivalent to the limbs of P_∞. The work of [Pet96] (or more generally [Wan+21]) guarantees that the Julia set of P_∞ is locally connected, and so any infinite sequence of limbs of P_∞ must shrink to a point. Therefore, for any $\varepsilon > 0$, all but finitely many outer limbs of f have diameter at most ε. By swapping the roles of 0 and ∞, we obtain the same result for inner limbs. □

Remark 5.2.4. The lemma above states that both ∂A_0 and ∂A_∞ are locally connected. In fact, the whole Julia set of f is locally connected. In case $(d_0, d_\infty) = (2, 2)$, this was proven by Petersen [Pet96, §4]. For arbitrary criticalities (d_0, d_∞), the availability of complex bounds (Theorem 4.4.16) facilitates a direct generalization of Petersen's proof.

For $\bullet \in \{0, \infty\}$, consider the Böttcher conjugacy $b_\bullet : (A_\bullet, \bullet) \to (\mathbb{D}, \bullet)$ between f and the power map $z^{d_\bullet}$. An *external ray* in $A_\bullet$ of angle $t \in \mathbb{R}/\mathbb{Z}$ is defined by

$$\left\{ b_\bullet^{-1}(re^{2\pi i t}) \,:\, 0 < r < 1 \right\},$$

and an *equipotential* in $A_\bullet$ of level λ is the analytic Jordan curve defined by

$$\left\{b_\bullet^{-1}(z) \,:\, |z| = e^{-\lambda}\right\}.$$

External rays and equipotentials form a pair of f-invariant transverse foliations of $A_\bullet$. According to Lemma 5.2.3, every external ray in $A_\bullet$ lands at a point on $\partial A_\bullet$. Every point x in $\partial A_\bullet$ is the landing point of exactly one external ray in $A_\bullet$, except when x is a critical point or its iterated preimage in which case it is the landing point of $d_\bullet$ external rays in $A_\bullet$.

Consider the map r_{prm} from §2.1.3, which encodes how rotation number is transformed under sector renormalization.

Lemma 5.2.5. *For any point $x \in \mathbf{H}$ that is not a pre-critical point of f, any $\varepsilon > 0$, and any sufficiently high $n \in \mathbb{N}$, there is a rotational pre-corona*

$$P = \left(f_- := f^{k_-} : U_- \to S,\ f_+ := f^{k_+} : U_+ \to S\right)$$

around x such that

(1) *P has rotation number $(r_{prm})^n(\theta)$;*

(2) *every external ray segment of P is within an external ray of P;*

(3) *the union $\bigcup_{\diamond \in \{-,+\}} \bigcup_{i=0}^{k_\diamond - 1} f^i(U_\diamond)$ lies in the ε-neighborhood of $\mathbf{H}$.*

Proof. For every integer $i \in \mathbb{Z}$, let $x_i := (f|_{\mathbf{H}})^i(x)$. By Lemma 2.1.11, for all $n \geq 1$, there exist return times $\mathbf{a}_n$, $\mathbf{b}_n$ such that the commuting pair

$$\left(f^{\mathbf{a}_n}|_{[x_{\mathbf{b}_n}, x_0]},\, f^{\mathbf{b}_n}|_{[x_0, x_{\mathbf{a}_n}]}\right)$$

is a sector pre-renormalization of $f|_{\mathbf{H}}$ with rotation number $(r_{\text{prm}})^n(\theta)$. (In short, the pair above is the first return map of f on the interval $[x_{\mathbf{b}_n}, x_{\mathbf{a}_n}] \subset \mathbf{H}$, and gluing the two ends of the interval via $f^{|\mathbf{b}_n - \mathbf{a}_n|}$ projects the pair to a critical quasicircle map with rotation number $r_{\text{prm}}^n(\theta)$. Refer to §2.1.3 for more details.)

Let $k_- := \mathbf{a}_n$ and $k_+ := \mathbf{b}_n$, and let us pick a small constant $\lambda > 0$. For $\bullet \in \{0, \infty\}$, denote by $E^\bullet$ the equipotential in $A_\bullet$ of level λ, and by $R_-^\bullet$, $R^\bullet$, and $R_+^\bullet$ the external rays in $A_\bullet$ that land at x_{k_+}, $x_{k_- + k_+}$, and x_{k_-} respectively. Then, the union $\bigcup_{\bullet \in \{0,\infty\}} R_\mp^\bullet \cup R^\bullet \cup E^\bullet$ encloses a rectangle $S_\pm$ containing the interval $[x_{k_\pm}, x_{k_- + k_+}] \subset \mathbf{H}$.

Let $I_- := [x_{k_+}, x_0]$ and $I_+ := [x_0, x_{k_-}]$. Precisely one of the two intervals, say I_- without loss of generality, contains a critical point of f^{k_-}. The rectangle $S_\pm$ lifts under $f^{k_\pm}$ to a topological disk $\Upsilon_\pm$ containing $I_\pm$, where $f^{k_-} : \Upsilon_- \to S_-$ is a degree d branched covering map and $f^{k_+} : \Upsilon_+ \to S_+$ is univalent. There are precisely $d-1$ disjoint disks $D_1, \ldots, D_{d-1}$ which are the lifts of S_+ under f^{k_-} that are touching Υ_- on the boundary and are disjoint from $\mathbf{H}$. Set

$$U_+ := \Upsilon_+, \quad U_- := \Upsilon_- \cup \bigcup_{j=1}^{d-1} D_j, \quad S := S_- \cup S_+.$$

See Figure 5.3 for an illustration. Then,

$$\left(f^{k_-} : U_- \to S, \quad f^{k_+} : U_+ \to S\right)$$

is a (d_0, d_∞)-critical pre-corona with rotation number $r^n_{\text{prm}}(\theta)$.

Let us embed the restriction of external rays of f in $S \backslash U$ where $U := U_- \cup U_+$. Notice that the boundaries of U_- and U_+ contain equipotential segments of different levels. Assume without loss of generality that the equipotential segments in U_- have smaller level. In order to satisfy (2), let us truncate a pair of small topological triangles near two vertices of the rectangle S_+, one where R^0_+ meets E^0 and the other where R^∞_+ meets E^∞. Refer to Figure 5.3. We will also truncate preimages of these triangles under f^{k_-} in U_-. Replace U and S with the new truncated domains. Then, every point in the legitimate boundary of U is now a landing point of an external ray segment, and (2) follows.

We claim that (3) follows from taking n to be sufficiently large and λ to be sufficiently small. Indeed, if $z \in U_\pm$ intersects an external ray landing at a point $w \in J(f) \cap U_\pm$, then the orbits of z and w remain close under iteration f^i for $i = 1, \ldots, k_\pm$. Suppose $z \in U_\pm$ is outside of $A_0 \cup A_\infty$. Then, it must lie within some filled limb $\hat{L}$ rooted at some pre-critical point $c_{-j} := \left(f|_{\mathbf{H}}\right)^{-j}(1)$ for some $j \geq 0$. If c_{-j} is not the unique critical point of f^{k_-}, then the forward images $\hat{L}$, $f(\hat{L})$, $\ldots$, $f^{k_\pm}(\hat{L})$ must remain small due to Lemma 5.2.3. If c_{-j} is the critical point of f^{k_-} in U_-, then we must have $0 < j < k_-$. In the latter case, the image $f^j(U_-)$ must remain in a small neighborhood of the critical point $c_0 = 1$ of f as we take λ to be small and n to be large. Therefore, the forward orbit z, $f(z)$, $\ldots$, $f^j(z)$ must be close to $\mathbf{H}$. □

Corollary 5.2.6. *Any (d_0, d_∞)-critical quasicircle map $g : \mathbf{H}_g \to \mathbf{H}_g$ with bounded type rotation number is corona renormalizable, that is, there is a (d_0, d_∞)-critical rotational pre-corona which is an iterate of g near $\mathbf{H}$.*

Proof. Given any (d_0, d_∞)-critical quasicircle map g of bounded type rotation number, Theorem D asserts that there is a global quasiconformal map ϕ conjugating g on some neighborhood W of its Herman curve with $f := F_c$. By Lemma 5.2.5, f admits a pre-corona P with range contained within $\phi(W)$. Then, g admits a (d_0, d_∞)-critical pre-corona of the form $\phi^{-1} \circ P \circ \phi$. □

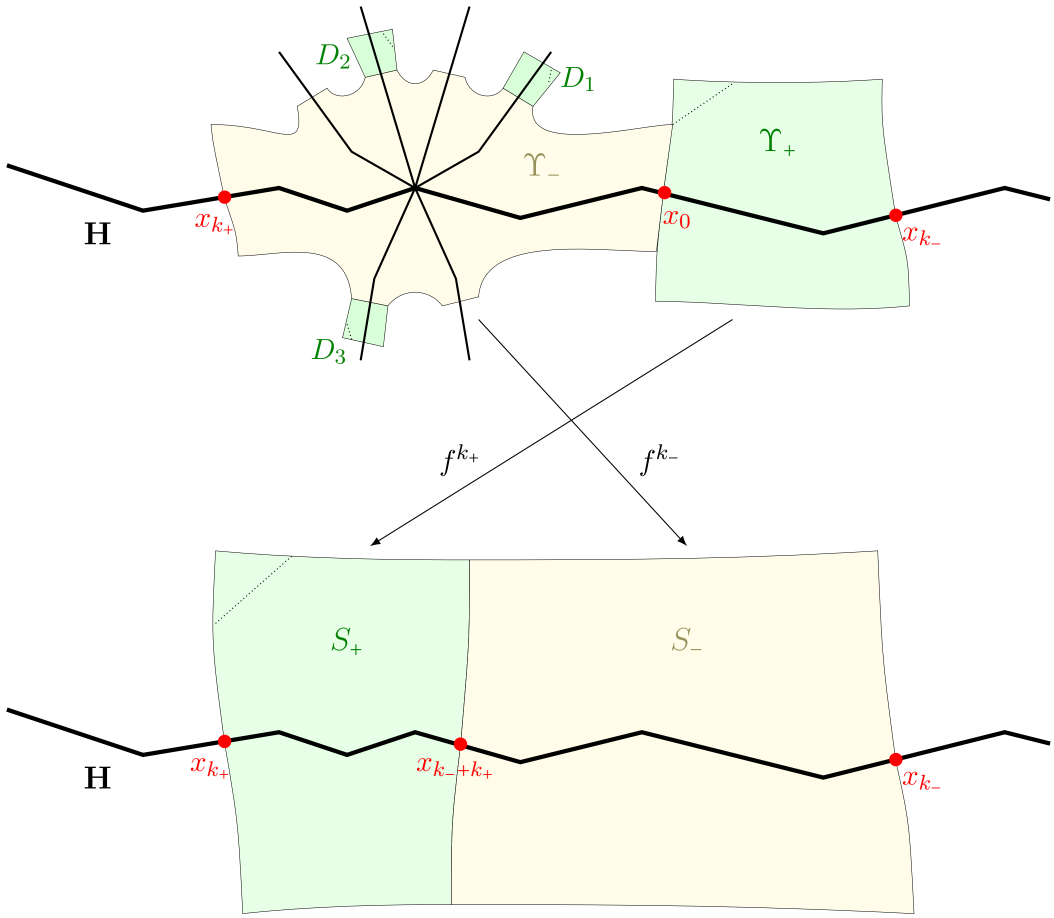

Figure 5.3: The construction of the pre-corona in the proof of Lemma 5.2.5 when $(d_0, d_\infty) = (3, 2)$. The triangle defined by the dotted line on the top left corner of S_+ and its preimages are to be removed.

5.2.2 Quasiconformal rigidity

Given a critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$ with critical point $c \in \mathbf{H}$, there is a unique conjugacy $h_f : (\mathbf{H}, c) \to (\mathbb{T}, 1)$ between f and the rigid rotation R_θ sending c to 1. We can endow $\mathbf{H}$ with the *combinatorial metric*, which is the pullback of the normalized Euclidean metric of $\mathbb{T}$ under h_f and thus the unique normalized f-invariant metric of $\mathbf{H}$. For any point $z \in \mathbf{H}$, the *combinatorial position* of z is the point $h_f(z)$ on the unit circle.

We say that two (d_0, d_∞)-critical rotational coronas f_1 and f_2 are *combinatorially equivalent* if

1. they have the same rotation number,

2. their marked points $m(f_1)$ and $m(f_2)$ have the same combinatorial position, and

3. for $\bullet \in \{0, \infty\}$, the external rays $R^\bullet(f_1)$ and $R^\bullet(f_2)$ have the same external angles.

In this subsection, we will prove quasiconformal rigidity of rotational coronas.

Theorem 5.2.7. *Two combinatorially equivalent (d_0, d_∞)-critical rotational coronas with bounded type rotation number are quasiconformally conjugate.*

The proof below is an application of the pullback argument. Let us make a couple of technical preparations. Let us first consider the model rational map f discussed in the previous subsection.

Definition 5.2.8. A *bubble chain* of f of generation $l \geq 1$ is an infinite sequence of bubbles $\{B_j\}_{j\geq 1}$ of f where B_1 has generation l and for all $j \geq 1$, B_j contains the root of B_{j+1} and the generation of B_j is strictly increasing in j. We say that a bubble chain $\{B_j\}_{j\geq 1}$

- ▹ is an *outer/inner* bubble chain if each B_j is an outer/inner bubble,
- ▹ is *periodic* of period p if there exists some $k \geq 1$ such that $f^p(B_{j+k}) = B_j$ for all $j \geq k$, and
- ▹ *lands* if the accumulation set $\bigcap_{j\geq 1} \overline{\cup_{k\geq j} B_k}$ is a single point, which we call the *landing point* of the bubble chain.

We say that a periodic point z of f is an *outer* (resp. *inner*) periodic point if its orbit is contained in the connected component of $\hat{\mathbb{C}} \backslash \mathbf{H}$ containing ∞ (resp. 0).

Let us fix a rotational pre-corona $P = (f_\pm : U_\pm \to S)$ of f (which exists thanks to Lemma 5.2.5).

Definition 5.2.9. We define the *non-escaping set* $K(P)$ of P to be the set of points whose orbit under $f_\pm$ that never escapes $\overline{U_\pm}$. By spreading around $K(P)$, we define the *local non-escaping set* of f relative to P by

$$K^{loc}(f) := \bigcup_{n\geq 0} f^n(K(P)).$$

The set $K^{loc}(f)$ is precisely the set of points which does not escape from the tiling $\mathbf{\Delta}_P$ associated to P.

Lemma 5.2.10. *The set $K^{loc}(f)$ is a connected compact set, and it is equal to*

- ▹ *the closure of the set of periodic points of f in $K^{loc}(f)$;*
- ▹ *the closure of the set of points of $K^{loc}(f)$ that are contained in $\bigcup_{n\geq 1} f^{-n}(\mathbf{H})$.*

For every outer (resp. inner) periodic point z in $K^{loc}(f)$, there is a unique maximal bubble chain in $K^{loc}(f)$ landing at z.

Proof. The first statement follows from the basic fact that as a rational map, the Julia set $J(f)$ can be characterized as either the set of points in $\hat{\mathbb{C}}$ that do not escape to 0 nor ∞ (which are the only non-repelling periodic points of f) or the closure of the set of repelling periodic points of f, or the closure of the iterated preimages of $\mathbf{H}$. The compactness of $K^{loc}(f)$ is clear, and the connectedness follows from the fact that, if we denote by $\mathbf{H}_P$ the invariant quasiarc of P, then for all $n \geq 1$, $P^{-n}(\mathbf{H}_P)$ is connected (in fact, it admits a tree structure). □

Let $g : U \to V$ be a rotational corona that is combinatorially equivalent to f, and let us denote the Herman quasicircle of g by $\mathbf{H}'$. By Theorem D, there is a quasiconformal conjugacy ϕ between g and f on some neighborhood W' of $\mathbf{H}'$ onto a neighborhood W of $\mathbf{H}$.

By Lemma 5.2.5, the pre-corona $P = (f_\pm : U_\pm \to S)$ of f can be assumed such that S is contained in W. The corona g also admits a pre-corona $P' = (g_\pm : U'_\pm \to S')$ contained in W' and it can be selected such that it is conjugate to P via ϕ. As such, we can define the non-escaping set $K(P')$ of P' in a similar way and spread it around to obtain the *local non-escaping set* $K^{loc}(g)$ of g relative to P'. The quasiconformal map ϕ induces a conjugacy between $g|_{K^{loc}(g)}$ and $f|_{K^{loc}(f)}$.

Let us define a *bubble* of g in $K^{loc}(g)$ to be the image under ϕ^{-1} of the intersection of a bubble of f with $K^{loc}(f)$. A *bubble chain* of g in $K^{loc}(g)$ is an infinite sequence of (non-empty) bubbles in $K^{loc}(g)$ defined in a similar way.

Let x be the marked point of g, and let R^∞ and R^0 be the outer and inner external rays of g landing at x. These rays make up the arc $\gamma_1(g)$.

Lemma 5.2.11. *Every outer (resp. inner) periodic point y of g in $K^{loc}(g)$ is the landing point of a unique maximal periodic outer (resp. inner) bubble chain $\{B_j\}_{j\geq 1}$ in $K^{loc}(g)$ and a unique periodic outer (resp. inner) external ray R_y, which has the same period as y.*

Proof. Suppose y is an outer periodic point of g in $K^{loc}(g)$. As a periodic point, y does not lie on any bubble in $K^{loc}(g)$. By Lemmas 5.2.3 and 5.2.10, y must be the landing point of a unique maximal outer bubble chain $\{B_j\}_{j\geq 1}$ in $K^{loc}(g)$. By maximality, the bubble B_1 of the lowest generation is rooted at a point on $\mathbf{H}'$.

Let p denote the period of y and let $k \in \mathbb{N}$ be the minimal number such that B_k has generation greater than p. By periodicity, the image of $\{B_j\}_{j\geq k}$ under g^p is also an outer bubble chain that is rooted at a point on $\mathbf{H}'$ and lands at y. By Lemma 5.2.10, the bubble chain $\{B_j\}_{j\geq 1}$ is equal to its image $\{g^p(B_j)\}_{j\geq k}$, and thus it is p-periodic.

Let us pick iterated preimages R_l and R_r of the external ray $R^\infty = R^\infty(g)$ landing at points $x_{l,0}$ and $x_{r,0}$ on B_1 respectively such that the union $B_1 \cup R_l \cup R_r \cup \partial V$ bounds a topological rectangle D_0 that contains y and is disjoint from $\mathbf{H}'$. Then, D_0 lifts under g^p to a rectangle D_{-1} containing y. Since the vertical sides of D_{-1} are external ray segments with a much smaller external angle difference compared to D_0, then D_{-1} is compactly contained in D_0. By Schwarz Lemma, $g^p : D_{-1} \to D_0$ is uniformly expanding with respect to the hyperbolic metric of D_0 and y is its unique repelling fixed point.

For every $n \in \mathbb{N}$, let D_{-n} be the lift of D_0 under g^{pn} containing y. Consider the lifts $R_{l,n}$ and $R_{r,n}$ of R_l and R_r under g^{pn} which touch the boundary of D_{-n}; these are external rays landing at points $x_{l,n}$ and $x_{r,n}$ respectively, which are vertices of D_{-n}. By uniform expansion, $x_{l,n}$ and $x_{r,n}$ converge to y and the external rays $R_{l,n}$ and $R_{r,n}$ converge to a limiting external ray R_y, which is a p-periodic outer external ray. By Lemma 5.2.3, R_y must land at y. □

Let c_0 denote the critical point of g and for $n \in \mathbb{Z}$, let $c_n := (g|_{\mathbf{H}'})^n(c_0)$.

Lemma 5.2.12. *For any pre-critical point $c_{-t} \in \mathbf{H}'$ of g, there exist an outer periodic point y_t^∞ and an inner periodic point y_t^0 in $K^{loc}(g)$ such that for $\bullet \in \{0, \infty\}$, the unique maximal bubble chain $\mathcal{B}_t$ landing at $y_t^\bullet$ is rooted at c_{-t}.*

Proof. We say that a bubble chain of f is in $K^{loc}(f)$ if its intersection with $K^{loc}(f)$ induces via ϕ a bubble chain in $K^{loc}(g)$. It is sufficient to prove the lemma in the case $g = f$.

Let us denote by $I_\varepsilon \subset \mathbf{H}$ the interval of combinatorial length ε centered at c_1. We will pick $\varepsilon > 0$ to be small enough such that the full preimage under f of I_ε is contained in the tiling $\mathbf{\Delta}_P$. Let us pick the first $s \in \mathbb{N}$ such that c_{-t-s} is contained in I_ε. Below, we will construct the desired outer periodic point y_t^∞, which will have period $p := s + t + 1$. The construction of y_t^0 can be done analogously.

First, let us pick a small closed interval neighborhood $J_0 \subset \mathbf{H}$ of c_{-t}. Let us arrange that the endpoints of J_0 are not in the grand orbit of the critical point of f, so there exists a unique pair of external rays R_l and R_r in the basin A_∞ that land on the pair of endpoints of J_0 respectively. Consider the open rectangle D_0 cut out by the union of $J_0 \cup R_l \cup R_r$ and an arc connecting R_l and R_r which is contained in the equipotential in A_∞ of some small level $\lambda > 0$.

Consider the intervals $J_{-j} := (f|_{\mathbf{H}})^{-j}(J_0)$ for $j \geq 1$. We assume that the combinatorial length of J_0 is small enough such that J_{-j} does not contain c_1 for all $j \in \{0, 1, \ldots, s\}$, and in particular J_{-s} is contained in $I_\varepsilon \backslash \{c_1\}$. Let us pick an outer bubble B of generation one. (There are $d_\infty - 1$ of such bubbles.) Let D_0' be the unique lift of D_0 under f^{s+1} such that $\partial D_0' \cap f^{-1}(I_\varepsilon) = B \cap f^{-1}(I_\varepsilon)$. For sufficiently small $\varepsilon > 0$ and $\lambda > 0$, we can guarantee that D_0' is contained in $\mathbf{\Delta}_P$.

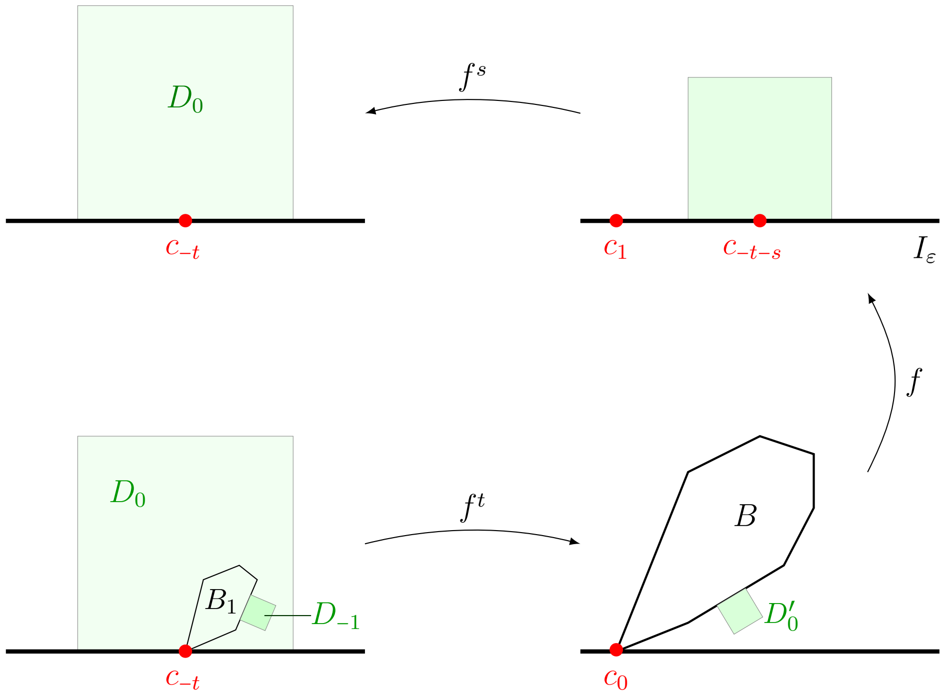

Figure 5.4: The construction of the map $f^p : D_{-1} \to D_0$ in the proof of Lemma 5.2.12.

Next, consider the outer bubble B_1 rooted at c_{-t} such that $f^t(B_1) = B$. Let D_{-1} be the lift of D'_0 under f^t that is attached to B_1. See Figure 5.4 for a reference. Since $D'_0 \subset \mathbf{\Delta}_P$, then $D_{-1} \subset \mathbf{\Delta}_P$ too. Also, since D_{-1} is compactly contained in D_0, then $f^p : D_{-1} \to D_0$ is uniformly expanding with respect to the hyperbolic metric of D_{-1}, and thus admits a unique repelling fixed point y_t^∞.

Let us construct the corresponding outer bubble chain landing at y_t^∞. For $j \geq 1$, we define the outer bubble B_{j+1} inductively to be the unique lift of B_j under f^p that is rooted at a point on $B_j \cap \overline{D_{-1}}$. By uniform expansion, the roots of B_j converge to y_t^∞. Thus, $\{B_j\}_{j\geq 1}$ is the unique outer bubble chain in $K^{loc}(f)$ that lands at y_t^∞ and is rooted at c_{-t}. □

For each pre-critical point c_{-t} of g, consider the outer and inner periodic bubble chains $\mathcal{B}_t^\infty$ and $\mathcal{B}_t^0$ in $K^{loc}(g)$ given by Lemma 5.2.12. For each $\bullet \in \{0, \infty\}$, the landing point of $\mathcal{B}_t^\bullet$ is also the landing point of a unique external ray $R_t^\bullet$ of g. Consider

$$\mathcal{T}_t := \mathcal{B}_t \cup R_t \qquad \text{where} \qquad \mathcal{B}_t := \mathcal{B}_t^\infty \cup \mathcal{B}_t^0 \text{ and } R_t := R_t^\infty \cup R_t^0. \tag{5.2.1}$$

Lemma 5.2.13 (Rational approximation of $\gamma_1(g)$)**.** *For every $\varepsilon > 0$, there exists a pair of pre-critical points $c_{-t_l}, c_{-t_r} \in \mathbf{H}'$ located on the left and right of x respectively such that $\mathcal{T}_{t_l}$ and $\mathcal{T}_{t_r}$ are both in the ε-neighborhood of $\gamma_1(g)$.*

Proof. Since pre-critical points are dense on $\mathbf{H}'$, there exists a pair of pre-critical points c_{-t_l} and c_{-t_r} on the left and right of x, where the moments t_l and t_r grow as we require them to be arbitrarily close to x. Due to Lemma 5.2.3, the bubble chains within $\mathcal{T}_{t_l}$ and $\mathcal{T}_{t_r}$ shrink as we get close to x. The outer (resp. inner) external rays within $\mathcal{T}_{t_l}$ and $\mathcal{T}_{t_r}$ are also close to R^∞ (resp. R^0) because their external angles are close to that of R^∞. □

We are now ready to run the pullback argument.

proof of Theorem 5.2.7. Let $g_1 : U_1 \to V_1$ and $g_2 : U_2 \to V_2$ be two combinatorially equivalent (d_0, d_∞)-critical rotational coronas with rotation number $\theta \in \Theta_{bdd}$. Let $f = f_{d_0,d_\infty,\theta} \in \mathcal{X}$ be the prototypical Example 4.4.5, and denote its Herman curve by $\mathbf{H}$. From the previous discussion, for $i \in \{1, 2\}$, there is a quasiconformal conjugacy ϕ_i between g_i and f on some neighborhood W_i of the Herman quasicircle $\mathbf{H}_i$ of g_i onto a neighborhood W of $\mathbf{H}$.

We fix a pre-corona $P = (f_\pm : U_\pm \to S)$ of f where S is contained in W, and for $i \in \{1, 2\}$, let $P_i = (g_\pm : U_{i,\pm} \to S_i)$ be the corresponding pre-corona of g_i conjugate to P via ϕ_i. We consider the local non-escaping set $K^{loc}(g_i)$ of g_i relative to P_i. The quasiconformal map $\phi_2 \circ \phi_1^{-1} : W_1 \to W_2$ restricts to a conjugacy $h : K^{loc}(g_1) \to K^{loc}(g_2)$ between g_1 and g_2.

For $i \in \{1, 2\}$ and $t \in \{t_l, t_r\}$, consider the sets $\mathcal{T}_t(g_i) = \mathcal{B}_t(g_i) \cup R_t(g_i)$ from Lemma 5.2.13 which approximate the critical arc $\gamma_1(g)$. By design, we can arrange such that for each $t \in \{t_l, t_r\}$, $\phi_2 \circ \phi_1^{-1}$ sends $\mathcal{B}_t(g_1)$ to $\mathcal{B}_t(g_2)$, and the outer/inner rays in $R_t(g_1)$ and $R_t(g_2)$ have the same external angles. For $i \in \{1, 2\}$, consider the union

$$Z_i = K^{loc}(g_i) \cup \bigcup_{n \geq 0} g_i^n (R_{t_l} \cup R_{t_r}).$$

Clearly, Z_i is forward invariant and $V_i \backslash Z_i$ consists of finitely many connected components. Since $R_t(g_1)$ and $R_t(g_2)$ have the same external angles, h extends to a quasiconformal map $h : V_1 \to V_2$ that is equivariant on $Z_i \cup \partial_{\mathrm{L}} U_1$.

Let us define a new domain $\hat{U}_1$ out of U_1 by replacing the forbidden boundary $\partial_{\mathrm{F}} U_1$ with some set $\partial_{\mathrm{F}} \hat{U}_1$ of curves slightly outside of $\partial_{\mathrm{F}} U_1$ such that the image $g_1(\partial_{\mathrm{F}} \hat{U}_1)$ is now contained inside of $\mathbf{H}_1 \cup \mathcal{T}_{t_l}(g_1) \cup \mathcal{T}_{t_r}(g_1)$. In the same manner, we replace U_2 with a slightly larger disk $\hat{U}_2$ such that $h|_{Z_1}$ lifts to a conjugacy between $g_1|_{\partial \hat{U}_1}$ and $g_2|_{\partial \hat{U}_2}$.

We can now run the pullback argument. Set $h_0 := h$ and we inductively construct quasiconformal maps $h_n : V_1 \to V_2$ such that

$$h_n(z) = \begin{cases} h_{n-1}(z), & \text{if } z \notin \hat{U}_1, \\ g_2^{-1} \circ h_{n-1} \circ g_1(z), & \text{if } z \in \hat{U}_1. \end{cases}$$

Each h_n has the same dilatation as h. Since $K^{loc}(g_1)$ is nowhere dense, h_n stabilizes and converges to a quasiconformal conjugacy between g_1 and g_2. □

5.3 Hyperbolic renormalization fixed point

From now on, let us fix a periodic type irrational $\theta \in \Theta_{per}$. In this section, we will construct the desired corona renormalization fixed point f_* and prove most of Theorem J. The remaining sections §5.4-5.8 are dedicated to proving that the local unstable manifold is one-dimensional.

5.3.1 Corona renormalization fixed point

We say that a rotational corona is *standard* if the arc γ_0 passes through the critical value. Similarly, we say that a rotational pre-corona is *standard* if it is a pre-corona around the critical value.

Theorem 5.3.1. *There exists a standard (d_0, d_∞)-critical rotational corona $f_* : U_* \to V_*$ with rotation number θ which admits a standard rotational pre-corona*

$$F_* = \left(f_*^{\mathbf{a}} : U_- \to S_*,\ f_*^{\mathbf{b}} : U_+ \to S_*\right)$$

together with a gluing map $\psi_* : S_* \to \overline{V_*}$ projecting F_* back to $f_* : U_* \to V_*$. Moreover, we have an improvement of domain: $\mathbf{\Delta}_{F_*} \Subset U_*$.

Proof. Consider the unique normalized (d_0, d_∞)-critical commuting pair

$$\zeta = (f_- : I_- \to I, f_+ : I_+ \to I)$$

with rotation number θ in the renormalization horseshoe from Theorem 4.6.7. Let us break down the quasiarc I into $I_- \cup I_+ = [f_+(0), 0] \cup [0, f_-(0)]$. Fix a positive integer $n \in \mathbb{N}$. There exists some $\mu \in \mathbb{D}$ independent of n such that there is a pre-renormalization $\zeta_n = (f_{n,-} : J_- \to J, f_{n,+} : J_+ \to J)$ of ζ on a subinterval $J \subset I$ that is conjugate to ζ via the linear map $L^n(z) = \mu^n z$. We will convert this renormalization fixed point in the category of commuting pairs to that in the category of critical quasicircle maps, and then project it to that in the category of rotational coronas.

Consider the gluing map $\phi_1 \coloneqq G_\zeta$ described in Proposition 4.4.4. Then, ϕ_1 projects the modified commuting pair $\zeta' \coloneqq \left(f_-|_{[f_+f_-(0),0]}, f_+f_-|_{[0,f_-(0)]},\right)$ into a (d_0, d_∞)-critical quasicircle map $g : \mathbf{H} \to \mathbf{H}$ having the same rotation number θ.

Denote by $c_0 \coloneqq \phi_1(0)$ the critical point of g, and let $c_k \coloneqq g^k(c_0)$ for all $k \in \mathbb{N}$. Consider the modification of ζ_n, which is ζ' rescaled by L^n, and project it to the dynamical plane of g via ϕ_1 to obtain a commuting pair $g_n = \left(g^{a_n}|_{[c_{b_n}, c_0]}, g^{b_n}|_{[c_0, c_{\mathbf{a}}]}\right)$ for some return times a_n and b_n. Then, $\psi_1 \coloneqq \phi_1 L^n \phi_1^{-1}$ is the gluing map projecting g_n back to g.

To make it standard, we will push g_n forward under one iterate of g. More precisely, we will consider $\psi_2 \coloneqq g \circ \psi_1 \circ g^{-1}$. It is well-defined because for every point z close to c_1, the preimage $g^{-1}(z)$ is a set of $d = d_0 + d_\infty - 1$ points close to c_0 whose images under ψ_1 remain close to c_0 and get mapped to the same point $\psi_2(z)$ under g. The new gluing map ψ_2 sends a small neighborhood of c_1 to a neighborhood of $\mathbf{H}$. Moreover, ψ_2 fixes the critical value c_1 and projects $\left(g^{a_n}|_{[c_{b_n+1},c_1]}, g^{b_n}|_{[c_1,c_{a_n+1}]}\right)$ back to g.

By Corollary 5.2.6, g admits a standard pre-corona P defined in a small neighborhood of c_1. The corresponding gluing map ϕ_2 projects P onto a (d_0, d_∞)-critical rotational corona $f_* : U_* \to V_*$. Since θ is periodic, we can prescribe f_* to have rotation number θ. The corresponding Herman quasicircle $\mathbf{H}_*$ of f_* is the image of (an interval in) $\mathbf{H}$ under ϕ_2.

Let us rescale the pre-corona P by ψ_2^{-1} to obtain yet another pre-corona P' in the dynamical plane of g with a much smaller domain compared to P. Project P' via ϕ_2 to obtain a pre-corona F_* of f_*. The map $\psi_* := \phi_2 \circ \psi_2 \circ \phi_2^{-1}$ will project the pre-corona F_* back to f_*. The improvement of domain property is satisfied once we take n to be sufficiently high. □

Corollary 5.3.2. *Let f_* and F_* be from the previous theorem. There exist a pair of small Banach neighborhoods $\mathcal{U}$ and $\mathcal{B}$ of f_* and a compact analytic corona renormalization operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ such that $\mathcal{R}f_* = f_*$ and the pre-renormalization of $\mathcal{R}f_*$ is F_*. Moreover, for any rotational corona f in $\mathcal{U}$ with the same rotation number θ, f is infinitely renormalizable and $\mathcal{R}^n f$ converges exponentially fast to f_*.*

Proof. The existence of $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ follows from Theorems 5.1.6 and 5.3.1. Exponential convergence is guaranteed by Theorem 4.6.6 (2) provided that $\mathcal{U}$ is a sufficiently small neighborhood of f_*. □

Let us again denote by G the Gauss map acting on on $(0,1)\backslash\mathbb{Q}$.

Lemma 5.3.3. *Consider any irrational $\theta' \in \Theta_{pre}$ where $G^k(\tau) = \theta$ for some $k \in \mathbb{N}$. For any Banach neighborhood $\mathcal{U}$ of f_* and any (d_0, d_∞)-critical quasicircle map f of rotation number θ', there is a compact analytic corona renormalization operator $\mathcal{R}_1 : N(f) \to \mathcal{U}$ on a Banach neighborhood $N(f)$ of f.*

Proof. By Theorem 4.6.7 (2), there is a high $m \in \mathbb{N}$ such that $\mathcal{R}^m f$ is a critical commuting pair of rotation number θ that is arbitrarily close to the critical commuting pair ζ_*. By quasiconformal rigidity, f admits a rotational pre-corona F which projects to a rotational corona g of rotation number θ close to f_*. By Theorem 5.1.6, there is a compact analytic renormalization operator $\mathcal{R}_1$ on a small neighborhood of f such that $\mathcal{R}_1(f) = g$. □

5.3.2 Renormalization tiling

Recall from Theorem 5.1.6 that every corona f in $\mathcal{U}$ has the same codomain V and critical arc γ_1 as the renormalization fixed point f_*.

Let us pick a positive integer n and a corona $f = f_0$ in the neighborhood

$$\mathcal{U}_n := \bigcap_{0 \le k \le n} \mathcal{R}^{-k}(\mathcal{U})$$

of f_*. In particular, f is n times renormalizable. For $k \le n$, denote by

- ▷ $f_k := \mathcal{R}^k f = [f_k : U_k \to V]$ the k^{th} renormalization of f,
- ▷ $\psi_k : S_k \to V$ the renormalization change of variables for f_{k-1}, and
- ▷ $\phi_k := \psi_k^{-1}$.

For $k \in \{0, 1, \ldots, n\}$, let us cut the dynamical plane of f_k along the critical arc γ_1 and obtain a pre-corona

$$F_k = (f_{k,\pm} : U_{k,\pm} \to V \backslash \gamma_1).$$

The map

$$\Phi_n := \phi_1 \circ \phi_2 \circ \ldots \phi_n$$

is well defined on $V \backslash \gamma_1$ and projects F_n to the dynamical plane of f as the pre-corona

$$F_n^{(0)} = \left(f_{n,\pm}^{(0)} : U_{n,\pm}^{(0)} \to S_n^{(0)} \right) \qquad \text{where} \qquad f_{n,-}^{(0)} = f_0^{\mathbf{a}_n} \text{ and } f_{n,+}^{(0)} = f_0^{\mathbf{b}_n}$$

for some return times $\mathbf{a}_n$ and $\mathbf{b}_n$. Let us also set $\Phi_0 := \mathrm{Id}$.

Let us divide $\overline{U_0}$ along the arcs γ_0 and γ_1 to obtain a tiling $\mathbf{\Delta}_0$ of $\overline{U_0}$ consisting of two tiles $\Delta_0(0)$ and $\Delta_0(1)$. We make the convention that $\Delta_0(0)$, γ_0, and $\Delta_0(1)$ are in counterclockwise order. The tiling $\mathbf{\Delta}_0$ is called the *zeroth tiling* associated to f_0.

Next, define the *n^{th} tiling* $\mathbf{\Delta}_n$ associated to f by spreading around $U_{n,\pm}^{(0)}$ via f. It consists of $f^i\left(U_{n,-}^{(0)}\right)$ for $i \in \{0, 1 \ldots, \mathbf{a}_n - 1\}$ and $f^j\left(U_{n,+}^{(0)}\right)$ for $j \in \{0, 1 \ldots, \mathbf{b}_n - 1\}$. Let us denote by $\Delta_n(0)$ the image of the zeroth tile $\Delta_0(0, f_n)$ of f_n under Φ_n, label the rest of the tiles in $\mathbf{\Delta}_n$ in counterclockwise order by $\Delta_n(i)$ for $i \in \{0, 1, \ldots, \mathbf{a}_n + \mathbf{b}_n - 1\}$. See Figure 5.5.

If f is rotational, then $\mathbf{\Delta}_n$ always forms an annular neighborhood of the Herman quasicircle of f. In general, the map f always acts almost like a rotation on the tiling $\mathbf{\Delta}_n$. There exists $\mathbf{p}_n \in \mathbb{N}_{\ge 1}$ such that f maps $\Delta_n(i)$ univalently onto $\Delta_n(i + \mathbf{p}_n)$ whenever $i \notin \{-\mathbf{p}_n, -\mathbf{p}_n + 1\}$. Moreover, f maps $\Delta_n(-\mathbf{p}_n) \cup \Delta_n(-\mathbf{p}_n + 1)$ back to $S_n^{(0)}$ almost as a degree d covering map branched at its critical point $c_0(f)$.

Lemma 5.3.4. *The operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ can be arranged such that the following holds. For $f \in \mathcal{U}_n$,*

1. *there is a holomorphic motion of $\partial \mathbf{\Delta}_0, \ldots, \partial \mathbf{\Delta}_n$ over $f \in \mathcal{U}_n$ that is equivariant with respect to $f : \partial \Delta_n(i) \to \partial \Delta_n(i + \mathbf{p}_n)$ for $i \notin \{-\mathbf{p}_n, -\mathbf{p}_n + 1\}$;*

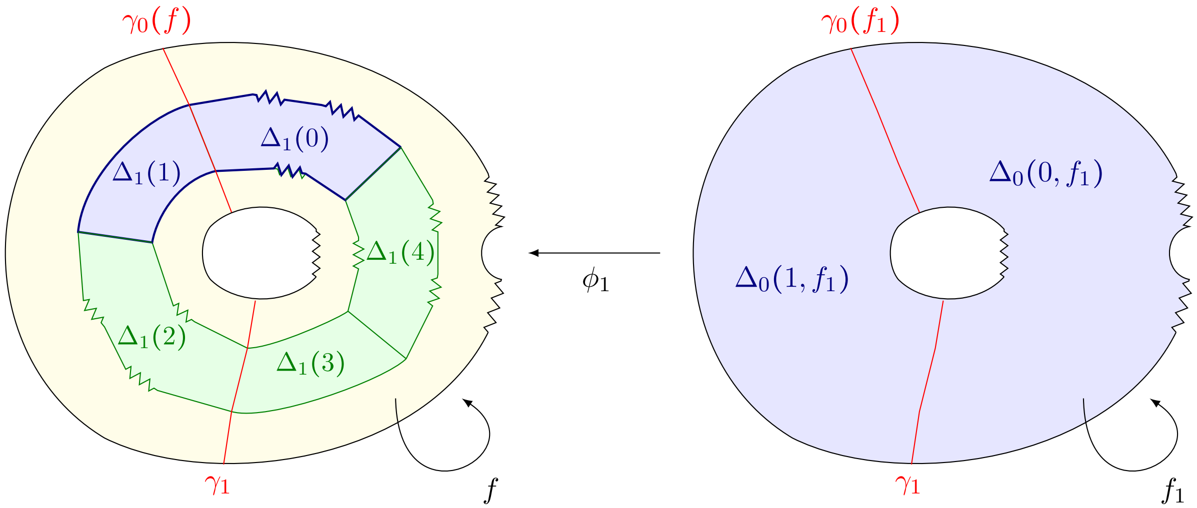

Figure 5.5: The construction of the first tiling $\mathbf{\Delta}_1$ when $(\mathbf{a}_n, \mathbf{b}_n) = (3, 2)$.

(2) *for every* $f \in \mathcal{U}_n$ *and* $1 \leq k \leq n$, $\mathbf{\Delta}_m \cup f(\mathbf{\Delta}_m) \Subset \mathbf{\Delta}_{m-1}$*;*

(3) *the tiling* $\mathbf{\Delta}_n(f)$ *is close to the Herman curve of* f_* *in Hausdorff topology.*

Proof. Let us first consider the case where $f = f_*$. By the improvement of domain property in Theorem 5.3.1, the diameters of the tiles in $\mathbf{\Delta}_n(f_*)$ must shrink to 0 as $n \to \infty$. Consider a tile $\Delta_1(i, f_*)$. There is some $t \geq 0$ and $j \in \{0, 1\}$ such that f_*^t sends $\Delta_1(j, f_*)$ onto $\Delta_1(i, f_*)$. By replacing $\mathcal{R}$ with some high iterate $\mathcal{R}^k$ if necessary, the map

$$\psi_* \circ f_*^{-t} : \Delta_1(i, f_*) \to \Delta_0(j, f_*)$$

expands the Euclidean metric by some high factor $C > 1$. Inductively, (2) and (3) hold for f_*.

By design, it is clear that $\partial\mathbf{\Delta}_0$ moves holomorphically over $f \in \mathcal{U}$. For $1 \leq k \leq n$, we push forward the holomorphic motion $\partial\mathbf{\Delta}_0(f_k)$ via Φ_k and spread it around dynamically to obtain a holomorphic motion of $\partial\mathbf{\Delta}_k(f)$ over $f \in \mathcal{U}_n$.

By continuity, every $f \in \mathcal{U}_n$ also satisfies the following property. For any tile $\Delta_n(i, f)$ in $\mathbf{\Delta}_n(f)$, there is some $t \geq 0$ and $j \in \{0, 1\}$ such that f^t sends $\Delta_n(j, f)$ onto $\Delta_n(i, f)$. We obtain a holomorphic motion of $\partial\mathbf{\Delta}_n(f)$ by pulling back the holomorphic motion of $\partial\mathbf{\Delta}_0(f_n)$ via maps of the form

$$\Psi_{n,i} := \Phi_n^{-1} \circ f^{-t} : \Delta_n(i, f) \to \Delta_0(j, f_n) \tag{5.3.1}$$

for each tile. This implies (1). Moreover, (2) follows from the observation that each $\Psi_{n,i}$ expands the Euclidean metric by a factor close to C^n. Moreover, (3) follows from (1) as well as the special case of (3) for $f = f_*$. $\square$

Let us extend the tiling $\mathbf{\Delta}_n$ of a subset of $\overline{U_0}$ to a full tiling of $\overline{U_0}$ as follows. Consider

$$\hat{\gamma}_0 := \gamma_0 \backslash f^{-1}(U_0) \quad \text{and} \quad \Gamma := \partial U_0 \cup \hat{\gamma}_0.$$

Observe that $\hat{\gamma}_0$ is a disjoint union of two subarcs $\hat{\gamma}_0^0$ and $\hat{\gamma}_0^\infty$ of γ_0 where each $\hat{\gamma}_0^\bullet$ connects the boundary component $\partial^\bullet U_0$ to $f^{-1}(U_0)$. Consider the maps $\Psi_{n,i}$ from (5.3.1).

Lemma 5.3.5. *When $\mathcal{U}$ is sufficiently small, the following holds for all $f \in \mathcal{U}$.*

(1) *$\Gamma(f_1)$ contains $\Psi_{1,i}(\partial \mathbf{\Delta}_1(f) \cap \partial \Delta_1(i,f))$ for every $i \in \{0, 1, \ldots, \mathbf{a}_n + \mathbf{b}_n - 1\}$. There is some i such that $\hat{\gamma}_0(f_1)$ is contained in $\Psi_{1,i}(\partial \mathbf{\Delta}_1(f) \cap \partial \Delta_1(i,f))$.*

(2) *$\Gamma(f)$ is disjoint from $\partial \mathbf{\Delta}_1(f)$.*

(3) *For $\bullet \in \{0, \infty\}$, there is an arc $\xi_0^\bullet$ such that both $\xi_0^\bullet \cup \hat{\gamma}_0^\bullet$ and $\xi_1^\bullet := f(\xi_0^\bullet)$ connect $\partial^\bullet U_0$ and $\partial \mathbf{\Delta}_1(f)$.*

Moreover, $\xi_0 := \xi_0^0 \cup \xi_0^\infty$ and $\xi_1 := \xi_1^0 \cup \xi_1^\infty$ can be chosen such that there is a holomorphic motion of

$$\Gamma \cup \xi_0 \cup \xi_1 \cup \mathbf{\Delta}_1$$

over $f \in \mathcal{U}$ that is equivariant with respect to

- ▹ *$f : \xi_0(f) \to \xi_1(f)$,*
- ▹ *$f : \Delta_1(i,f) \to \Delta_1(i + \mathbf{p}_1, f)$ for $i \neq \{-\mathbf{p}_1, -\mathbf{p}_1 + 1\}$, and*
- ▹ *$\Psi_{1,i} : \partial \mathbf{\Delta}_1(f) \cap \Delta_1(i,f) \to \Gamma(f_1)$ for all i.*

Proof. Every tile $\Delta_1(i,f)$ is a rectangle. Clearly, each $\Psi_{1,i}$ maps the horizontal sides of $\Delta_1(i,f)$ into ∂U_1. Let us label the vertical sides of $\Delta_1(i,f)$ by $l(i)$ and $r(i)$ such that each $l(i)$ intersects the side $r(i+1)$ of the next tile. Then, the intersection $\partial \mathbf{\Delta}_1(f) \cap \partial \Delta_1(i,f)$ is the union of the horizontal sides of $\Delta_1(i,f)$ and the symmetric difference $l(i) \vartriangle r(i+1)$ between touching vertical sides across all i's.

It is clear that $l(i) \neq r(i+1)$ for at least one i. For such i, either $l(i)$ is the preimage of $\gamma_0(f_1)$ under $\Psi_{1,i}$ and $r(i+1)$ is the preimage of the arc $\gamma_1(f_1)$ under $\Psi_{1,i+1}$, or vice versa. In this case, $l(i) \vartriangle r(i+1)$ will be mapped by $\Phi_{1,i}$ or $\Phi_{1,i+1}$ onto $\hat{\gamma}_0(f_1)$. This implies (1).

Item (2) follows directly from Lemma 5.3.4. Moreover, (2) allows us to find for each $\bullet \in \{0, \infty\}$ a proper arc $\xi_0^\bullet$ in $U_0 \backslash (\hat{\gamma}_0 \cup \mathbf{\Delta}_1)$ in a small neighborhood of γ_0 that connects the tip of $\hat{\gamma}_0^\bullet$ to a point on $\partial \Delta_1(i,f)$ for some $i \neq \{-\mathbf{p}_1, -\mathbf{p}_1 + 1\}$. This yields (3).

In Lemma 5.3.4, we already established the equivariant holomorphic motion of $\partial \mathbf{\Delta}_0 \cup \partial \mathbf{\Delta}_1$. By lifting via $\Phi_{1,i}$, this motion immediately extends to an equivariant motion of Γ. We then

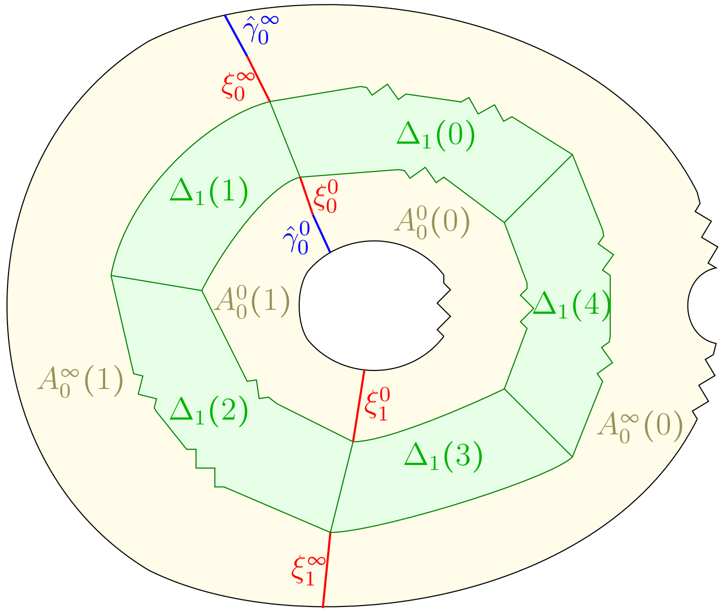

Figure 5.6: The first full renormalization tiling of U_0.

lift the motion of $\mathbf{\Delta}_0(f_1)$ via $\Psi_{1,i}$ to obtain an equivariant motion of $\partial\mathbf{\Delta}_1 \cup \Gamma$. Finally, by applying the λ-lemma, we extend this motion to $\Gamma \cup \xi_0 \cup \xi_1 \cup \mathbf{\Delta}_1$. $\square$

For $n \in \mathbb{N}$, we define the n^{th} *full renormalization tiling* of U_0 to be the union of the tilings $\mathbf{\Delta}_n$ and $\mathbf{A}_k$ for $k = 0, 1, \ldots, n-1$ where the latter is constructed as follows. Each $\mathbf{A}_k$ is a disjoint union of two tilings $\mathbf{A}_k^0$ and $\mathbf{A}_k^\infty$ where the former is closer to $\partial^0 U_0$ and the latter is closer to $\partial^\infty U_0$. For each $\bullet \in \{0, \infty\}$,

- ▷ $\mathbf{A}_0^\bullet$ is the connected component of $\overline{\mathbf{\Delta}_0 \backslash \mathbf{\Delta}_1}$ that touches $\partial^\bullet U_0$ on the boundary, and it is split by $\hat{\gamma}_0^\bullet \cup \xi_0^\bullet \cup \xi_1^\bullet$ into two tiles $A_0^\bullet(0)$, $A_0^\bullet(1)$. Again, we make the convention that $A_0^\bullet(0)$, $\hat{\gamma}_0^\bullet \cup \xi_0^\bullet$, $A_0^\bullet(1)$ are in counterclockwise order.

- ▷ $\mathbf{A}_k^\bullet$ is the connected component of $\overline{\mathbf{\Delta}_k \backslash \mathbf{\Delta}_{k+1}}$ that touches $\partial^\bullet \mathbf{\Delta}_k$ on the boundary, and it has tiles $\{A_k^\bullet(i)\}_{i=0,1,\ldots,\mathbf{a}_k+\mathbf{b}_k-1}$ obtained by spreading via forward iterates of f the tiles $A_k^\bullet(j, f) \coloneqq \Phi_k(A_0^\bullet(j, f_k))$ for $j \in \{0, 1\}$ and labeled in counterclockwise order.

The first full renormalization tiling is illustrated in Figure 5.6.

Definition 5.3.6. A *quasiconformal combinatorial pseudo-conjugacy of level* n between f and $f_\star$ is a quasiconformal map $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that sends $\overline{U_0}$ to $\overline{U_\star}$ and preserves the n^{th} renormalization tiling as follows.

(1) The map h sends $\Delta_n(i,f)$ to $\Delta_n(i,f_*)$ for all i, and is equivariant on $\Delta_n(i,f)$ for all $i \notin \{-\mathbf{p}_n, -\mathbf{p}_n + 1\}$;

(2) For all $\bullet \in \{0,\infty\}$ and $k \in \{0,1,\dots n-1\}$, h sends $A_k^\bullet(i,f)$ to $A_k^\bullet(i,f_*)$ for all i, and is equivariant on $A_k^\bullet(i,f)$ for all $i \notin \{-\mathbf{p}_k, -\mathbf{p}_k+1\}$.

Theorem 5.3.7 (Combinatorial pseudo-conjugacy). *Consider $f \in \mathcal{U}_n$ and let*

$$D := \max_{0\le k\le n} \operatorname{dist}(f_k, f_*).$$

There is a K_D-quasiconformal combinatorial pseudo-conjugacy h of level n between f and f_* such that

$$\sup_{z\in\mathbf{\Delta}_n(f)} |h(z) - z| \le M_D.$$

Moreover, $K_D \to 1$ and $M_D \to 0$ as $D \to 0$.

Proof. Recall that each tile $A_k^\bullet(i,f)$ admits some $t \in \mathbb{N}$ and $j \in \{0,1\}$ such that $\Psi_{k,i} := \Phi_k^{-1} \circ f^{-t}$ univalently maps $A_k^\bullet(i,f)$ onto $A_0^\bullet(j,f_k)$. By Lemma 5.3.5, we have a holomorphic motion of the first full renormalization tiling over $\mathcal{U}$. Let us pull back this motion via maps of the form $\Psi_{n,i}$ to obtain a holomorphic motion of the full n^{th} renormalization tiling. By equivariance and λ-lemma, this holomorphic motion induces the desired quasiconformal map h. The dilatation K_D of h is bounded by the dilatation of the motion at $f_0, f_1, \dots f_n$, which depends only on D, where $K_D \to 1$ as $D \to \infty$. The upper bound M_D follows from the continuity of the holomorphic motion and the compactness of quasiconformal maps. □

Corollary 5.3.8. *There is some $\varepsilon > 0$ such that the following holds. Suppose $f \in \mathcal{U}$ is infinitely renormalizable and $\mathcal{R}^n f$ is in the ε-neighborhood of f_* for all $n \in \mathbb{N}$. Then, f is a rotational corona with rotation number θ.*

Proof. By Theorem 5.3.7, we have a $K(\varepsilon)$-quasiconformal combinatorial pseudo-conjugacy h_n of level n between f and f_* for all $n \in \mathbb{N}$. By the compactness of K-quasiconformal maps, h_n converges in subsequence to a quasiconformal map $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, and h^{-1} must be a conjugacy on the Herman quasicircle $\mathbf{H}_*$ of f_*. The image $h^{-1}(\mathbf{H}_*)$ is a Herman quasicircle of f containing the critical point $c_0(f)$ and separating the boundaries of the domain of f. It follows that f must be a rotational corona with rotation number θ. □

5.3.3 Towards hyperbolicity

Theorem 5.3.9. *The renormalization operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ is hyperbolic at the fixed point f_* with a finite positive dimensional local unstable manifold $\mathcal{W}^u_{\text{loc}}$. If $\mathcal{U}$ is sufficiently small, the* *local stable manifold $\mathcal{W}^s_{\text{loc}}$ of f_* consists of the set of (d_0, d_∞)-critical rotational coronas in $\mathcal{U}$ with rotation number θ.*

Proof. Consider a corona f near f_* lying on the local stable manifold $\mathcal{W}^s_{\text{loc}}$. For sufficiently small $\mathcal{U}$, $\mathcal{R}^n f$ is in the ε-neighborhood of f_* for all $n \in \mathbb{N}$. By Corollary 5.3.8, f must be a rotational corona with rotation number θ.

Let us consider the derivative $D\mathcal{R}_{f_*}$ of the renormalization operator at the fixed point f_*. By the compactness of $\mathcal{R}$, the number of neutral and repelling eigenvalues is finite. We claim that neutral eigenvalues do not exist and repelling eigenvalues must exist.

Suppose for a contradiction that there are neutral eigenvalues. By Small Orbits Theorem 2.3.1, there exists an infinitely renormalizable corona f such that its forward orbit lies entirely in the ε-neighborhood of f_* and it satisfies

$$\lim_{n\to\infty} \frac{1}{n} \log \operatorname{dist}(\mathcal{R}^n f, f_*) = 0. \tag{5.3.2}$$

By Corollary 5.3.8, f must be a rotational corona with the same rotation number θ as f_*. By Corollary 5.3.2, renormalizations $\mathcal{R}^n f$ converge to f_* exponentially fast, which contradicts (5.3.2). Hence, neutral eigenvalues do not exist.

Consider the family of rational maps F_c from Proposition 4.3.8. By combinatorial rigidity, there is a unique parameter c_* such that F_{c_*} admits a Herman quasicircle with the same rotation number as f_*. By Lemma 5.3.3, there is an analytic renormalization operator $\mathcal{R}_1$ on a neighborhood of F_{c_*} such that $\mathcal{R}_1 F_{c_*}$ is a rotational corona with rotation number θ that is sufficiently close to f_*. For any parameter $c \neq c_*$ sufficiently close to c_*, $\mathcal{R}_1 F_c$ is also sufficiently close to f_*. By the uniqueness of c_*, the parameter c can be picked such that F_c is postcritically finite, and so $\mathcal{R}_1 F_c$ is not a rotational corona.

Suppose for a contradiction that $D\mathcal{R}_{f_*}$ has no repelling eigenvalues. Then, $\mathcal{W}^s_{\text{loc}}$ is an open neighborhood of f_* and contains $\mathcal{R}_1 F_c$. However, the non-rotationality of $\mathcal{R}_1 F_c$ would contradict Corollary 5.3.8. □

5.4 Transcendental extension

From now on, we will consider the corona renormalization operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ together with its hyperbolic fixed point $f_* : U_* \to V_*$ constructed in Section 5.3.

Definition 5.4.1. A map $g : A \to B$ is said to be *σ-proper* if there exist exhaustions A_n, B_n of A, B respectively such that for all n, $g : A_n \to B_n$ is a proper map; equivalently, every connected component of the preimage of a compact set under g is compact.

In [McM98], McMullen proved the existence of maximal σ-proper extensions of holomorphic commuting pairs associated to renormalizations of quadratic Siegel disks. This is generalized in [DLS20, Theorem 5.5] where pre-pacmen on the local unstable manifold are shown to admit maximal σ-proper extension. In this section, we will show that our case is no different. We will study coronas in the local unstable manifold $\mathcal{W}^u_{\text{loc}}$ of f_*, which we will identify as a parameter space (of unknown dimension) of transcendental holomorphic maps onto $\mathbb{C}$.

5.4.1 Maximal σ-proper extension

Consider a corona $f : U \to V$ lying in the local unstable manifold $\mathcal{W}^u_{\text{loc}}$ of f_*. Since f is infinitely anti-renormalizable, it comes with a backward tower of corona renormalizations $\{f_k : U_k \to V\}_{k \leq 0}$, where each f_k embeds into U_{k-1} as a pre-corona $F_k = (f_{k,\pm} : U_{k,\pm} \to S_k)$ consisting of iterates of f_{k-1}. Let $\psi_k : S_k \to V$ be the renormalization change of variables realizing the renormalization of f_{k-1} and let $\phi_k := \psi_k^{-1} : V \to S_k$.

Let us normalize our coronas such that they have a critical value at 0. For each $k \leq 0$, consider the translation $T_k(z) = z - c_1(f_k)$ and denote

$$U_k^\natural = T_k(U_k), \quad V_k^\natural = T_k(V), \quad U_{k,\pm}^\natural = T_{k-1}(U_{k,\pm}), \quad S_k^\natural = T_{k-1}(S_k).$$

The translations T_k's normalize our maps f_k, F_k, and ϕ_k into

$$f_k^\natural : U_k^\natural \to V_k^\natural, \quad F_k^\natural := (f_{k,\pm}^\natural : U_{k,\pm}^\natural \to S_k^\natural), \quad \phi_k^\natural : V_k^\natural \to S_k^\natural$$

respectively. Consider the linear map

$$A_*(z) := \mu_* z$$

where $\mu_* := (\phi_*^\natural)'(0) \in \mathbb{D}$ is the self-similarity factor of f_*.

Lemma 5.4.2. *The limit*

$$h_f^\natural(z) := \lim_{k \to -\infty} A_*^k \circ \phi_{k+1}^\natural \circ \ldots \circ \phi_1^\natural \circ \phi_0^\natural(z)$$

defines a univalent map on a neighborhood D of 0 where D is independent of f.

Proof. As $\phi_k^\natural \to \phi_*^\natural$ exponentially fast, so is the derivative $\mu_k := (\phi_k^\natural)'(0)$ towards μ_*. There are positive constants ε and δ such that $\varepsilon < 1 - |\mu_*|$ and for all $|z| < \delta$ and $k \leq 0$, we have $|\phi_k^\natural(z)| \leq (|\mu_*| + \varepsilon)|z|$. Therefore, for all $|z| < \delta$ and $k \leq 0$,

$$|\phi_{k+1}^\natural \circ \ldots \circ \phi_0^\natural(z)| \leq (|\mu_*| + \varepsilon)^{-k}|z|.$$

The sequence $h^{(k)}(z) := A_*^k \circ \phi^\natural_{k+1} \circ \ldots \circ \phi^\natural_0 \circ \phi^\natural_0(z)$ indeed converges to a univalent map on $\{|z| < \delta\}$ since

$$\frac{h^{(k-1)}(z)}{h^{(k)}(z)} = \frac{\phi^\natural_k\left(\phi^\natural_{k+1} \circ \ldots \circ \phi^\natural_0(z)\right)}{\mu_* \phi^\natural_{k+1} \circ \ldots \circ \phi^\natural_0(z)} = \frac{\mu_k}{\mu_*} + O(|\phi^\natural_{k+1} \circ \ldots \circ \phi^\natural_0(z)|) \to 1$$

exponentially fast as $k \to -\infty$. □

For $k \leq 0$, let $h^\natural_k := h^\natural_{f_k}$ and denote its rescaling by $h^\#_k := A^k_* \circ h^\natural_k$. The following properties are easy to verify.

Proposition 5.4.3. *For $k \leq 0$,*

$$h^\natural_{k-1} \circ \phi^\natural_i = A_* \circ h^\natural_k \qquad \textit{and} \qquad h^\natural_0 = h^\#_k \circ \phi^\natural_{k+1} \circ \ldots \circ \phi^\natural_0.$$

Moreover, $h^\natural_0$ extends to a univalent map on the interior of $V^\natural_0 \backslash \gamma^\natural_1$.

The maps $h^\natural_k$ act as linear coordinates under which renormalization change of variables are simply linear maps. Objects in linear coordinates will be written in bold:

$$\mathbf{U}_{k,\pm} := h^\natural_k(U^\natural_{k,\pm}), \quad \mathbf{S}_k := h^\natural_k(S^\natural_k), \quad \mathbf{F}_k := (\mathbf{f}_{k,\pm} : \mathbf{U}_{k,\pm} \to \mathbf{S}_k).$$

Often, we will also work with the rescaled linear coordinates $h^\#_k$ in which we add the symbol "#" as follows:

$$\mathbf{U}^\#_{k,\pm} := h^\#_k(U^\natural_{k,\pm}), \quad \mathbf{S}^\#_k := h^\#_k(S^\natural_k), \quad \mathbf{F}^\#_k := \left(\mathbf{f}^\#_{k,\pm} : \mathbf{U}^\#_{k,\pm} \to \mathbf{S}^\#_k\right).$$

By design, it is clear that for all $k \leq 0$,

$$\mathbf{f}^\#_{k,\pm} = A^k_* \circ \mathbf{f}_{k,\pm} \circ A^{-k}_*. \tag{5.4.1}$$

Lemma 5.4.4. *There is a matrix of positive integers $\mathbf{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ such that for every negative integer k,*

$$\mathbf{f}^\#_{k+1,-} = \left(\mathbf{f}^\#_{k,-}\right)^{m_{11}} \circ \left(\mathbf{f}^\#_{k,+}\right)^{m_{12}} \quad \textit{and} \quad \mathbf{f}^\#_{k+1,+} = \left(\mathbf{f}^\#_{k,-}\right)^{m_{21}} \circ \left(\mathbf{f}^\#_{k,+}\right)^{m_{22}}.$$

Proof. The action of renormalization restricted to the Herman quasicircle of f_* is a sector renormalization, and in particular an iterate of prime renormalization. See §2.1.3. The existence of such a matrix $\mathbf{M}$ follows from §2.1.4. □

Theorem 5.4.5 (Maximal extension)**.** *Assume $\mathcal{U}$ is a sufficiently small Banach neighborhood of f_*. For every $f \in \mathcal{W}^u_{\text{loc}}$ and every $k \leq 0$, the maps $\mathbf{f}^\#_{k,\pm}$ described above extend to σ-proper branched coverings $\mathbf{X}^\#_{k,\pm} \to \mathbb{C}$, where $\mathbf{X}^\#_{k,\pm}$ are simply connected domains in $\mathbb{C}$.*

Remark 5.4.6. Actually, $\mathbf{X}^{\#}_{k,\pm}$ are dense subsets of $\mathbb{C}$. For the renormalization fixed point f_*, this will follow from Corollary 5.6.32 (2). For general $f \in \mathcal{W}^u_{\mathrm{loc}}$, this property will be apparent after we establish Theorem 5.7.5 on the holomorphic motion of $\partial\mathbf{X}^{\#}_{k,\pm}$.

Proof. For every $k \leq 0$, the composition $\phi_{k+1} \circ \ldots \circ \phi_0$ embeds the pre-corona $F_0 = (f_{0,\pm} : U_{0,\pm} \to V \backslash \gamma_1)$ to the dynamical plane of f_k as a pair of iterates

$$\left(f_k^{\mathbf{a}_k} : U^{(k)}_{0,-} \to V^{(k)}_0,\ f_k^{\mathbf{b}_k} : U^{(k)}_{0,+} \to V^{(k)}_0 \right). \tag{5.4.2}$$

Since ϕ_k is contracting at the critical value, the diameter of $U^{(k)}_{0,\pm} \to V^{(k)}_0$ shrinks to 0 as $k \to -\infty$.

To proceed, we need the following technical lemma.

Lemma 5.4.7. *Assume $\mathcal{U}$ is a sufficiently small Banach neighborhood of f_*. There is an open disk D around the critical value $c_1(f_*)$ of f_* such that for all sufficiently large $n \in \mathbb{N}$, $t \in \{\mathbf{a}_n, \mathbf{b}_n\}$, and $f \in \mathcal{R}^{-n}(\mathcal{U})$, then $f^t(c_1(f))$ is contained in D and D can be pulled back by f^t to a disk $D_0 \subset U_f \backslash \gamma_1$ containing $c_1(f)$ on which $f^t : D_0 \to D$ is a branched covering.*

This lemma initially appears in [DLS20, Key Lemma 4.8] in the context of quadratic Siegel pacmen. Due to its length, the proof will be supplied later in §5.4.2. The lemma tells us that for sufficiently large $k \ll 0$, the disk D contains the set $\{c_{1+\mathbf{a}_k}(f_k), c_{1+\mathbf{b}_k}(f_k)\}$ and the pair in (5.4.2) extends to a commuting pair of branched coverings

$$\left(f_k^{\mathbf{a}_k} : W^{(k)}_{-} \to D,\ f_k^{\mathbf{b}_k} : W^{(k)}_{+} \to D \right), \tag{5.4.3}$$

where $W^{(k)}_{\pm} \cup D$ are disks in $V \backslash \gamma_1$ containing $c_1(f_k)$. By conjugating with $h^{\#}_k \circ T_k$, we transform this pair into the commuting pair of branched coverings

$$\mathbf{f}_{0,\pm} : \mathbf{W}^{(k)}_{\pm} \to \mathbf{D}^{(k)}$$

where

$$\mathbf{W}^{(k)}_{\pm} := h^{\#}_k \circ T_k(W^{(k)}_{\pm}) \qquad \text{and} \qquad \mathbf{D}^{(k)} := h^{\#}_k \circ T_k(D).$$

For all sufficiently large t and $m \leq 0$, $\mathbf{D}^{(tm)}$ is compactly contained in $\mathbf{D}^{(tm-t)}$, and

$$\operatorname{mod}\left(\mathbf{D}^{(tm-t)} \backslash \overline{\mathbf{D}^{(tm)}} \right) \succ 1.$$

As such,

$$\bigcup_{k<0}^{\infty} \mathbf{D}^{(k)} = \mathbb{C}.$$

The maps $\mathbf{f}_{0,\pm}$ extend to σ-proper branched coverings from $\mathbf{X}_{0,\pm} := \bigcup_{k<0} \mathbf{W}^{(k)}_{\pm}$ onto $\mathbb{C}$. It is clear from the construction that $\mathbf{X}_{0,\pm}$ is a simply connected domain. $\square$

The proof of the theorem above actually gives us something stronger, which we will use later in Section §5.7.2.

Lemma 5.4.8 (Stability of σ-branched structure)**.** *Assume $\mathcal{U}$ is a sufficiently small Banach neighborhood of f_*. For every $f \in \mathcal{W}^u_{\text{loc}}$, there are sequences of nested disks*

$$\mathbf{W}_\pm^{(-1)} \subset \mathbf{W}_\pm^{(-2)} \subset \mathbf{W}_\pm^{(-3)} \subset \dots \qquad \textit{and} \qquad \mathbf{D}^{(-1)} \subset \mathbf{D}^{(-2)} \subset \mathbf{D}^{(-3)} \subset \dots$$

where

$$\bigcup_{k<0} \mathbf{W}_\pm^{(k)} = \mathbf{X}_{0,\pm} \qquad \textit{and} \qquad \bigcup_{k<0} \mathbf{D}^{(k)} = \mathbb{C},$$

such that for every $k < 0$,

(1) *each of $\mathbf{W}_\pm^{(k)}$ and $\mathbf{D}^{(k)}$ depends continuously on f;*

(2) *the maps $\mathbf{f}_{0,\pm} : \mathbf{W}_\pm^{(k)} \to \mathbf{D}^{(k)}$ are proper branched coverings of fixed finite degree;*

(3) *critical points of $\mathbf{f}_{0,\pm} : \mathbf{W}_\pm^{(k)} \to \mathbf{D}^{(k)}$ move holomorphically over $f \in \mathcal{U}$.*

Proof. The construction of such disks is similar to the proof of the previous theorem. We add the following modification. By Theorem 5.3.7, we can replace the disk D with a slightly smaller disk $D(f_0, k)$ depending continuously on f_0 such that for all $i \leq \max\{\mathbf{a}_k, \mathbf{b}_k\}$,

$$c_i(f_*) \in D(f_*, k) \quad \text{if and only if} \quad c_i(f_k) \in D(f_0, k).$$

Under this replacement, the domains of branched coverings $(f_k^{\mathbf{a}_k}, f_k^{\mathbf{b}_k})$ from (5.4.3) become

$$\mathbf{f}_{0,\pm} : W_\pm(f_0, k) \to D(f_0, k),$$

which depend continuously on f_0. By conjugating with $h_k^\# \circ T_k$, we obtain the commuting pair $\mathbf{f}_{0,\pm} : \mathbf{W}_\pm^{(k)} \to \mathbf{D}^{(k)}$ with the desired property. □

5.4.2 Key lemma for transcendental extension

Let us now discuss the proof of Lemma 5.4.7.

Fix a small neighborhood D of the critical value $c_1(f_*)$ of the renormalization fixed point f_*. Fix $n \in \mathbb{N}$ and a large constant $s \in \mathbb{N}$. We will denote by $\mathbf{a}_n$ and $\mathbf{b}_n$ the n^{th} renormalization return times. Consider a corona f that is $m := n + s$ times renormalizable such that $f_i := \mathcal{R}^i f$ is close to f_* for all $i \in \{1, \dots, m\}$. We will denote the critical orbit by $c_j(f) := f^j(c_0(f))$. Our goal is to show that for $t \in \{\mathbf{a}_n, \mathbf{b}_n\}$, $c_{1+t}(f)$ is contained in D and there is a branched covering map $f^t : (D_0, c_1(f)) \to (D, c_{1+t}(f))$. The proof we present below is similar to the

Key Lemma in [DLS20], which is to ensure that pullbacks of D must avoid the forbidden boundary.

Let h be a level m combinatorial pseudo-conjugacy between f and f_*, and consider the renormalization tiling $\mathbf{\Delta}_m(f) := h^{-1}(\mathbf{\Delta}_m(f_*))$ defined in §5.3.2. Recall that f maps $\Delta_m(f,i)$ univalently onto $\Delta_m(f,i+\mathbf{p}_n)$ whenever $i \notin \{-\mathbf{p}_m, -\mathbf{p}_m+1\}$, and on $\Delta_m(f,-\mathbf{p}_m) \cup \Delta_m(f,-\mathbf{p}_m+1)$, f is almost a degree d covering map branched at its critical point $c_0(f)$ onto its image, which contains $\Delta_m(f,0) \cup \Delta_m(f,1)$. By Theorem 5.3.7, h is close to the identity map and $\mathbf{\Delta}_m(f)$ approximates the Herman quasicircle $\mathbf{H}_*$ of f_*.

In the dynamical plane of f_*, for sufficiently large $n \gg 0$, both $c_{1+\mathbf{a}_n}(f_*)$ and $c_{1+\mathbf{b}_n}(f_*)$ are contained in D because it is sufficiently close to $c_1(f_*)$. Let us fix $t \in \{\mathbf{a}_n, \mathbf{b}_n\}$. Since s is picked to be large,

$$t \leq \max\{\mathbf{a}_n, \mathbf{b}_n\} < \min\{\mathbf{a}_m, \mathbf{b}_m\} - 1.$$

Therefore, the orbit $\{c_j(f_*)\}_{j=1,2,\dots t+1}$ avoids both $\Delta_m(-\mathbf{p}_m, f_*)$ and $\Delta_m(-\mathbf{p}_m+1, f_*)$. Since h is close to the identity, it follows that $c_{1+t}(f)$ is also contained in D.

Let

$$D_1, \ D_2, \ \dots, \ D_{t+1} := D$$

denote the lift of D along the orbit $c_1(f)$, $c_2(f)$, $\dots$, $c_{1+t}(f)$. We would like to show that for $i \in \{1,2,\dots,t\}$, the disk D_i does not intersect $\partial_{\mathrm{F}} U_f$ so that $f : D_i \to D_{i+1}$ is a branched covering.

5.4.2.1 A new tiling $\mathbf{\Lambda}_m$

We say that a subset I of $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$ is an *interval* if it is a sequence of consecutive elements of $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$ of cardinality less than $\mathbf{p}_m$. For any interval I in $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$, we will use the notation

$$\Delta_m(I) := \bigcup_{i \in I} \Delta_m(i)$$

and

$$f^{-1}I := \begin{cases} I - \mathbf{p}_m & \text{if } I \cap \{\mathbf{p}_m, \mathbf{p}_m+1, 0, 1\} = \varnothing, \\ (I - \mathbf{p}_m) \cup \{-\mathbf{p}_m, -\mathbf{p}_m+1\} & \text{if } I \cap \{0,1\} \neq \varnothing, \\ (I - \mathbf{p}_m) \cup \{0,1\} & \text{if } I \cap \{\mathbf{p}_m, \mathbf{p}_m+1\} \neq \varnothing. \end{cases}$$

The following property holds.

Claim 1. For any interval I in $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$, the lift of $\Delta_m(I)$ under $f|_{\mathbf{\Delta}_m}$ is contained in $\Delta_m(f^{-1}I)$.

First, consider the dynamical plane of $f_m := \mathcal{R}^m f : U_m \to V$. Let us define the tiling $\mathbf{\Lambda}_0(f_m) := \{\Lambda_0(i, f_m)\}_{i \in \{0,1\}}$, which is a skinnier version of $\mathbf{\Delta}_0(f_m)$, as follows. For $i \in \{0,1\}$, we define $\Lambda_0(i, f_m)$ to be the closure of the connected component of $f_m^{-1}(U_m)\setminus(\gamma_0(f_m)\cup\gamma_1)$ contained in $\Delta_0(i, f_m)$. Let us embed it via Φ_m to the dynamical plane of f and spread it around via iterates of f to obtain the tiling $\mathbf{\Lambda}_m = \mathbf{\Lambda}_m(f)$.

Similar to Claim 1, we have:

Claim 2. For any interval I in $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$, we have

$$\Lambda_m(I) = \mathbf{\Lambda}_m \cap \Delta_m(I)$$

and the lift of $\Lambda_m(I)$ under $f|_{\mathbf{\Lambda}_m}$ is contained in $\Lambda_m(f^{-1}I)$.

The problem with the tiling $\mathbf{\Delta}_m$ is that for $j \in \{1, \ldots, t\}$, even when $D_{j+1}\cap\mathbf{\Delta}_m$ is contained in $\Delta_m(I)$ for some interval I, it is possible that $D_j \cap \mathbf{\Delta}_m$ is not contained in $\Delta_m(f^{-1}I)$. However, this issue does not occur for the tiling $\mathbf{\Lambda}_m$.

Claim 3. For any interval I in $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$, any $j \in \{0, 1, \ldots, \min\{\mathbf{a}_m, \mathbf{b}_m\} - 1\}$, and any subset $T \subset V$,

$$T \cap \mathbf{\Delta}_m \subset \Delta_m(I) \implies f^{-j}(T) \cap \mathbf{\Lambda}_m \subset \Lambda_m(f^{-j}I).$$

Proof. By construction, the tiling $\mathbf{\Lambda}$ has the property that $f^j(\mathbf{\Lambda}_m) \subset \mathbf{\Delta}_m$ for all $j < \min\{\mathbf{a}_m, \mathbf{b}_m\}$. Let I, j, and T be as in the hypothesis and suppose $T \cap \mathbf{\Delta}_m$ is contained in $\Delta_m(I)$. Consider a point z in $\mathbf{\Lambda}_m$ such that $f^j(z)$ is contained in T. Clearly, $f^j(z)$ is in $\Delta_m(I)$, and by Claim 1, z is contained in $\mathbf{\Delta}_m(f^{-j}I)$. By Claim 2, the point z is indeed contained in $\Lambda_m(f^{-j}I)$. □

Consider the smallest interval I_{t+1} in $\mathbb{Z}/\mathbf{q}_m\mathbb{Z}$ such that

$$\{0, 1\} \subset I_{t+1} \quad \text{and} \quad D_{t+1} \cap \mathbf{\Delta}_m(f) \subset \Delta_m(I_{t+1}).$$

For $j \in \{1, \ldots t\}$, let $I_j := f^{-(t+1-j)}I_t$. It is assumed that $D \cap \mathbf{H}_*$ is roughly a level less than n combinatorial interval, so, since $m > n$, $|I_j|$ is large for all j.

Let us fix some positive integer η where $\eta \ll t$. This will be taken to be the maximum of the periods $\eta_k^\bullet$ introduced in the next subsection.

Claim 4. For $j \in \{1, 2, \ldots t + 1\}$,

(1) $|I_j|/\mathbf{q}_m$ is small and $\Delta_m(I_j, f_*) \cap \mathbf{H}_*$ has a small combinatorial length;

(2) if $j \leq t - 2 - \eta$, the intervals I_j, I_{j+1}, $\ldots$, $I_{j+\eta+3}$ are pairwise disjoint;

(3) if $1 \leq j \leq \eta + 2$, then I_j is disjoint from $\{-\mathbf{p}_m, -\mathbf{p}_m + 1\}$.

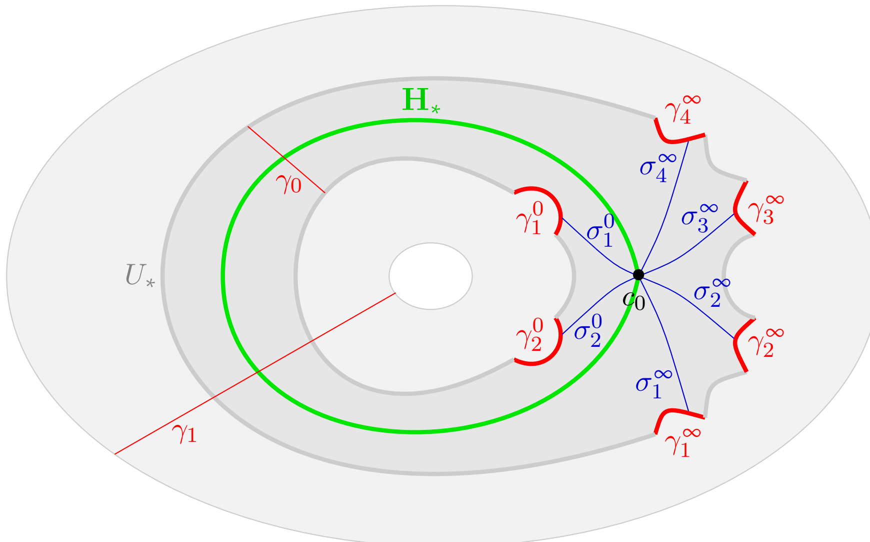

Figure 5.7: The spines of f_* of generation one when $(d_0, d_\infty) = (2, 3)$.

Proof. Since the rotation number is of bounded type, the combinatorial length of the intersection of $\mathbf{H}_*$ with every tile of $\mathbf{\Delta}_m(f_*)$ is comparable. Since D is assumed to be small and s is taken to be sufficiently large, (1) follows.

Since (2) is combinatorial in nature, it suffices to prove (2) in the dynamical plane of f_*, which is obvious from the irrational rotational action of f_* on $\mathbf{H}_*$. If (3) does not hold, then for some integer $j \in [2, \eta+3]$, the interval I_j intersects $\{0,1\}$, but this contradicts (2) and the fact that I_1 must intersect $\{0,1\}$. □

5.4.2.2 Spines and pseudo-spines

Let us first consider the dynamical plane of f_*. Recall that the preimage of $f_*^{-1}(\gamma_1)\backslash\gamma_0$ consists of arcs

$$\gamma_1^0, \ldots, \gamma_{2(d_0-1)}^0 \subset \partial^0 U_*, \quad \gamma_1^\infty, \ldots, \gamma_{2(d_\infty-1)}^\infty \subset \partial^\infty U_*.$$

The strict preimage $f^{-1}(\mathbf{H}_*)\backslash\mathbf{H}_*$ is a bouquet of pairwise disjoint arcs

$$\sigma_1^0, \ \ldots, \ \sigma_{2(d_0-1)}^0, \quad \sigma_1^\infty, \ \ldots, \ \sigma_{2(d_\infty-1)}^\infty$$

where each $\sigma_i^\bullet$ connects $c_0(f_*)$ to a point on $\gamma_i^\bullet$. We call each of $\sigma_i^\bullet$ a *spine* of f_* of generation one. In general, a *spine* of generation $g \geq 1$ is a lift of under f_*^{g-1} of a spine of generation one, and its *root* is the endpoint that is a critical point of f_*^g.

A *spine chain* of generation g is an infinite sequence of spines

$$\Sigma = (S_1, S_2, S_3, \ldots)$$

of increasing generation such that S_1 has generation g and for all $i \geq 1$, the root of S_{i+1} is contained in S_i. We say that a spine chain $\mathcal{S}$ is *periodic* with period p if for all $i \geq 1$, $f_*^p(S_{i+1}) = S_i$.

The following is a direct consequence of Lemma 5.2.3 and Theorem 5.2.7.

Proposition 5.4.9. *Every spine chain of f_* lands at a unique point. Different spine chains admits different landing points. The landing point of a periodic spine chain of period p is a repelling periodic point of period p, and it is also the landing point of exactly one periodic external ray of period p.*

When f is rotational with bounded type rotation number, the notion of spines of f can be formulated analogously and the proposition above holds. Below, we will formulate an analog of bubbles for arbitrary coronas f which are sufficiently close to f_*. This is achieved by replacing $\mathbf{H}_*$ with $\mathbf{\Lambda}_m(f)$.

For f, $\bullet \in \{0,1\}$, and $i \in \{1,\dots,2(d_\bullet - 1)\}$, we define the *pseudo-spine* $\mathbb{S}_i^\bullet$ of generation one to be the closure of the connected component of $f^{-1}(\mathbf{\Lambda}_m)\backslash\mathbf{\Lambda}_m$ that intersects with the spine $\sigma_i^\bullet$ of f_*. Each $\mathbb{S}_i^\bullet$ is connected and

$$\mathbb{S}_i^\bullet \cap \mathbf{\Lambda}_m \subset \Lambda_m(\{-\mathbf{p}_m, -\mathbf{p}_m + 1\}), \qquad f(\mathbb{S}_i^\bullet) \subset \mathbf{\Lambda}_m.$$

We say that every pseudo-spine of generation one is attached to $\mathbf{\Lambda}(\{-\mathbf{p}_m, -\mathbf{p}_m + 1\})$. In general, a *pseudo-spine* of generation $g \geq 1$ is a lift under f^{g-1} of a pseudo-spine of generation one.

Let us fix a large integer $M \gg 1$. We will assume that f is sufficiently close to f_* depending on M.

Claim 5. Every spine S of f_* of generation up to M is approximated by a pseudo-spine $\mathbb{S}$ of f such that

1. $\mathbb{S}$ is close to S and $f|_{\mathbb{S}}$ is close to $f_*|_S$,
2. if S is attached to another spine S', then $\mathbb{S}$ is attached to the pseudo-spine corresponding to S';
3. if S is attached to $\mathbf{H}_*$, then $\mathbb{S}$ is attached to $\Lambda_m(I)$ for some interval I disjoint from $\{0,1\}$.

Proof. This is because $\mathbf{\Lambda}_m$ compactly contains and well approximates $\mathbf{H}_*$. □

Let us fix $\bullet \in \{0,1\}$ and $k \in \{1,\dots,2(d_\bullet - 1)\}$. Let us construct a periodic spine chain

$$\Sigma_k^\bullet = (S_1, S_2, S_3, \dots)$$

for f_* that is very close to $\gamma_k^\bullet$. First, we set $S_1 := \sigma_k^\bullet$. Let us pick $\eta_k^\bullet \geq 1$ such that the pre-critical point $c_{-\eta_k^\bullet+1}(f)$ is close to the critical arc γ_1. Let $c_k^\bullet$ be the preimage of $c_{-\eta_k^\bullet+1}(f)$ located on $\sigma_k^\bullet$ close to $\gamma_k^\bullet$. Then, we set S_2 to be the unique spine rooted at $c_k^\bullet$ that is the lift of S_1 under $f^{\eta_k^\bullet}$. The other spines are then defined by induction, forming a periodic spine chain of period $\eta_k^\bullet$.

Let $x_k^\bullet(f_*)$ be the landing point of $\Sigma_k^\bullet$. It is a repelling periodic point of period $\eta_k^\bullet$ and it is also the landing point of a periodic external ray $R_k^\bullet(f_*)$. Since f is close to f_*, periodic rays $R_k^\bullet(f)$ and repelling periodic points $x_k^\bullet(f)$ exist in the dynamical plane of f.

Let us define a periodic pseudo-spine chain

$$\mathbb{\Sigma}_k^\bullet = (\mathbb{S}_1, \mathbb{S}_2, \mathbb{S}_3, \dots) \tag{5.4.4}$$

for f landing at $x_k^\bullet(f)$ as follows. Assume $M \gg \eta_k^\bullet$ and let $M' \in \mathbb{N}$ satisfy $\eta_k^\bullet M' \leq M$. For $2 \leq j \leq M'$, we set $\mathbb{S}_j$ to be the pseudo-spine approximating S_j. This can be arranged so that $\mathbb{S}_{M'}$ is within the linearization domain of the repelling periodic point $x_k^\bullet(f)$, and so inductively we define $\mathbb{S}_{M'+j+1}$ to be the unique lift of $\mathbb{S}_{M'+j}$ under $f^{\eta_k^\bullet}$ that is even closer to $x_k^\bullet(f)$.

5.4.2.3 Enlargements of D_j

Let us inductively define enlargements $\mathcal{D}_j$ and $\mathcal{D}_j'$ of D_j as follows. First, we set $\mathcal{D}_{t+1} = \mathcal{D}_{t+1}' := D$. For $j \leq t$, we set

- ▷ $\mathcal{D}_j'$ = the connected component of $f^{-1}(\mathcal{D}_{j+1})$ containing D_j;
- ▷ $\mathcal{D}_j$ = the smallest topological disk containing $\mathcal{D}_j'$ and the interior of $\Lambda_m(I_j)$.

For all j, we have $D_j \subset \mathcal{D}_j' \subset \mathcal{D}_j$.

Claim 6. For $j \in \{1, 2, \dots, t+1\}$, $\mathcal{D}_j \cap \mathbf{\Lambda}_m$ is connected and its closure is $\Lambda_m(I_j)$.

Proof. This follows from the observation that, due to Claim 3, $D_j \cap \mathbf{\Lambda}_m \subset \Lambda_m(I_j)$ for all j. □

We will assume D to be small enough such that it is disjoint from the periodic rays $f^i(R_k^\bullet)$ for all $i \in \{0, \dots, t\}$, $\bullet \in \{0, \infty\}$, and $k \in \{1, 2, \dots, 2(d_\bullet - 1)\}$.

Claim 7. For $j \in \{1, 2, \dots, t\}$, the disk $\mathcal{D}_j$ is disjoint from all the periodic rays of the form $f^i(R_k^\bullet)$ for all $i \in \{0, \dots, j-1\}$.

Proof. This claim follows from induction. If $\mathcal{D}_j$ intersects $f^i(R_k^\bullet)$, then $\mathcal{D}_j'$ intersects $f^i(R_k^\bullet)$ and so $\mathcal{D}_{j+1}$ intersects $f^{i+1}(R_k^\bullet)$. □

5.4.2.4 Proof of Lemma 5.4.7

Let $\mathbf{\Lambda}'_m$ denote the union of all pseudo-spines of f of generation one. Recall that the constant η is set to be the maximum of the periods $\eta^\bullet_k$ of the pseudo-spine chains $\mathbb{\Sigma}^\bullet_k$. To finally show that $f^t : D_1 \to D$ is a branched covering, we will prove by induction the following statements for $j = 1, \dots, t+1$.

(a) $\mathcal{D}_j$ intersects $\mathbf{\Lambda}'_m$ if and only if I_j contains $\{-\mathbf{p}_m, -\mathbf{p}_{m+1}\}$;

(b) If $\mathcal{D}_j$ intersects $\mathbf{\Lambda}'_m$, then the intersection is in a small neighborhood of c_0;

(c) If $\mathcal{D}_j$ intersects $\mathbf{\Lambda}'_m$ for $j < t$, then $j < t - \eta$ and $\mathcal{D}_{j+1}, \dots \mathcal{D}_{j+\eta+1}$ are all disjoint from $\mathbf{\Lambda}'_m$;

(d) If $\mathcal{D}_j$ intersects a pseudo-spine chain $\mathbb{\Sigma}^\bullet_k$ from (5.4.4), then the intersection is within $\mathbf{\Lambda}'_m$;

(e) $\mathcal{D}_j$ is an open disk disjoint from the forbidden boundary $\partial_{\mathrm{F}} U_f$.

Suppose (a)–(e) hold for $j+1$, $j+2$, $\dots t+1$. Let us show that they hold for j.

Suppose I_j contains $\{-\mathbf{p}_m, -\mathbf{p}_{m+1}\}$. Then, $\mathcal{D}_{j+1}$ contains either $\Lambda_m(\{-1,0,1\})$ or $\Lambda_m(\{0,1,2\})$, and so the lift $\mathcal{D}'_j$ of $\mathcal{D}_{j+1}$ contains the critical point $c_0(f)$ and intersects $\mathbf{\Lambda}'_m$.

Suppose I_j is disjoint from $\{-\mathbf{p}_m, -\mathbf{p}_{m+1}\}$. Then, $\mathcal{D}_{j+1}$ does not contain the critical value $c_1(f)$ and every point in $\mathcal{D}_{j+1}$ has at most one preimage under f in $\mathcal{D}'_j$. By Claim 6, the preimage of $\mathcal{D}_{j+1} \cap \mathbf{\Lambda}_m$ under $f|_{\mathcal{D}'_j}$ must be contained in $\mathbf{\Lambda}_m$. It follows that $\mathcal{D}'_j$ is disjoint from $\mathbf{\Lambda}'_m$. Since $\mathcal{D}'_j \cup \Lambda_m(I_j)$ does not surround $\mathbf{\Lambda}'_m$, then $\mathcal{D}_j$ is also disjoint from $\mathbf{\Lambda}'_m$.

We just proved (a). Item (b) follows from Claim 6 and the fact that $\Lambda_m(I_{j+1})$ is a small neighborhood of $c_1(f)$ as a result of Claim 4 (i). Item (c) then follows from Claim 4 (2).

Item (e) follows from (b) and (d). Indeed, if $\mathcal{D}_j$ were to intersect $\partial_{\mathrm{F}} U_f$, then by Claim 7, it must intersect some pseudo-spine chain $\mathbb{\Sigma}^\bullet_k$ from (5.4.4) and because of (d), its intersection is contained in $\mathbf{\Lambda}'_m$. In particular, $\mathcal{D}_k$ can only intersect $\mathbf{\Lambda}'_m$ in a small neighborhood of c_0, which implies that $\mathcal{D}_k$ cannot intersect $\partial_{\mathrm{F}} U_f$.

It remains to prove (d). By continuity, we can assume that (d) holds whenever $j \geq t - \eta$. Let us assume that $j < t - \eta$ and suppose for a contradiction that (d) fails, that is, there is a pseudo-spine chain $\mathbb{\Sigma}^\bullet_k = (\mathbb{S}_1, \mathbb{S}_2, \dots)$ such that $\mathcal{D}_j$ intersects $\mathbb{S}_i$ where $i \geq 2$.

We claim that $\mathcal{D}_j$ intersects $\mathbb{S}_2$. Indeed, suppose otherwise that the smallest possible $i \geq 2$ such that $\mathcal{D}_j$ intersects $\mathbb{S}_i$ satisfies $i > 2$. Since $\mathcal{D}'_j \cap \Lambda_m(I_j)$ is disjoint from the ray $R^\bullet_k$, then the subchain $\mathbb{\Sigma}^{(i)} = (\mathbb{S}_i, \mathbb{S}_{i+1}, \dots)$ intersects $\mathcal{D}'_j$ and its image $f(\mathbb{\Sigma}^{(i)})$ intersects $\mathcal{D}_{j+1}$. By periodicity of $\mathbb{\Sigma}^\bullet_k$, the chain $\mathbb{\Sigma}^{(i-1)}$ intersects $\mathcal{D}_{j+\eta^\bullet_k}$, which is a contradiction to (d) for index $j + \eta^\bullet_k$.

The argument from the previous paragraph results in the intersection of $\mathcal{D}_{j+\eta_k^\bullet}$ and $\mathbb{S}_1$ being non-empty. By (a), the interval $I_{j+\eta_k^\bullet}$ contains $\{-\mathbf{p}_m, -\mathbf{p}_{m+1}\}$, so for $l \in \{1, 2, \dots, \eta_k^\bullet\}$, $f^l(\mathbb{S}_2)$ is attached to $\Lambda_m(I_{j+l})$. Moreover, since the critical value $c_1(f)$ is not contained in $\mathcal{D}_{j+l} \cap \mathbf{\Lambda}_m$, every point in $\mathcal{D}_{j+l}$ has at most one preimage in $\mathcal{D}'_{j+l-1}$.

Consider the lift $\mathbb{S}'_2$ of $f(\mathbb{S}_2)$ under f that is attached to $\Lambda_m(I_j)$. Since $c_1(f)$ is not contained nor surrounded by $\mathcal{D}_{j+1} \cap f(\mathbb{S}_2)$, the lift E of $f(\mathcal{D}_j \cap \mathbb{S}_2)$ under $f|_{\mathcal{D}'_j}$ agrees with the lift under $f|_{\mathbb{S}'_2}$. Therefore, E would be contained in $\mathbb{S}'_2$, not $\mathbb{S}_2$, which is impossible. This concludes the proof of (d).

5.5 Dynamics of cascades

In §5.4.1, we have established that every corona f in the local unstable manifold $\mathcal{W}^u_{\text{loc}}$ of the fixed point f_* of the corona renormalization operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ induces the sequence $\{\mathbf{F}^\#_n = (\mathbf{f}^\#_{n,\pm} : \mathbf{X}^\#_{k,\pm} \to \mathbb{C})\}_{n \le 0}$ of pairs of σ-proper branched coverings. We will reinterpret such a sequence as a global transcendental cascade and formulate a Fatou-Julia theory for cascades.

5.5.1 Cascades

Consider the anti-renormalization matrix $\mathbf{M}$ from Lemma 5.4.4. Let us denote by $\mathbf{t} > 1$ and $1/\mathbf{t}$ the eigenvalues of $\mathbf{M}$.

For every positive integer n, let us define $\mathbf{F}^\#_n = (\mathbf{f}^\#_{n,\pm})$ inductively by the relation

$$(\mathbf{f}^\#_{n,-})^a \circ (\mathbf{f}^\#_{n,+})^b = (\mathbf{f}^\#_{n-1,-})^{a'} \circ (\mathbf{f}^\#_{n-1,+})^{b'} \tag{5.5.1}$$

for any $a, b, a', b' \in \mathbb{Z}_{\ge 0}$ satisfying $(a' \ b') = (a \ b)\mathbf{M}$. Then, $\left\{\mathbf{F}^\#_n\right\}_{n \in \mathbb{Z}}$ forms a sequence of commuting σ-proper holomorphic maps acting on the same dynamical plane.

Let us identify the local unstable manifold $\mathcal{W}^u_{\text{loc}}$ with the space $\boldsymbol{\mathcal{W}}^u_{\text{loc}}$ of pairs of σ-proper maps $\mathbf{F} = (\mathbf{f}_{0,\pm})$ associated to each $f \in \mathcal{W}^u_{\text{loc}}$. We extend our renormalization operator beyond $\boldsymbol{\mathcal{W}}^u_{\text{loc}}$ by setting

$$\mathcal{R}^n \mathbf{F}_0 = \mathbf{F}_n := A_*^{-n} \mathbf{F}^\#_n A_*^n,$$

(compare with (5.4.1)) and extend $\boldsymbol{\mathcal{W}}^u_{\text{loc}}$ to a global unstable manifold $\boldsymbol{\mathcal{W}}^u$ by adding $\mathbf{F}_n$ for all $n \ge 0$ and $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u_{\text{loc}}$. The complex manifold structure of $\boldsymbol{\mathcal{W}}^u_{\text{loc}}$ naturally extends to $\boldsymbol{\mathcal{W}}^u$ and the renormalization operator $\mathcal{R}$ now acts on $\boldsymbol{\mathcal{W}}^u$ as a biholomorphism with a unique fixed point $\mathbf{F}_*$, which is repelling.

In the rest of this dissertation, we will show that $\mathcal{R} : \boldsymbol{\mathcal{W}}^u \to \boldsymbol{\mathcal{W}}^u$ is holomorphically conjugate to an expanding linear map on $\mathbb{C}$. To do so, we will study each map on $\boldsymbol{\mathcal{W}}^u$ as a cascade of global transcendental maps.

Definition 5.5.1. We define the space $\mathbf{T}$ of *power-triples* to be the quotient of the semigroup $\mathbb{Z}\times\mathbb{Z}^2_{\geq 0}$ under the equivalence relation $\sim$ where $(n,a,b)\sim(n-1,a',b')$ if and only if $(a'\ b')=(a\ b)\mathbf{M}$.

We will equip $\mathbf{T}$ with the binary operation $+$ defined by

$$(n,a,b)+(n,a',b')=(n,a+a',b+b').$$

With respect to $+$, $\mathbf{T}$ has a unique identity element $0:=(n,0,0)$. For $P,Q\in\mathbf{T}$, let us denote by $P\geq Q$ if for all sufficiently large $n\ll 0$, there exist $a,b,a',b'\in\mathbb{N}$ such that $P=(n,a,b)$, $Q=(n,a',b')$, $a\geq a'$, and $b\geq b'$.

By Lemma 2.1.15, $(\mathbf{T},+,\geq)$ can be identified with a sub-semigroup of $(\mathbb{R}_{\geq 0},+,\geq)$. Moreover, $\mathbf{T}$ inherits a well-defined scalar multiplication by powers of $\mathbf{t}$ as follows. For every $(n,a,b)\in\mathbf{T}$ and integer k,

$$\mathbf{t}^k(n,a,b)=(n+k,a,b).$$

For every $\mathbf{F}\in\boldsymbol{\mathcal{W}}^u$ and every power-triple $P=(n,a,b)$, we will use the notation

$$\mathbf{F}^P:=\left(\mathbf{f}^{\#}_{n,-}\right)^a\circ\left(\mathbf{f}^{\#}_{n,+}\right)^b.$$

Each $\mathbf{F}^P$ is a σ-proper map from its domain $\mathrm{Dom}\left(\mathbf{F}^P\right)$ onto $\mathbb{C}$. We denote by $\mathbf{F}^{\geq 0}$ the cascade $\left(\mathbf{F}^P\right)_{P\in\mathbf{T}}$ associated to $\mathbf{F}$.

Lemma 5.5.2. *For every $\mathbf{F}\in\boldsymbol{\mathcal{W}}^u$, $P\in\mathbf{T}$, and $n\in\mathbb{Z}$,*

$$\mathbf{F}_0^P=\left(\mathbf{F}^{\#}_{-n}\right)^{\mathbf{t}^nP}.$$

In particular, when $\mathbf{F}=\mathbf{F}_*$,

$$\mathbf{F}_*^P=A_*^{-n}\circ\mathbf{F}_*^{\mathbf{t}^nP}\circ A_*^n. \tag{5.5.2}$$

5.5.2 Critical points and periodic points

Let us pick

$$\mathbf{F}=[\mathbf{f}_\pm:\mathbf{U}_\pm\to\mathbf{S}]:=[\mathbf{f}_{0,\pm}:\mathbf{U}_{0,\pm}\to\mathbf{S}_0]\in\boldsymbol{\mathcal{W}}^u_{\mathrm{loc}}$$

and let $\mathbf{F}_n:=\mathcal{R}^n\mathbf{F}$ for all $n\in\mathbb{Z}$. Within the cascade $\mathbf{F}^{\geq 0}$, $\mathbf{f}_\pm$ is the first return map of points in $\mathbf{U}_\pm$ back to $\mathbf{S}$. In particular, $\mathbf{U}_-\cup\mathbf{U}_+$ is disjoint from $\mathbf{F}^P(\mathbf{U}_-)$ for all $P<(0,1,0)$ and $\mathbf{F}^P(\mathbf{U}_+)$ for all $P<(0,0,1)$.

Definition 5.5.3. We define the *zeroth renormalization tiling* $\boldsymbol{\Delta}_0=\boldsymbol{\Delta}_0(\mathbf{F})$ associated to $\mathbf{F}^{\geq 0}$ to be the tiling consisting of $\Delta_0(0):=\overline{\mathbf{U}_+}$ and $\Delta_0(1):=\overline{\mathbf{U}_-}$, as well as $\mathbf{F}^P(\Delta_0(0))$ for all $P<(0,0,1)$ and $\mathbf{F}^P(\Delta_0(1))$ for all $P<(0,1,0)$. We label the tiles in left-to-right order as

$\Delta_0(i)$ for $i \in \mathbb{Z}$. For all $n \in \mathbb{Z}_{<0}$, we define the n^{th} *renormalization tiling* to be the rescaling of the zeroth tiling for $\mathbf{F}_n$, namely

$$\mathbf{\Delta}_n(\mathbf{F}) := A_*^n \left(\mathbf{\Delta}_0(\mathbf{F}_n)\right).$$

In $\boldsymbol{\mathcal{W}}^u_{\text{loc}}$, $\mathbf{F}$ is sufficiently close to $\mathbf{F}_*$ and the tiling $\mathbf{\Delta}_0(\mathbf{F})$ moves holomorphically in $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u_{\text{loc}}$. In general, for $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, the n^{th} tiling $\mathbf{\Delta}_n(\mathbf{F})$ is well-defined for all sufficiently large $n \ll 0$. Each tile $\Delta_n(i)$ is a compact disk in $\mathbb{C}$.

Definition 5.5.4. Consider $[f : U_f \to V] \in \mathcal{W}^u_{\text{loc}}$ and the associated pre-corona $\mathbf{F} = [\mathbf{f}_\pm : \mathbf{U}_\pm \to \mathbf{S}] \in \boldsymbol{\mathcal{W}}^u_{\text{loc}}$. Given a subset Z of U_f, the *full lift* $\mathbf{Z}$ of Z to the dynamical plane of $\mathbf{F}$ is defined as

$$\mathbf{Z} := \bigcup_{0 \le P < (0,0,1)} \mathbf{F}^P(\mathbf{Z}_0) \cup \bigcup_{0 \le P < (0,1,0)} \mathbf{F}^P(\mathbf{Z}_1),$$

where $\mathbf{Z}_0$ and $\mathbf{Z}_1$ are the embedding of $Z \cap \Delta_0(0,f)$ and $Z \cap \Delta_0(1,f)$ to the dynamical plane of $\mathbf{F}$ respectively.

In particular, we will define the *Herman curve* $\mathbf{H}$ of $\mathbf{F}_*$ to be the full lift of the Herman quasicircle of f_*. Observe that $\mathbf{H}$ is an A_*-invariant quasiarc.

Let us fix $\mathbf{F}$ in $\boldsymbol{\mathcal{W}}^u$. For every $x \in \mathbb{C}$ and $T \in \mathbf{T}$, we denote the finite orbit of x up to time T by

$$\text{orb}_x^T(\mathbf{F}) := \left\{\mathbf{F}^P(x) \,:\, 0 \le P \le T\right\}.$$

Definition 5.5.5. For $P \in \mathbf{T}_{>0}$, let us denote by $\text{CP}\left(\mathbf{F}^P\right)$ the set of critical points of $\mathbf{F}^P$ and by $\text{CV}\left(\mathbf{F}^P\right)$ the set of critical values of $\mathbf{F}^P$. We say that a point x is

- ▹ a *critical point* of $\mathbf{F}^{\ge 0}$ if it is in $\text{CP}\left(\mathbf{F}^P\right)$ for some $P \in \mathbf{T}_{>0}$,
- ▹ a *critical value* of $\mathbf{F}^{\ge 0}$ if it is in $\text{CV}\left(\mathbf{F}^P\right)$ for some $P \in \mathbf{T}_{>0}$, and
- ▹ a *periodic point* of $\mathbf{F}^{\ge 0}$ if there is some $P \in \mathbf{T}_{>0}$ such that $\mathbf{F}^P(x) = x$.

Lemma 5.5.6. *For $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, critical points of $\mathbf{F}^{\ge 0}$ satisfy the following properties.*

(1) *A point x is a critical point of $\mathbf{F}^{\ge 0}$ if and only if $\mathbf{F}^P(x) = 0$ for some $P \in \mathbf{T}_{>0}$.*

(2) *For $P \in \mathbf{T}_{>0}$,*

$$\text{CP}\left(\mathbf{F}^P\right) = \bigcup_{0 < S \le P} \mathbf{F}^{-S}\{0\} \quad \textit{and} \quad \text{CV}\left(\mathbf{F}^P\right) = \{\mathbf{F}^S(0) \,:\, 0 \le S < P\}.$$

(3) *There is some* $K_{\mathbf{F}} \in \mathbf{T}_{>0}$ *such that for every power-triple* $P < K_{\mathbf{F}}$*, every critical point of* $\mathbf{F}^P$ *has local degree* d*. If* 0 *is not periodic, this is still true for* $P \geq K_{\mathbf{F}}$*. In general, for every* $P \in \mathbf{T}$*, there is some* $k \in \mathbb{N}$ *such that the local degree of every critical point of* $\mathbf{F}^P$ *is at most* k*.*

Let $T := \min\{(0,1,0),(0,0,1)\}$*. If* $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u_{\mathrm{loc}}$*, then for every* $P < T$*,*

(4) $\mathrm{CV}(\mathbf{F}^P)$ *is a subset of* $\boldsymbol{\Delta}_0(\mathbf{F})\backslash\mathbf{S} \cup \{0\}$ *which moves holomorphically with* $\mathbf{F}$*, and*

(5) *every critical point of* $\mathbf{F}^P$ *has local degree* d*.*

Proof. Pick a bounded domain $\mathbf{D} \Subset \mathbb{C}$ and select a connected component $\mathbf{D}'$ of $\mathbf{F}^{-P}(\mathbf{D})$. Suppose $\mathbf{F}$ represents $f \in \mathcal{W}^u_{\mathrm{loc}}$, and recall that the anti-renormalizations of f when the critical value is normalized to be at 0 are denoted by $f_n^{\natural}$, $n \leq 0$. Recall that for sufficiently large $n \ll 0$, the map $\mathbf{F}^P : \mathbf{D}' \to \mathbf{D}$ can be identified via $h_n^{\#}$ with $(f_n^{\natural})^{s_n} : D' \to D$ for some domains $D', D \Subset \mathbb{C}$ and some $s_n \geq 0$. Therefore, x is a critical point of $\mathbf{F}^P$ if and only if $(h_n^{\#})^{-1}(x)$ is a critical point of $(f_n^{\natural})^{s_n}$, which happens precisely when $\mathbf{F}^S(x) = 0$ for some $S \leq P$. This leads to (1) and (2).

Suppose $[\mathbf{F} : \mathbf{U}_{\pm} \to \mathbf{S}]$ is in $\boldsymbol{\mathcal{W}}^u_{\mathrm{loc}}$ and fix $P \leq T$. For all $S < P$, $\mathbf{F}^S(0)$ is contained in some tile $\Delta_0(i,\mathbf{F})$ that is disjoint from $\mathbf{S}$. This implies (4). Also, (5) follows from the fact that for every critical point x of $\mathbf{F}^P$, $\mathrm{orb}_x^P(\mathbf{F})$ passes through the critical value 0 exactly once.

If $\mathbf{F}$ is not close to $\mathbf{F}_*$, then we can take some $n \ll 0$ such that $\mathcal{R}^n\mathbf{F}$ is in $\boldsymbol{\mathcal{W}}^u_{\mathrm{loc}}$. Then, (3) follows from (4) and (5) by taking $K_{\mathbf{F}}$ to be $\mathbf{t}^n T$ and k to be such that $P < (k-1)K_{\mathbf{F}}$. □

Lemma 5.5.7 (Discreteness)**.** *For any* $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$ *and any bounded open subset* D *of* $\mathbb{C}$*, there is some* $Q \in \mathbf{T}_{>0}$ *such that for all* $\mathbf{G} \in \boldsymbol{\mathcal{W}}^u$ *close to* $\mathbf{F}$ *and whenever* $P' < P < Q$*,*

(1) $\mathbf{G}^P$ *is well-defined and univalent on* D*, and*

(2) $\mathbf{G}^P(D)$ *is disjoint from* $\mathbf{G}^{P'}(D)$*.*

For every $x \in \mathbb{C}$ *and* $T \in \mathbf{T}$*,* $\mathrm{orb}_x^T(\mathbf{F})$ *is discrete in* $\mathbb{C}$*.*

Proof. There exist some integers $m \leq 0$ and $j \in \{0,1\}$ such that D is compactly contained in some level m tile $\Delta_m(j,\mathbf{G})$ associated to $\mathbf{G}$ for all $\mathbf{G}$ close to $\mathbf{F}$. Set $Q := \mathbf{t}^m \min\{(0,1,0),(0,0,1)\}$. For $P < Q$, the tile $\Delta_m(j,\mathbf{G})$ is mapped by $\mathbf{G}^P$ univalently onto to some other tile $\Delta_m(i,\mathbf{G})$ of level m disjoint from $\Delta_m(0,\mathbf{G}) \cup \Delta_m(1,\mathbf{G})$. This implies (1) and (2).

Let us fix $x \in \mathbb{C}$ and $T \in \mathbf{T}$. Let us pick any point y in the closure of $\mathrm{orb}_x^T(\mathbf{F})$, and pick a small open neighborhood D of y. From the first part, $\mathbf{F}^P(D)$ is disjoint from D for all sufficiently small $P \in \mathbf{T}_{>0}$. This implies that only finitely many points in $\mathrm{orb}_x^T(\mathbf{F})$ are contained in D. □

By a straightforward compactness argument, the lemma above has the following consequence.

Corollary 5.5.8 (Proper discontinuity)**.** *For any $P \in \mathbf{T}$, any compact subset $\mathbf{Y}$ of* $\mathrm{Dom}(\mathbf{F}^P)$*, and any bounded subset $\mathbf{X}$ of $\mathbb{C}$, there are at most finitely many power-triples $T \leq P$ such that $\mathbf{F}^T(\mathbf{Y})$ intersects $\mathbf{X}$.*

Corollary 5.5.9. *Every critical point x of $\mathbf{F}^{\geq 0}$ admits a minimal $P \in \mathbf{T}_{>0}$, called the generation of x, such that $\mathbf{F}^P(x) = 0$.*

Proof. By definition, there is some $P \in \mathbf{T}_{>0}$ such that $\mathbf{F}^P(x) = 0$. By Lemma 5.5.7, $\mathrm{orb}_x^P(\mathbf{F})$ is discrete, so there are at most finitely many power-triples S such that $S < P$ and $\mathbf{F}^S(x) = 0$. □

Corollary 5.5.10. *Every periodic point of $\mathbf{F}^{\geq 0}$ has a minimal period.*

Proof. Suppose x is a periodic point of $\mathbf{F}^{\geq 0}$. The set $\mathbf{T}_x := \{P \in \mathbf{T} : \mathbf{F}^P(x) = x\}$ of periods of x is a sub-semigroup of $\mathbf{T}$. Pick a small neighborhood D of x. By Lemma 5.5.7, there is some $Q \in \mathbf{T}_{>0}$ such that for all $0 < P < Q$, $\mathbf{F}^P(D)$ is disjoint from D and thus $P \notin \mathbf{T}_x$. This implies that $\mathbf{T}_x$ is finitely generated, and in particular, of the form $\{nS\}_{n\in\mathbb{N}}$, where $S \in \mathbf{T}_{>0}$ is the minimal period. □

5.5.3 The escaping sets

Consider $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$.

Definition 5.5.11. Given $P \in \mathbf{T}$, the P^{th} *escaping set* of $\mathbf{F}$ is

$$\mathbf{I}_{\leq P}(\mathbf{F}) := \mathbb{C} \backslash \mathrm{Dom}\left(\mathbf{F}^P\right).$$

The *finite-time escaping set* of $\mathbf{F}$ is the union

$$\mathbf{I}_{<\infty}(\mathbf{F}) := \bigcup_{P \in \mathbf{T}} \mathbf{I}_{\leq P}(\mathbf{F}),$$

the *infinite-time escaping set* of $\mathbf{F}$ is

$$\mathbf{I}_{\infty}(\mathbf{F}) := \{z \in \mathbb{C} \backslash \mathbf{I}_{<\infty}(\mathbf{F}) : \mathbf{F}^P(z) \to \infty \text{ as } P \to \infty\},$$

and the *full escaping set* of $\mathbf{F}$ is

$$\mathbf{I}(\mathbf{F}) := \mathbf{I}_{<\infty}(\mathbf{F}) \cup \mathbf{I}_{\infty}(\mathbf{F}).$$

Lemma 5.5.12. *For any $P \in \mathbf{T}$, every connected component of $\mathbf{I}_{\leq P}(\mathbf{F})$ is unbounded.*

Proof. There exists some $n \leq 0$ such that $\mathbf{F}_n := \mathcal{R}^n \mathbf{F}$ is in $\mathcal{W}^u_{\mathrm{loc}}$. Since the domains of $\mathbf{f}_{n,\pm}$ are simply connected, then $\mathrm{Dom}\,(\mathbf{F}_n^P)$ is simply connected for all $P \in \mathbf{T}$ and so the claim is true for $\mathbf{F}_n$. Since $\mathbf{F}$ is just a rescaling of $\mathbf{F}_n$, the claim is also true for $\mathbf{F}$. □

In Section 5.6, we will thoroughly study the structure of the finite-time escaping set of the fixed point $\mathbf{F}_*$. In Section 5.7, we will show that when $\mathbf{F}$ is hyperbolic, the finite and infinite-time escaping sets do not carry any invariant line field.

It is clear from the definition that the boundary of $\mathbf{I}_{\leq P}(\mathbf{F})$ coincides with the boundary of $\mathrm{Dom}\,(\mathbf{F}^P)$. Points on $\partial \mathbf{I}_{\leq P}(\mathbf{F})$ can be regarded as essential singularities of $\mathbf{F}^P$. The following lemma is an analog of Picard's theorem.

Lemma 5.5.13. *For every $P \in \mathbf{T}_{>0}$ and any sufficiently small Euclidean disk D centered at a point in $\partial \mathbf{I}_{\leq P}(\mathbf{F})$, the image $\mathbf{F}^P(D')$ of any connected component D' of $D \cap \mathrm{Dom}\,(\mathbf{F}^P)$ is dense in $\mathbb{C}$.*

This lemma is a direct consequence of σ-properness of $\mathbf{F}^P$. The keen reader may refer to [DL23, Lemma 6.5] for a detailed proof.

Corollary 5.5.14. *For every $\mathbf{F} \in \mathcal{W}^u$, $P \in \mathbf{T}_{>0}$, and $x \in \mathbb{C}$, the boundary of $\mathbf{I}_{\leq P}(\mathbf{F})$ is the set of accumulation points of $\mathbf{F}^{-P}(x)$.*

We will later show that $\mathbf{I}_{<\infty}(\mathbf{F})$ has no interior and its closure coincides with the "Julia set" of $\mathbf{F}$, which we will define in the next subsection. This corollary is an analog of the basic result in holomorphic dynamics which states that iterated preimages are dense in the Julia set. The proof below is similar to [DL23, Corollary 6.7].

Proof. By Lemma 5.5.7, there exists a disk neighborhood B of x such that $B \backslash \{x\}$ is disjoint from $\mathrm{CV}\,(\mathbf{F}^P)$. Then, every connected component B' of $\mathbf{F}^{-P}(B)$ contains at most one critical point and the degree of $\mathbf{F}^P : B' \to B$ is at most some uniform constant. Let $\Omega \subset B$ be an even smaller disk neighborhood of x such that $\mathrm{mod}\left(B \backslash \overline{\Omega}\right) \asymp 1$. Any lift $\Omega' \subset B'$ of Ω under $\mathbf{F}^P$ is also a disk with $\mathrm{mod}(B' \backslash \overline{\Omega'}) \asymp 1$.

Let us pick a connected component D of $\mathrm{Dom}\,(\mathbf{F}^P)$, a point $y \in \partial D$, and a small $\varepsilon > 0$. By Lemma 5.5.13, there is a connected component $\Omega' \subset D$ of $\mathbf{F}^{-P}(\Omega)$ that is of distance at most ε away from y. Since $\mathrm{mod}\left(B' \backslash \overline{\Omega'}\right) \asymp 1$, then Ω' has a small diameter depending on ε. Since Ω' contains point in $\mathbf{F}^{-P}(x)$, the assertion follows. □

5.5.4 Fatou-Julia theory

Let us formulate a Fatou-Julia theory for our dynamical systems $\mathbf{F}$ in $\boldsymbol{\mathcal{W}}^u$ and state a few analogues of basic results in classical holomorphic dynamics.

Definition 5.5.15. The *Fatou set* $\mathfrak{F}(\mathbf{F})$ of $\mathbf{F}$ is the set of points z which admit a small neighborhood $D \subset \mathbb{C}\backslash\mathbf{I}_{<\infty}(\mathbf{F})$ such that $\{\mathbf{F}^P|_D\}_{P\in\mathbf{T}}$ forms a normal family. The *Julia set* $\mathfrak{J}(\mathbf{F})$ of $\mathbf{F}$ is the complement $\mathbb{C}\backslash\mathfrak{F}(\mathbf{F})$.

Clearly, $\mathfrak{J}(\mathbf{F})$ contains the closure of $\mathbf{I}_{<\infty}(\mathbf{F})$.

We say that a connected component D of $\mathfrak{F}(\mathbf{F})$ is *periodic* if there is some $P \in \mathbf{T}_{>0}$ such that $\mathbf{F}^P(D) = D$. The smallest such P is called the *period* of D. Moreover, we say that D is *pre-periodic* if there is some $Q \in \mathbf{T}$ such that $\mathbf{F}^Q(D)$ is periodic. The smallest such Q is called the *pre-period* of D. (These quantities exist due to Lemma 5.5.7. Compare with Corollary 5.5.10.)

Proposition 5.5.16. *For all $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, every connected component of the Fatou set $\mathfrak{F}(\mathbf{F})$ is simply connected.*

In particular, $\mathbf{F}$ does not admit any Herman rings.

Proof. This is a standard application of the maximum modulus principle. Pick any Jordan domain D such that ∂D is contained in $\mathfrak{F}(\mathbf{F})$. For all $P \in \mathbf{T}$, $\mathbf{F}^P|_D$ attains maximum on Γ, thus $\{\mathbf{F}^P|_D\}_P$ forms a normal family. □

Definition 5.5.17. The *postcritical set* of $\mathbf{F}$ is

$$\mathfrak{P}(\mathbf{F}) := \overline{\{\mathbf{F}^P(0) \,:\, P \in \mathbf{T}\}}.$$

The postcritical set is the smallest forward invariant closed set such that

$$\mathbf{F}^P : \mathrm{Dom}\left(\mathbf{F}^P\right)\backslash\mathbf{F}^{-P}\left(\mathfrak{P}(\mathbf{F})\right) \to \mathbb{C}\backslash\mathfrak{P}(\mathbf{F})$$

is an unbranched covering map which is a local isometry with respect to the hyperbolic metrics.

In the case of $\mathbf{F} = \mathbf{F}_*$, equation (5.5.2) implies self-similarity of the corresponding dynamical sets.

Lemma 5.5.18. *The linear map A_* preserves $\mathfrak{F}(\mathbf{F}_*)$, $\mathfrak{J}(\mathbf{F}_*)$, $\mathbf{I}_{<\infty}(\mathbf{F}_*)$, $\mathbf{I}_\infty(\mathbf{F}_*)$, and $\mathfrak{P}(\mathbf{F}_*)$. For all $P \in \mathbf{T}_{>0}$, $A_*(\mathbf{I}_{\le P}(\mathbf{F}_*)) = \mathbf{I}_{\le \mathbf{t}P}(\mathbf{F}_*)$.*

Given a periodic point x of (minimal) period P of some $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, we say that x is *superattracting / attracting / parabolic / Siegel / Cremer / repelling* if x is a *superattracting / attracting / parabolic / Siegel / Cremer / repelling* fixed point of $\mathbf{F}^P$.

Proposition 5.5.19. *Suppose* $\mathbf{F}$ *admits a periodic point* x *of some period* P.

(1) *If* x *is attracting or parabolic, then the critical orbit* $\{\mathbf{F}^T(0)\}_{T\in\mathbf{T}}$ *converges to the periodic orbit* $\mathrm{orb}_0^P(\mathbf{F})$.

(2) *If* x *is Cremer, then* $x \in \mathfrak{P}(\mathbf{F})$.

(3) *If* x *is Siegel, then the boundary of the Siegel disk of* $\mathbf{F}^P$ *about* x *is contained in* $\mathfrak{P}(\mathbf{F})$.

Proof. (1) follows from a standard analytic continuation argument: if the forward orbit of a periodic Fatou component containing an attracting (resp. superattracting or parabolic) cycle does not containing 0, then the local linearizing (resp. Böttcher or Fatou) coordinates can be extended to a conformal map onto the whole plane, which is impossible. See [Mil06, Lemma 8.5] for details.

Suppose x is not in $\mathfrak{P}(\mathbf{F})$ and is not repelling. From (1), x is either Cremer or Siegel. Let us first prove (2) by showing that x must be Siegel. For all $T \in \mathbf{T}$, let us denote by D_T the connected component of $\mathrm{Dom}(\mathbf{F}^T)\backslash\mathbf{F}^{-T}(\mathfrak{P}(\mathbf{F}))$ containing x. Suppose first that D_P is properly contained in D_0. Then, $\mathbf{F}^P : D_P \to D_0$ is strictly expanding with respect to the hyperbolic metric of D_0, which implies that x must be repelling. Suppose instead $D_P = D_0$. Then, $\{\mathbf{F}^{nP}|_{D_0}\}_{n\in\mathbb{N}}$ is a normal family of automorphisms of a hyperbolic Riemann surface. By Denjoy-Wolff, the fixed point x must be Siegel.

Denote by Z the Siegel disk centered at x. If there exists some minimal $T \in \mathbf{T}$ where $\mathbf{F}^T(0)$ intersects Z, then the intersection $\mathfrak{P}(\mathbf{F}) \cap Z$ is a single $\mathbf{F}^P$-invariant curve on Z. Suppose for a contradiction that $\mathbb{C}\backslash\mathfrak{P}(\mathbf{F})$ intersects the boundary ∂Z. Then, a component E_0 of $\mathbb{C}\backslash\mathfrak{P}(\mathbf{F})$ contains some neighborhood of ∂Z. For $n \in \mathbb{N}$, let E_{nP} be the connected component of $\mathrm{Dom}(\mathbf{F}^{nP})\backslash\mathbf{F}^{-nP}(\mathfrak{P}(\mathbf{F}))$ containing $E_0 \cap Z_0$. There are again two cases. If $E_P = E_0$, then $\{\mathbf{F}^{nP}|_{E_0}\}_{n\in\mathbb{N}}$ forms a normal family and E_0 must be contained in the Fatou set, which is a contradiction. If E_P is a proper subset of E_0, then $\mathbf{F}^P : E_P \to E_0$ is strictly expanding with respect to the hyperbolic metric of E_0, which would contradict the fact that $\mathbf{F}^P$ restricts to a self diffeomorphism of any invariant curve in $Z \cap E_0$. □

5.6 The external structure of $\mathbf{F}_*$

Consider the dynamics of $\mathbf{F} = \mathbf{F}_*$ corresponding to the fixed point f_* of the renormalization operator. We denote by $\mathbf{H}$ the Herman curve of $\mathbf{F}$, which is defined to be the full lift of the

Herman curve of f_*. The action of $\mathbf{F}$ along $\mathbf{H}$ can be described as follows. For $a \in \mathbb{C}$, we denote the translation map by a by $T_a(z) := z + a$.

Lemma 5.6.1. *There is a quasisymmetric map $h : (\mathbf{H}, 0) \to (\mathbb{R}, 0)$ that conjugates the cascade $(\mathbf{F}^P|_{\mathbf{H}})_{P\in\mathbf{T}}$ with the cascade of translations $(T^P)_{P\in\mathbf{T}}$ defined by $T^{(n,a,b)} := T_{\mathbf{t}^{-n}(b\mathbf{v}-a\mathbf{u})}$, where $\mathbf{u}, \mathbf{v} > 0$ and $\theta = \frac{\mathbf{u}}{\mathbf{u}+\mathbf{v}}$.*

Proof. The pre-corona F_* associated to f_* admits an invariant quasiarc which projects to the Herman curve of f_*. In linear coordinates, this corresponds to an invariant quasiarc $\mathbf{H}_0$ of $(\mathbf{f}_{0,\pm} : \mathbf{U}_\pm \to \mathbf{S})$ which passes through 0 and connects $\mathbf{f}_{0,+}(0)$ and $\mathbf{f}_{0,-}(0)$. The dynamics of $\mathbf{f}_{0,\pm}$ along $\mathbf{H}_0$ is quasisymetrically conjugate to a pair of translations $\left(T_{-\theta}|_{[0,1-\theta]}, T_{1-\theta}|_{[-\theta,0]}\right)$ on the real interval $[-\theta, 1-\theta]$. Set $\mathbf{u} = -\theta$ and $\mathbf{v} = 1-\theta$. As we extend $\mathbf{f}_{0,\pm}$ to its maximal σ-proper extension via A_*, the quasisymmetric conjugacy h between $(\mathbf{f}_{0,-}, \mathbf{f}_{0,+})$ and $(T_{-\mathbf{u}}, T_{-\mathbf{v}})$ extends to the whole lift $\mathbf{H}$ of $\mathbf{H}_0$. The claim holds because the pairs $(\mathbf{f}_{0,-}, \mathbf{f}_{0,+})$ and $(T_{-\mathbf{u}}, T_{-\mathbf{v}})$ generate the cascades $\mathbf{F}^{\geq 0}|_{\mathbf{H}}$ and $T^{(n,a,b)} := T_{\mathbf{t}^{-n}(b\mathbf{v}-a\mathbf{u})}$ via iteration and rescaling according to (5.5.2) and §2.1.4. □

In this section, we will comprehensively describe the dynamics of $\mathbf{F}$ beyond $\mathbf{H}$. We study the structure of preimages of $\mathbf{H}$ in §5.6.1–5.6.2, then the structure of the finite-time escaping set $\mathbf{I}_{<\infty} := \mathbf{I}_{<\infty}(\mathbf{F})$ in §5.6.3–5.6.4, and lastly the dynamical puzzles cut out by subsets of $\mathbf{I}_{<\infty}$ in §5.6.5.

5.6.1 Lakes

Let us label the components of $\mathbb{C}\backslash\mathbf{H}$ by $\mathbf{O}^0$ and $\mathbf{O}^\infty$, which we will refer to as the *oceans* of $\mathbf{F}$. The two oceans will be distinguished as follows. For $\bullet \in \{0, \infty\}$ and for any point x in $\mathbf{S} \cap \mathbf{O}^\bullet$ close to 0, we assume that there are $d_\bullet$ preimages of x under $\mathbf{f}_{0,\pm} : \mathbf{U}_\pm \to \mathbf{S}$ that are located near the critical point and inside of $\mathbf{O}^\bullet$.

Definition 5.6.2. A *lake* $\mathbf{O}$ *of generation* $P \in \mathbf{T}$ is a connected component of $\mathbf{F}^{-P}(\mathbf{O}^\bullet)$ for some $\bullet \in \{0, \infty\}$. Its *coast* is defined by $\partial^c\mathbf{O} := \partial\mathbf{O} \cap \operatorname{Dom}(\mathbf{F}^P)$.

Lemma 5.6.3 (Chessboard rule)**.** *For every $P \in \mathbf{T}_{>0}$ and $\bullet \in \{0, \infty\}$, the preimage $\mathbf{F}^{-P}(\mathbf{H})$ is a tree in* $\operatorname{Dom}(\mathbf{F}^P)$ *and $\mathbf{F}^{-P}(\mathbf{O}^\bullet)$ is disjoint union of lakes $\bigcup_{i\in\mathbb{N}} \mathbf{O}_i$ of generation P such that*

(1) *each lake $\mathbf{O}_i$ is an unbounded non-separating disk in* $\operatorname{Dom}(\mathbf{F}^P)$*;*

(2) *for $j \neq i$, the intersection $\partial^c\mathbf{O}_i \cap \partial^c\mathbf{O}_j$ is either empty or a singleton consisting of a critical point of $\mathbf{F}^P$.*

Proof. The whole lemma follows immediately from σ-properness of the cascade, e.g. [DL23, Lemma 5.1], and the fact that $\mathrm{CV}(\mathbf{F})$ is contained in $\mathbf{H}$. $\square$

Given any lake $\mathbf{O}$ of some generation $P \in \mathbf{T}_{>0}$, the map $\mathbf{F}^P$ sends $\mathbf{O}$ univalently onto an ocean, and its coast homeomorphically onto $\mathbf{H}$. In general, when $0 < P < Q$, a lake of generation Q is contained in a lake of generation P, and $\mathbf{F}^{Q-P}$ conformally sends any lake of generation Q onto a lake of generation P.

Lemma 5.6.4. *For every $P \in \mathbf{T}_{>0}$, there is a unique critical point $C_P \in \mathbf{H}$ of $\mathbf{F}^{\geq 0}$ of generation P and a pairwise disjoint collection of lakes*

$$_1\mathbf{O}^0_P, \ \ldots, \ _{2d_0-3}\mathbf{O}^0_P, \ _1\mathbf{O}^\infty_P, \ \ldots, \ _{2d_\infty-3}\mathbf{O}^\infty_P, \tag{5.6.1}$$

of generation P together with a bouquet of pairwise-disjoint open quasiarcs

$$_1\mathbf{H}^0_P, \ \ldots, \ _{2d_0-2}\mathbf{H}^0_P, \ _1\mathbf{H}^\infty_P, \ \ldots, \ _{2d_\infty-2}\mathbf{H}^\infty_P, \tag{5.6.2}$$

rooted at C_P such that for each $\bullet \in \{0,\infty\}$ and $j \in \{1,\ldots,2d_\bullet - 3\}$,

(1) *the coast of $_j\mathbf{O}^\bullet_P$ is $_j\mathbf{H}^\bullet_P \cup \{C_P\} \cup {}_{j+1}\mathbf{H}^\bullet_P$;*

(2) *$_j\mathbf{O}^\bullet_P$ is contained in $\mathbf{O}^\bullet$;*

(3) *$_j\mathbf{O}^\bullet_P$ is mapped conformally by $\mathbf{F}^P$ onto $\mathbf{O}^\bullet$ if j is is even, and onto $\mathbb{C}\backslash\overline{\mathbf{O}^\bullet}$ if j is odd.*

Proof. The existence and uniqueness of C_P is due to the fact that $\mathbf{F}^P$ restricts to a homeomorphism on $\mathbf{H}$. From the previous lemma, $\mathbf{F}^{-P}(\mathbf{H})$ is a tree. The quasiarcs $_j\mathbf{H}^\bullet_P$'s are precisely the components of $\mathbf{F}^{-P}(\mathbf{H})\backslash\{C_P\}$, and the lakes $_j\mathbf{O}^\bullet_P$'s in (5.6.2) are precisely the connected components of $\mathrm{Dom}\,(\mathbf{F}^P)\backslash\mathbf{F}^{-P}(\mathbf{H})$ which touch $\mathbf{H}$ at exactly one point, which is C_P. For all $S < P$, the image of each quasiarc $_j\mathbf{H}^\bullet_P$ under $\mathbf{F}^S$ is disjoint from 0. Therefore, $\mathbf{F}^P$ maps each of $_j\mathbf{H}^\bullet_P$ onto a component of $\mathbf{H}\backslash\{0\}$ homeomorphically. They can be enumerated such that the three claims above hold because C_P has inner and outer criticalities d_0 and d_∞ respectively. $\square$

Each quasiarc in (5.6.2) is called a *spine* of C_P. The spines in (5.6.2) are labelled in counterclockwise order about C_P.

Let us pick a pair of power-triples $P, Q \in \mathbf{T}_{>0}$. For any $\bullet \in \{0,\infty\}$ and any $j \in \{1,\ldots,d_\bullet - 1\}$, the union of two consecutive spines $_{2j-1}\mathbf{H}^\bullet_P \cup {}_{2j}\mathbf{H}^\bullet_P$ are mapped homeomorphically by $\mathbf{F}^P$ onto $\mathbf{H}\backslash\{0\}$ and so it contains a unique critical point $_jC^\bullet_{P,Q}$ of generation $P+Q$. Attached to this critical point is a bouquet of lakes

$$_{j,1}\mathbf{O}^{\bullet,0}_{P,Q}, \ \ldots, \ _{j,2d_0-3}\mathbf{O}^{\bullet,0}_{P,Q}, \ _{j,1}\mathbf{O}^{\bullet,\infty}_{P,Q}, \ \ldots, \ _{j,2d_\infty-3}\mathbf{O}^{\bullet,\infty}_{P,Q},$$

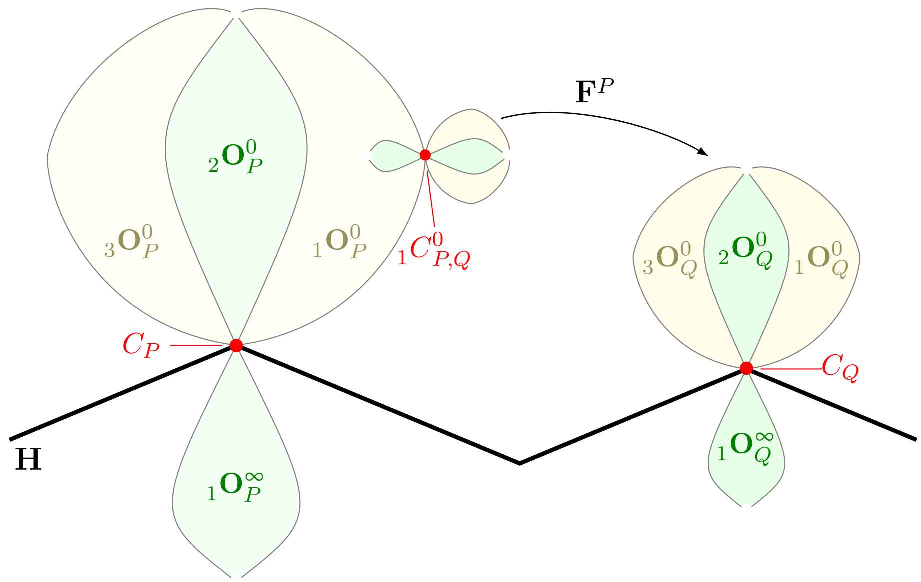

Figure 5.8: The structure of lakes attached to critical points C_P, C_Q, and ${}_1C^0_{P,Q}$ when $(d_0, d_\infty) = (3, 2)$.

of generation $P + Q$ together with spines

$$_{j,1}\mathbf{H}^{\bullet,0}_{P,Q},\quad \dots,\quad {}_{j,2d_0-2}\mathbf{H}^{\bullet,0}_{P,Q},\ {}_{j,1}\mathbf{H}^{\bullet,\infty}_{P,Q},\quad \dots,\quad {}_{j,2d_\infty-2}\mathbf{H}^{\bullet,\infty}_{P,Q},$$

meeting at ${}_jC^\bullet_{P,Q}$ such that ${}_{j,k}\mathbf{O}^{\bullet,\circ}_{P,Q}$ has coast

$$\partial^c\, {}_{j,k}\mathbf{O}^{\bullet,\circ}_{P,Q} = {}_{j,k}\mathbf{H}^{\bullet,\circ}_{P,Q} \cup \left\{ {}_jC^\bullet_{P,Q} \right\} \cup {}_{j,k+1}\mathbf{H}^{\bullet,\circ}_{P,Q}$$

and is mapped univalently by $\mathbf{F}^P$ onto ${}_k\mathbf{O}^\circ_Q$.

Consider a tuple $S = (P_1, \dots, P_{m+1}) \in \mathbf{T}^{m+1}_{>0}$ of $m + 1$ power-triples for some $m \in \mathbb{N}$. We denote the sum by

$$|S| \coloneqq \sum_{i=1}^{m+1} P_i.$$

Given $\blacksquare = (\bullet_1, \dots, \bullet_m) \in \{0, \infty\}^m$ and $J = (j_1, \dots j_m)$ where $j_i \in \{1, \dots, d_{\bullet_i} - 1\}$ for all i, we inductively define a critical point ${}_JC^\blacksquare_S$ of generation $|S|$. Attached to this critical point are lakes ${}_{J,i}\mathbf{O}^{\blacksquare,\bullet}_S$ for $\bullet \in \{0, \infty\}$ and $i \in \{1, \dots, 2d_\bullet - 3\}$, and spines ${}_{J,j}\mathbf{H}^{\blacksquare,\bullet}_S$ for $\bullet \in \{0, \infty\}$ and $j \in \{1, \dots, 2d_\bullet - 2\}$.

Definition 5.6.5. We say that a lake $\mathbf{O}$ is a *middle lake* if it is of the form ${}_{J,j}\mathbf{O}^{\blacksquare,\bullet}_S$ described above. The finite tuple S is called the *itinerary* of $\mathbf{O}$.

Consider a lake $\mathbf{O}$ of generation $P \in \mathbf{T}_{>0}$. Let $Q \in \mathbf{T}$ be the smallest power-triple such that the coast of $\mathbf{O}$ touches $\mathbf{F}^{-Q}(\mathbf{H})$.

Lemma 5.6.6 (Left and right coasts)**.** *The intersection between $\partial^c\mathbf{O}$ and $\mathbf{F}^{-Q}(\mathbf{H})$ is either a singleton or a closed quasiarc, and the complement $\partial^c\mathbf{O}\backslash\mathbf{F}^{-Q}(\mathbf{H})$ consists of two non-empty open quasiarcs $\partial^c_l\mathbf{O}$ and $\partial^c_r\mathbf{O}$.*

Proof. It is sufficient to consider the case when $Q = 0$. The intersection between $\partial^c\mathbf{O}$ and $\mathbf{F}^{-Q}(\mathbf{H})$ is connected because of the tree structure of $\mathbf{F}^{-P}(\mathbf{H})$. For any point z in $\partial^c\mathbf{O} \cap \mathbf{H}$, every component of $\mathbf{H}\backslash\{z\}$ contains infinitely many critical points of generation at most P, and each of these points is a branch point of the tree $\mathbf{F}^{-P}(\mathbf{H})$. Since $\partial^c\mathbf{O} \cap \mathbf{H}$ does not contain such branch points, the claim follows. □

We call $\partial^c_l\mathbf{O}$ and $\partial^c_r\mathbf{O}$ the *left and right coasts* of $\mathbf{O}$ respectively, and we always assume that $\partial^c_l\mathbf{O}$, $\partial^c\mathbf{O} \cap \mathbf{F}^{-Q}(\mathbf{H})$, and $\partial^c_r\mathbf{O}$ are oriented counterclockwise relative to $\mathbf{O}$.

The closure of the left coast of $\mathbf{O}$ admits a maximal sequence of critical points $c_{l,1}$, $c_{l,2}$, $c_{l,3}$, ... of $\mathbf{F}^P$, labelled in increasing order of generation. We define the *left itinerary* of $\mathbf{O}$ to be the sequence $I_l := (P_{l,1}, P_{l,2}, \ldots)$ where each $P_{l,i}$ is the generation of $c_{l,i}$. Similarly, we define the *right itinerary* I_r of $\mathbf{O}$.

Lemma 5.6.7. *Consider a lake $\mathbf{O}$ of generation $P \in \mathbf{T}_{>0}$ with left and right itineraries $I_l = (P_{l,1}, P_{l,2}, \ldots)$ and $I_r = (P_{r,1}, P_{r,2}, \ldots)$ respectively.*

(1) $\sup_i P_{l,i} = \sup_j P_{r,j} = P$.

(2) *If I_l (resp. I_r) is finite, the left (resp. right) coast of $\mathbf{O}$ contains a spine attached a critical point ${}_JC^{\blacksquare}_S$ of generation $|S| = P$.*

(3) *If both I_l and I_r are finite, then $\mathbf{O}$ is a middle lake attached to the critical point ${}_JC^{\blacksquare}_S$.*

(4) *Either I_l or I_r is a finite sequence.*

Proof. Suppose for a contradiction that $\sup_i P_{l,i} < P$, so then there is some $P' \in \mathbf{T}$ such that $\sup_i P_{l,i} < P' < P$. Then, $\mathbf{F}^{P'}(\mathbf{O})$ is a lake of positive generation with an empty left coast, which is impossible due to Lemma 5.6.6. Therefore, (1) holds.

Suppose I_l is finite. By (1), there exists a critical point c_l of generation P on $\overline{\partial^c_l\mathbf{O}}$. Removing c_l splits the coast into two open quasiarcs, one of which contains no critical points of $\mathbf{F}^P$ and is thus a spine attached to c_l. This implies (2). Suppose I_r is also finite, so there also exists a critical point c_r of generation P on $\overline{\partial^c_r\mathbf{O}}$. The complement of the interval $[c_l, c_r] \subset \partial^c\mathbf{O}$ is now a pair of spines of generation P attached to c_l and c_r respectively. Recall that $\mathbf{F}^P$ sends each of these spines to a component of $\mathbf{H}\backslash\{0\}$. However, since $\mathbf{F}^P : \partial^c\mathbf{O} \to \mathbf{H}$ is a homeomorphism, we see that $c_l = c_r$ and $\mathbf{O}$ is a middle lake. Hence, (3) holds.

Let us now prove (4). We will again assume without loss of generality that the coast of $\mathbf{O}$ touches $\mathbf{H}$. Let us pick a point y in $\partial^c\mathbf{O} \cap \mathbf{H}$. If the open interval $(y, C_P) \subset \mathbf{H}$ does not contain any critical point of generation $\leq P$, then either $\partial^c_l\mathbf{O}$ or $\partial^c_r\mathbf{O}$ is rooted at C_P and contains no other critical points of generation $\leq P$. Otherwise, by Lemma 5.5.6, there are only finitely many critical points of generation $\leq P$ within (y, C_P), and they have some maximum generation $R < P$. We then apply the previous argument to the lake $\mathbf{F}^R(\mathbf{O})$ and the interval $(\mathbf{F}^R(y), C_{P-R}) \subset \mathbf{H}$. $\square$

Consider a critical point ${}_JC^{\blacksquare}_S$ of $\mathbf{F}^{\geq 0}$. There exist lakes

$$ {}_{J,l}\mathbf{O}^{\blacksquare,0}_S, \quad {}_{J,r}\mathbf{O}^{\blacksquare,0}_S, \quad {}_{J,l}\mathbf{O}^{\blacksquare,\infty}_S, \quad {}_{J,r}\mathbf{O}^{\blacksquare,\infty}_S \tag{5.6.3} $$

of generation $|S|$ such that

(i) they are disjoint from all the middle lakes rooted at ${}_JC^{\blacksquare}_S$;

(ii) for $\bullet \in \{0, \infty\}$, the right coast of ${}_{J,l}\mathbf{O}^{\blacksquare,\bullet}_S$ contains the spine ${}_{J,2d_\bullet-2}\mathbf{H}^{\blacksquare,\bullet}_S$ and the left coast of ${}_{J,r}\mathbf{O}^{\blacksquare,\bullet}_S$ contains the spine ${}_{J,1}\mathbf{H}^{\blacksquare,\bullet}_S$;

(iii) if $j, j' \in \{l, r\}$ and $j \neq j'$, the coasts of ${}_{J,j}\mathbf{O}^{\blacksquare,0}_S$ and ${}_{J,j'}\mathbf{O}^{\blacksquare,\infty}_S$ intersect on a non-degenerate closed interval in $\mathbf{F}^{-|S|}(\mathbf{H})$ with endpoint ${}_JC^{\blacksquare}_S$.

We will call the lakes in (5.6.3) the *left/right side lakes* of ${}_JC^{\blacksquare}_S$.

Observe that by (ii),

$$ {}_{J,r}\mathbf{O}^{\blacksquare,0}_S,\ {}_{J,1}\mathbf{O}^{\blacksquare,0}_S,\ \ldots,\ {}_{J,2d_0-3}\mathbf{O}^{\blacksquare,0}_S,\ {}_{J,l}\mathbf{O}^{\blacksquare,0}_S,\ {}_{J,r}\mathbf{O}^{\blacksquare,\infty}_S,\ {}_{J,1}\mathbf{O}^{\blacksquare,\infty}_S,\ \ldots,\ {}_{J,2d_\infty-3}\mathbf{O}^{\blacksquare,\infty}_S,\ {}_{J,l}\mathbf{O}^{\blacksquare,\infty}_S $$

are in counterclockwise order about ${}^{\blacksquare}_JC_S$ and the closure of their union is a neighborhood of ${}^{\blacksquare}_JC_S$. By Lemma 5.6.7 (4), the left itinerary of ${}_{J,l}\mathbf{O}^{\blacksquare,\bullet}_S$ and the right itinerary of ${}_{J,r}\mathbf{O}^{\blacksquare,\bullet}_S$ are infinite. The following is a consequence of Lemma 5.6.7 (2)–(4).

Corollary 5.6.8. *Every lake $\mathbf{O}$ is either a middle lake or a side lake of a critical point ${}_JC^{\blacksquare}_S$. In other words, $\mathbf{O}$ is of the form ${}_{J,j}\mathbf{O}^{\blacksquare,\bullet}_S$ where $j \in \{l, 1, \ldots, 2d_\bullet - 3, r\}$.*

Given some tuple $S = (P_1, \ldots, P_k) \in \mathbf{T}^k_{>0}$, we can perform scalar multiplication by $\mathbf{t}$ and denote $\mathbf{t}S := (\mathbf{t}P_1, \ldots, \mathbf{t}P_k)$. The following is a direct consequence of (5.5.2).

Lemma 5.6.9. *For any middle or side lake ${}_{J,j}\mathbf{O}^{\blacksquare,\bullet}_S$ rooted at a critical point ${}_JC^{\blacksquare}_S$,*

$$ A_*\left({}_JC^{\blacksquare}_S\right) = {}_JC^{\blacksquare}_{\mathbf{t}S} \quad \textit{and} \quad A_*\left({}_{J,j}\mathbf{O}^{\blacksquare,\bullet}_S\right) = {}_{J,j}\mathbf{O}^{\blacksquare,\bullet}_{\mathbf{t}S}. $$

Proof. Recall from (5.5.2) that A_* conjugates $\mathbf{F}^P$ and $\mathbf{F}^{\mathbf{t}P}$ for any $P \in \mathbf{T}_{>0}$. Since A_* preserves $\mathbf{H}$, then $A_*(C_P) = C_{\mathbf{t}P}$ and thus $A_*\left({}_j\mathbf{O}_P^\bullet\right) = {}_j\mathbf{O}_{\mathbf{t}P}^\bullet$ for all $\bullet \in \{0,\infty\}$ and $j \in \{l, 1, \ldots, 2d_\bullet - 3, r\}$.

Suppose a spine ${}_j\mathbf{H}_P^\bullet$ attached to C_P contains some critical point ${}_iC_{P,Q}^\bullet$ where $i = \lceil \frac{j}{2} \rceil$. Since $A_*({}_iC_{P,Q}^\bullet)$ is contained in ${}_j\mathbf{H}_{\mathbf{t}P}^\bullet$ and is a critical point of generation $\mathbf{t}(P+Q)$, then it is equal to ${}_iC_{\mathbf{t}P,\mathbf{t}Q}^\bullet$. The rest follows by induction. $\square$

5.6.2 Limbs

Definition 5.6.10. A *limb* ${}_J\mathbf{L}_S^\blacksquare$ is the union of the spine ${}_J\mathbf{H}_S^\blacksquare$ together with all spines of the form ${}_{J,j_1,\ldots,j_k}\mathbf{H}_{S,P_1,\ldots,P_k}^{\blacksquare,\bullet_1,\ldots,\bullet_k}$. The *generation* of ${}_J\mathbf{L}_S^\blacksquare$ is $|S|$.

By Lemma 5.6.9, the linear map A_* sends each limb ${}_J\mathbf{L}_S^\blacksquare$ onto another limb ${}_J\mathbf{L}_{\mathbf{t}S}^\blacksquare$.

Lemma 5.6.11. *Every limb is bounded in $\mathbb{C}$.*

The proof we present below is identical to [DL23, Lemma 5.10].

Proof. Recall the rescaled pre-corona $\mathbf{F}_n^\# = \left(\mathbf{f}_{n,\pm}^\# : \mathbf{U}_{n,\pm}^\# \to \mathbf{S}_n^\#\right)$ where $\mathbf{S}_n^\# := A_*^n(\mathbf{S})$ for all $n \in \mathbb{Z}$. Since $\mathbf{S}$ is compactly contained in $A_*^{-1}(\mathbf{S})$, then $\bigcup_{n\in\mathbb{Z}} \mathbf{S}_n^\# = \mathbb{C}$. For every integer $n \in \mathbb{Z}$, there is a gluing map $\rho_n : \mathbf{S}_n^\# \to V$ projecting $\mathbf{F}_n^\#$ to the corona $f : U \to V$.

Let us fix a large $n \ll 0$. Consider open rectangles

$$X_0 := \rho_n\left(\mathbf{S}_0^\#\right) \quad \text{and} \quad X_1 := \rho_n\left(\mathbf{S}_{-1}^\#\right)$$

living in the dynamical plane of f. Denote by $\mathbf{H}_*$ the Herman curve of f, and consider the interval $I := X_0 \cap \mathbf{H}_*$ and pick a slightly smaller interval $J \subset I$.

Claim 1. There is some $M \in \mathbb{N}$ such that the following holds. For any connected component W of $X_1 \backslash \mathbf{H}_*$, any $m \geq M$, and any point $x \in J$ with $f^m(x) \in \partial W$, the univalent lift W_{-m} of W under f^m along the orbit $x, \ldots, f^m(x)$ is contained in X_0.

Proof. Let Y^0 and Y^∞ denote the inner and outer components of $\mathbb{C}\backslash\mathbf{H}_*$. Assume without loss of generality that W is contained in Y^∞. Since $f^i(x) \in \mathbf{H}_*$ for all $i \geq 0$, then the lift W_{-m} is also contained in Y^∞. We will first claim that W_{-m} is well-defined and $f^m : W_{-m} \to W$ is univalent by ensuring that W_{-k} is disjoint from $\partial_{\mathrm{F}} U$ for all $k \geq 0$.

Let us pick two outer external rays R_l and R_r landing at a pair of points on $\mathbf{H}_*$ such that R_l is slightly on the left of W and R_r is slightly on the right of W. Since $n \ll 0$, the difference δ between the external angles of R_l and R_r is small. For $k = 1, \ldots, m$, let $R_{l,-k}$ and $R_{r,-k}$ be the preimages of R_l and R_r under f^k such that they are slightly on the left and right of W_{-k} respectively.

By definition, for every arc γ_j^∞ on the forbidden boundary $\partial_{\mathrm{F}} U$ of U, the part that gets mapped to $\gamma_1 \cap Y^\infty$ is an external ray of some definite distance from $\mathbf{H}_*$. The difference between the external angles of $R_{l,-k}$ and $R_{r,-k}$ is δ/d_∞^k, which is much smaller than δ. Therefore, W_{-k} is disjoint from $\partial_{\mathrm{F}} U$ for all k and so $f^m : W_{-m} \to W$ is univalent.

For sufficiently large m, W_{-m} is within a small neigborhood of $\mathbf{H}_*$ and it is sandwiched between the rays $R_{l,-m}$ and $R_{r,-m}$, whose external angles differ by a small constant. By local connectivity (Lemma 5.2.3), W_{-m} must be contained in a small neighborhood of J, and thus $W_{-m} \subset X_0$. □

The composition $\rho_n \circ A_*^{-n}$ identifies $\mathbf{S}_n^\#$ with X_0. Let $\mathbf{J}_n := A_*^n \circ \rho_n^{-1}(J)$.

Claim 2. There is a power-triple $R \in \mathbf{T}_{>0}$ such that $\mathbf{F}^R(\mathbf{J}_0) \subset \mathbf{J}_{-1}$ and for every point x on $\mathbf{J}_0$, if $\mathbf{F}^P(x) \in \mathbf{S}_{-1}^\#$ for some $P \geq R$, then there is an open subset $\mathbf{\Omega}_P$ of $\mathbf{S}_0^\# \backslash \mathbf{H}$ such that $x \in \partial \mathbf{\Omega}_P$ and $\mathbf{F}^P$ maps $\mathbf{\Omega}_P$ conformally to $\mathbf{S}_{-1}^\# \backslash \mathbf{H}$.

Proof. Since the action of $\mathbf{F}^{\geq 0}$ on $\mathbf{H}$ is combinatorially modelled by the cascade of translations $(T^P)_{P \in \mathbf{T}}$ on $\mathbb{R}$, there is an arbitrarily large $R \in \mathbf{T}$ such that $\mathbf{F}^R(\mathbf{J}_0) \subset \mathbf{J}_{-1}$. Suppose $x \in \mathbf{J}_0$ and $\mathbf{F}^P(x) \in \mathbf{S}_{-1}^\#$ for some $P \geq R$. Since $\mathbf{f}_{-1,\pm}$ is the first return map of the cascade $\mathbf{F}^{\geq 0}$ back to $\mathbf{S}_{-1}^\#$, then $\mathbf{F}^P$ is the m^{th} iterate of the pair $\mathbf{f}_{-1,\pm}$ for some $m \in \mathbb{N}$. If R is chosen to be large enough, then $m \geq M$ and the claim now follows from Claim 1. □

By self-similarity, Claim 2 also holds if we replace $\mathbf{J}_0$, $\mathbf{J}_{-1}$, P, and R with $\mathbf{J}_n$, $\mathbf{J}_{n-1}$, $\mathbf{t}^n P$, and $\mathbf{t}^n R$ respectively.

Claim 3. There is a power-triple $Q \in \mathbf{T}_{>0}$ such that for every $n \ll 0$ and every point $x \in \mathbf{J}_0$, if $\mathbf{F}^P(x) \in \mathbf{S}_n^\#$ for some $P \geq Q$, then there is an open subset $\mathbf{\Omega}_0$ of $\mathbf{S}_0^\# \backslash \mathbf{H}$ such that $x \in \partial \mathbf{\Omega}_0$ and $\mathbf{F}^P$ maps $\mathbf{\Omega}_0$ conformally to $\mathbf{S}_n^\# \backslash \mathbf{H}$.

Proof. Let us fix a large negative integer n and choose $Q \in \mathbf{T}_{>0}$ such that

$$Q > R + R/\mathbf{t} + R/\mathbf{t}^2 + R/\mathbf{t}^3 + \dots.$$

Consider a point $x_0 := x \in \mathbf{J}_0$ such that $\mathbf{F}^P(x) \in \mathbf{S}_n^\#$ for some $P \geq Q$. For $j \in \{0, -1, -2, \dots, n+2\}$, set $P_j := \mathbf{t}^j R$ and $x_{j-1} := \mathbf{F}^{P_j}(x_j)$ inductively. Then, we set

$$P_{n+1} := P - P_0 - P_{-1} - \dots - P_{n+2} \quad \text{ and } x_n := \mathbf{F}^{P_{n+1}}(x_{n+1}).$$

Clearly, $P_{n+1} \geq \mathbf{t}^{n+1} R$. By Claim 2, there exists an open set $\mathbf{\Omega}_{n+1} \subset \mathbf{S}_{n+1}^\# \backslash \mathbf{H}$ such that $x_{n+1} \in \partial \mathbf{\Omega}_{n+1}$ and $\mathbf{F}^{P_{n+1}}$ maps $\mathbf{\Omega}_{n+1}$ conformally to $\mathbf{S}_n^\# \backslash \mathbf{H}$. Inductively, for $j \in \{0, -1, \dots, n+2\}$, we construct open sets $\mathbf{\Omega}_j \subset \mathbf{S}_j^\# \backslash \mathbf{H}$ such that $x_j \in \partial \mathbf{\Omega}_j$ and $\mathbf{F}^{P_j}$ maps $\mathbf{\Omega}_j$ conformally to $\mathbf{\Omega}_{j-1}$. Therefore, $\mathbf{F}^P$ maps $\mathbf{\Omega}_0$ conformally to $\mathbf{S}_n^\# \backslash \mathbf{H}$. □

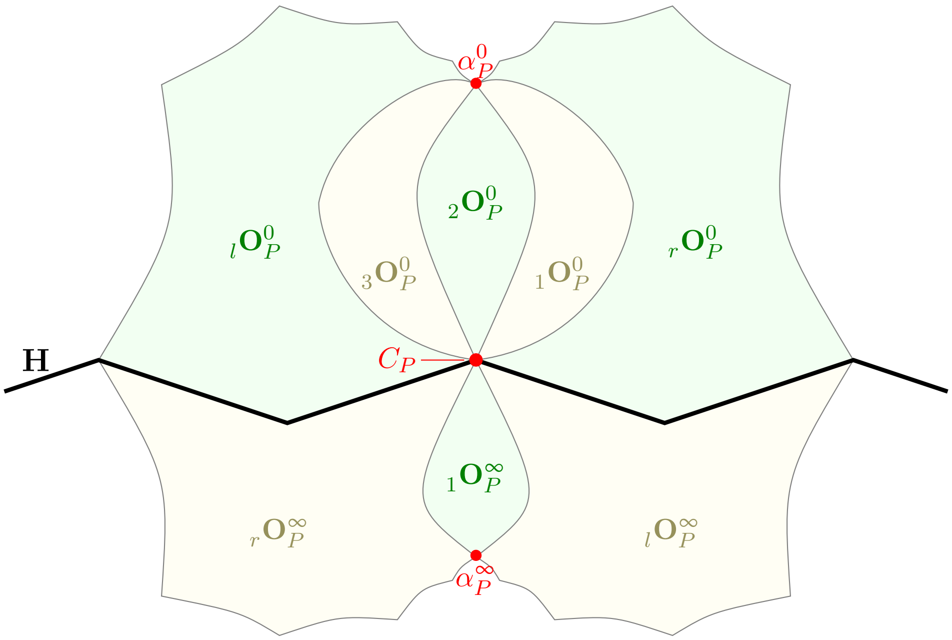

Figure 5.9: The configuration of middle and side lakes rooted at C_P when $(d_0, d_\infty) = (3, 2)$. Their coasts land at α_P^0 and α_P^∞.

To prove the lemma, it is sufficient to consider a limb L of some generation $K \in \mathbf{T}_{>0}$ rooted at the critical point C_K on $\mathbf{H}$. Choose a large $T \in \mathbf{T}$ such that $T \geq Q + K$ and that the critical point C_T is on $\mathbf{J}_0$. There exists some limb L' rooted at C_T such that $\mathbf{F}^{T-K}(L') = L$. Then, the connected component of $\mathbf{S}_n^\# \cap \overline{L}$ containing C_K can be lifted by $\mathbf{F}^{T-K}$ into $\mathbf{S}_0^\#$. As $n \ll 0$ is arbitrary, the lifts of $\mathbf{S}_n^\# \cap \overline{L}$ exhaust L' and so L' is contained in $\mathbf{S}_0^\#$. Hence, L is bounded. $\square$

5.6.3 Alpha-points

For $P \in \mathbf{T}_{>0}$, let $\mathbf{I}_{\leq P} := \mathbf{I}_{\leq P}(\mathbf{F})$ be the P^{th} escaping set of $\mathbf{F}$.

Lemma 5.6.12. *Every critical point ${}_J C_S^{\blacksquare}$ admits a pair of points ${}_J\alpha_S^{\blacksquare,0}$ and ${}_J\alpha_S^{\blacksquare,\infty}$ with the following properties. For any $\bullet \in \{0, \infty\}$ and $j \in \{l, 1, \ldots, 2d_\bullet - 3, r\}$, both the left and the right coasts of ${}_{J,j}\mathbf{O}_S^{\blacksquare,\bullet}$ land at ${}_J\alpha_S^{\blacksquare,\bullet}$ and*

$$\partial\, {}_{J,j}\mathbf{O}_S^{\blacksquare,\bullet} \Big\backslash \partial^c\, {}_{J,j}\mathbf{O}_S^{\blacksquare,\bullet} = \left\{ {}_J\alpha_S^{\blacksquare,\bullet} \right\}.$$

In particular, every lake is a disk and each of the spines ${}_{J,j}\mathbf{H}_S^{\blacksquare,\bullet}$ attached to ${}_J C_S^{\blacksquare}$ is a quasiarc connecting its common root ${}_J C_S^{\blacksquare}$ to a common landing point ${}_J\alpha_S^{\blacksquare,\bullet}$. See Figure 5.9 for an illustration. We call ${}_J\alpha_S^{\blacksquare,0}$ and ${}_J\alpha_S^{\blacksquare,\infty}$ the *inner* and *outer alpha-points* corresponding to ${}_J C_S^{\blacksquare}$. Moreover, we say that ${}_J\alpha_S^{\blacksquare,\bullet}$ is *the alpha-point* of any lake of the form ${}_{J,j}\mathbf{O}_S^{\blacksquare,\bullet}$.

Proof. By Corollary 5.6.8, for every lake $\mathbf{O}$, there is some $Q \in \mathbf{T}$ such that $\mathbf{F}^Q(\mathbf{O})$ is either a side lake or a middle lake attached to some critical point on $\mathbf{H}$. Therefore, it is sufficient to prove the lemma for lakes of the form ${}_j\mathbf{O}^\bullet_P$ where $\bullet \in \{0, \infty\}$ and $j \in \{l, 1, \ldots, 2d_\bullet - 3, r\}$.

Suppose ${}_j\mathbf{O}^\bullet_P$ is a middle lake. Then, it is contained in some side lake ${}_k\mathbf{O}^\bullet_{P-P/\mathbf{t}}$ of generation $P - P/\mathbf{t}$ where $k \in \{l, r\}$. Consider the conformal map

$$\mathbf{G} := \mathbf{F}^{\mathbf{t}P-P} \circ A_* : {}_k\mathbf{O}^\bullet_{P-P/\mathbf{t}} \to \mathbf{O}^\bullet.$$

Observe that $\mathbf{G}$ expands the hyperbolic metric of the ocean $\mathbf{O}^\bullet$, and $\mathbf{G}$ sends ${}_j\mathbf{O}^\bullet_P$ onto itself. Since $\overline{{}_j\mathbf{O}^\bullet_P} \cap \mathbf{I}_{\le P}$ is a $\mathbf{G}$-invariant compact subset of $\mathbf{O}^\bullet$, then it must be a singleton $\{\alpha^\bullet_P\}$ consisting of the unique repelling fixed point of $\mathbf{G}$.

It remains to show that for $j \in \{l, r\}$, the intersection $\overline{\partial^c_j\, {}_j\mathbf{O}^\bullet_P} \cap \mathbf{I}_{\le P}$ is also a compact subset of $\mathbf{O}^\bullet$. By invariance under $\mathbf{G}$, this will again imply that $\overline{\partial^c_j\, {}_j\mathbf{O}^\bullet_P} \cap \mathbf{I}_{\le P}$ is the same singleton $\{\alpha^\bullet_P\}$, and we are done.

Let us assume without loss of generality that $j = l$. Denote the left itinerary of ${}_l\mathbf{O}^\bullet_P$ by $(Q_1, Q_2, Q_3, \ldots)$. The left coast of ${}_l\mathbf{O}^\bullet_P$ starts with a segment of the spine ${}_1\mathbf{H}^\bullet_{Q_1}$ connecting C_{Q_1} and ${}_1C^\bullet_{Q_1,Q_2}$. Let us pick a pair of power-triples $R_-, R_+ \in \mathbf{T}_{>0}$ such that the critical points ${}_1C^\bullet_{Q_1,R_-}$ and ${}_1C^\bullet_{Q_1,R_+}$ form a small open interval neighborhood $J \subset {}_1\mathbf{H}^\bullet_{Q_1}$ of ${}_1C^\bullet_{Q_1,Q_2}$. Let $B_\pm$ be the spines of generation $Q_1 + R_\pm$ attached to ${}_1C^\bullet_{Q_1,R_\pm}$ that are combinatorially closest to ${}_{1,1}\mathbf{H}^{\bullet,\bullet}_{Q_1,Q_2}$. Let $R := Q_1 + \max\{R_+, R_-\}$. By Lemma 5.5.12, every connected component of $\mathbf{I}_{\le R}$ is unbounded and thus the union $J \cup B_+ \cup B_- \cup \mathbf{I}_{\le R}$ separates $\partial^c_l\, {}_l\mathbf{O}^\bullet_P \setminus {}_i\mathbf{H}^\bullet_{Q_1}$ from $\mathbf{H}$. This observation implies that $\overline{\partial^c_l\, {}_l\mathbf{O}^\bullet_P} \cap \mathbf{I}_{\le P}$ is indeed compactly contained in $\mathbf{O}^\bullet$. □

The alpha-points ${}_J\alpha^{\blacksquare,\bullet}_S$ can be viewed as preimages of infinity under the map $\mathbf{F}^{|S|}$. They are unique in the following sense.

Lemma 5.6.13. *Two alpha-points ${}_J\alpha^{\blacksquare,\bullet}_S$ and ${}_{J'}\alpha^{\square,\circ}_{S'}$ coincide if and only if $J = J'$, $\blacksquare = \square$, $\bullet = \circ$, and $S = S'$.*

Proof. Suppose ${}_J\alpha^{\blacksquare,\bullet}_S = {}_{J'}\alpha^{\square,\circ}_{S'}$. Clearly, $|S| = |S'|$. Let us write $S = (P_1, \ldots, P_m)$ and $S' = (Q_1, \ldots, Q_k)$, and pick a power-triple $R \in \mathbf{T}$ such that

$$\max\{P_1 + \ldots + P_{m-1}, Q_1 + \ldots + Q_{k-1}\} < R < |S|.$$

Pushing forward by $\mathbf{F}^R$ yields a pair of alpha-points $\alpha^\bullet_{|S|-R}$ and $\alpha^\circ_{|S'|-R}$ where, since they are equal, $\bullet = \circ$. If $(J, \blacksquare, S) \neq (J', \square, S')$, then this would imply that $\alpha^\bullet_{|S|-R}$ is a critical point of $\mathbf{F}^R$, which is not the case. □

Consequently, if two disjoint spines touch at a common alpha-point, then they are rooted at a common critical point. This guarantees a more precise tree structure of $\mathbf{F}^{-P}(\mathbf{H})$ in terms of spines. For convenience, we will call $\mathbf{H}$ the unique spine of generation 0.

Corollary 5.6.14. *Consider two distinct spines ${}_{J,j}\mathbf{H}_S^{\blacksquare,\bullet}$ and ${}_{J',j'}\mathbf{H}_{S'}^{\square,\circ}$ with $|S| \geq |S'|$.*

(1) *If the intersection $\overline{{}_{J,j}\mathbf{H}_S^{\blacksquare,\bullet}} \cap \overline{{}_{J',j'}\mathbf{H}_{S'}^{\square,\circ}}$ is non-empty, then it is either the singleton $\{{}_J C_S^{\blacksquare}\}$ or the set $\{{}_J C_S^{\blacksquare}, {}_J\alpha_S^{\blacksquare,\bullet}\}$. The former case happens if and only if ${}_{J',j'}\mathbf{H}_{S'}^{\square,\circ}$ contains ${}_J C_S^{\blacksquare}$, and the latter case happens if and only if $(J, \blacksquare, \bullet, S) = (J', \square, \circ, S')$.*

(2) *There is a unique sequence of pairwise different spines*

$$B_1 = {}_{J,j}\mathbf{H}_S^{\blacksquare,\bullet}, \ B_2, \ \ldots, \ B_{n-1}, \ B_n = {}_{J',j'}\mathbf{H}_{S'}^{\square,\circ}$$

such that $\overline{B_i}$ intersects $\overline{B_{i'}}$ if and only if $|i - i'| \leq 1$.

Given an alpha-point $\alpha = {}_J\alpha_S^{\blacksquare,\bullet}$, we define

- ▷ a *finite skeleton landing at* α to be the union of a spine ${}_{J,j}\mathbf{H}_S^{\blacksquare,\bullet}$ together with the unique closed quasiarc in $\mathbf{F}^{-|S|}(\mathbf{H})$ connecting ${}_J C_S^{\blacksquare}$ to 0;
- ▷ an *infinite skeleton landing at* α to be the union of $\partial_k^c\, {}_{J,k}\mathbf{O}_S^{\blacksquare,\bullet}$ for some $k \in \{l, r\}$ together with the unique closed quasiarc in $\mathbf{F}^{-|S|}(\mathbf{H})$ connecting the root of $\partial_k^c\, {}_{J,k}\mathbf{O}_S^{\blacksquare,\bullet}$ to 0.

In short, skeletons landing at α are the shortest paths from 0 to α within the tree of preimages of $\mathbf{H}$. There are exactly $d_\bullet$ skeletons landing at α, and precisely two of them are finite.

The set of skeletons admit a total order "<" defined as follows. Let us fix a ray γ in $\mathbf{H}$ connecting 0 to ∞. Given two distinct skeletons $\mathfrak{S}$ and $\mathfrak{S}'$,

- ▷ we write $\mathfrak{S} < \mathfrak{S}'$ if γ, $\mathfrak{S}$, and $\mathfrak{S}'$ have a counterclockwise orientation around the quasiarc $\mathfrak{S} \cap \mathfrak{S}'$, and
- ▷ we say that $\mathfrak{S}$ and $\mathfrak{S}'$ are <-*separated* if there is another skeleton $\mathfrak{S}''$ such that either $\mathfrak{S} < \mathfrak{S}'' < \mathfrak{S}'$ or $\mathfrak{S}' < \mathfrak{S}'' < \mathfrak{S}$.

We say that two alpha-points α and α' in the same ocean $\mathbf{O}^\bullet$ are <-*separated* if

- ▷ there exists an alpha point $\alpha'' \in \mathbf{O}^\bullet$ with generation lower than that of α and α', and
- ▷ there exist skeletons $\mathfrak{S}$, $\mathfrak{S}'$, $\mathfrak{S}''$ landing at α, α', α'' respectively such that $\mathfrak{S}$ and $\mathfrak{S}'$ are <-separated by $\mathfrak{S}''$.

Let us introduce another partial order on the set of alpha-points. Given two alpha-points α and α' in the same ocean,

- ▷ we write $\alpha \prec \alpha'$ if α' is contained in the closure of a lake attached to α, and

- ▷ we say that α and α' are $\prec$*-separated* if α and α' are contained in two distinct lakes with a common alpha-point.

The following proposition describes the relation between "$\prec$" and "$<$".

Proposition 5.6.15. *Consider two distinct alpha-points α and α' of generations P and P' inside of the ocean $\mathbf{O}^\bullet$ for some $\bullet \in \{0, \infty\}$. Assume $P \leq P'$. The following are equivalent.*

(1) *$\alpha \prec \alpha'$;*

(2) *α and α' are not $<$-separated by another alpha-point α'' of generation $< P$;*

(3) *α and α' are not $\prec$-separated.*

Proof. Suppose (1) holds. Then, α is the alpha-point of a lake $\mathbf{O}$ containing α', which implies (3). Meanwhile, (2) follows from the observation that any alpha-point α'' $<$-separating α and α' must be contained in a proper sub-lake of $\mathbf{O}$, which necessarily has generation higher than P.

Suppose (1) does not hold, so α' is located outside of every lake with alpha-point α. Let us pick any skeleton $\mathfrak{S}'$ landing at α', and let $\mathfrak{S}_l$ and $\mathfrak{S}_r$ denote the left and right infinite skeletons landing at α respectively. The assumption implies that either $\mathfrak{S}_l$ $<$-separates $\mathfrak{S}_r$ and $\mathfrak{S}'$, or $\mathfrak{S}_r$ $<$-separates $\mathfrak{S}_l$ and $\mathfrak{S}'$. Without loss of generality, let us assume the latter.

Denote by $(c_{r,1}, c_{r,2}, \ldots)$ the infinite sequence of critical points of $\mathbf{F}^P$ of increasing generation that is found along $\mathfrak{S}_r$. Let $\alpha_{r,i}$ denote the alpha-point that is the landing point of the unique spine attached to $c_{r,i}$ that intersects $\mathfrak{S}_k$. It has generation $P_{r,i}$ where $P_{r,i} < P$ and $P_{r,i} \to P$ as $i \to \infty$. Since the intersection $\mathfrak{S}_r \cap \mathfrak{S}'$ is a compact subset of $\mathrm{Dom}\,(\mathbf{F}^P)$, then for any sufficiently large $i \gg 0$ and any skeleton $\mathfrak{S}_{r,i}$ landing at $\alpha_{r,i}$, $\mathfrak{S}_r \cap \mathfrak{S}'$ is a proper subset of $\mathfrak{S}_r \cap \mathfrak{S}_{r,i}$. Therefore, $\mathfrak{S}_{r,i}$ $<$-separates $\mathfrak{S}_r$ and $\mathfrak{S}'$, and so α and α' are $<$-separated by $\alpha_{r,i}$.

We have just shown that (1) and (2) are equivalent. Suppose (1) and (2) do not hold. We will now prove that (3) also does not hold.

Let us consider the unique spine B such that α and α' are contained in the closure of different components of $(\mathfrak{S} \cup \mathfrak{S}') \backslash B$. We claim that the generation Q of B is less than P. Indeed, if $Q = P$, then α is the landing point of B and so there exists a lake with alpha-point α which contains both $\mathfrak{S}' \backslash \mathfrak{S}$ and α'. However, this would instead imply (1).

Let $\hat{\mathbf{O}}$ and $\hat{\mathbf{O}}'$ denote the pair of lakes of generation Q such that their coast contains B and $\mathfrak{S} \backslash \mathfrak{S}' \subset \hat{\mathbf{O}}$ and $\mathfrak{S}' \backslash \mathfrak{S} \subset \hat{\mathbf{O}}'$. If $\hat{\mathbf{O}}$ and $\hat{\mathbf{O}}'$ are distinct, they lie on different sides of B and so α and α' are $\prec$-separated by the landing point of B.

Now, suppose instead that $\hat{\mathbf{O}} = \hat{\mathbf{O}}'$. Consider the roots c and c' of $\mathfrak{S} \backslash B$ and $\mathfrak{S}' \backslash B$ respectively. Within the closed interval $[c, c'] \subset B$ (possibly degenerate if $c = c'$), we can find a unique critical point c'' of the smallest generation P'' such that $Q < P'' \le P$. In fact, $P'' \ne P$ because if otherwise, $\mathfrak{S}'\backslash\mathfrak{S}$ would have been contained in a lake attached to c, and so $\alpha \prec \alpha'$ instead. Since $[c, c']$ does not contain any critical point of generation lower than P'', then $\mathfrak{S}\backslash\mathfrak{S}'$ and $\mathfrak{S}'\backslash\mathfrak{S}$ are contained in distinct lakes of generation P'' attached to c''. Thus, the alpha-point $\alpha'' \in \hat{\mathbf{O}}$ corresponding to c'' must $\prec$-separate α and α'. □

5.6.4 External chains

Let us pick a power-triple $P \in \mathbf{T}_{>0}$ and $\bullet \in \{0, \infty\}$. Let $\mathbf{O}^\bullet(P)$ denote the unique lake of generation P inside of the ocean $\mathbf{O}^\bullet$ that contains 0 on its boundary. Then, the coast of $\mathbf{O}^\bullet(P)$ intersects $\mathbf{H}$ on some interval $J \subset \mathbf{H}$ containing 0 on its interior. (In fact, J is independent of $\bullet$.) Let us denote by $\alpha^\bullet(P)$ the unique alpha-point in $\partial\mathbf{O}^\bullet(P)$. By self-similarity,

$$\mathbf{O}^\bullet(\mathbf{t}^n P) = A_*^n(\mathbf{O}^\bullet(P)) \qquad \text{for all } n \in \mathbb{Z}$$

and

$$\bigcup_{n<0} \mathbf{O}^\bullet(\mathbf{t}^n P) = \mathbf{O}^\bullet. \tag{5.6.4}$$

Let us denote by $\mathbf{I}^\bullet_{\le P}$ the intersection $\mathbf{I}_{\le P} \cap \mathbf{O}^\bullet$ for $\bullet \in \{0, \infty\}$.

Lemma 5.6.16. *For every $\bullet \in \{0, \infty\}$ and $P, Q \in \mathbf{T}_{>0}$ with $P < Q$,*

(1) *$\mathbf{I}^\bullet_{\le P}$ is connected;*

(2) *$\mathbf{I}^\bullet_{\le Q}\backslash\mathbf{I}^\bullet_{\le P}$ is bounded;*

(3) *every connected component of $\mathbf{I}^\bullet_{\le Q}\backslash\mathbf{I}^\bullet_{\le P}$ is a lift of a component of $\mathbf{I}_{\le Q-P}$ under $\mathbf{F}^P$; it is contained in a unique lake $\mathbf{O}$ of generation P and its boundary contains the alpha-point of $\mathbf{O}$.*

Proof. Consider a component I of $\mathbf{I}^\bullet_{\le P}$. It intersects $\mathbf{O}^\bullet(\mathbf{t}^k P)$ for some maximal $k \in \mathbb{Z}$. By Lemma 5.5.12, since I intersects $\mathbf{O}^\bullet(\mathbf{t}^n P)$ for all $n \le k$, then it contains the alpha-point $\alpha^\bullet(\mathbf{t}^n P)$ which is the alpha-point of $\mathbf{O}^\bullet(\mathbf{t}^n P)$ for all $n \le k$. Therefore, $\mathbf{I}^\bullet_{\le P}$ is connected.

Let us consider a connected component X of $\mathbf{I}^\bullet_{\le Q}\backslash\mathbf{I}^\bullet_{\le P}$. Since X avoids $\alpha^\bullet(\mathbf{t}^n P)$ for all $n \ll 0$, it must be contained inside of the lake $\mathbf{O}^\bullet(\mathbf{t}^k P)$ for all $n \ll 0$, and so X is bounded. Since X avoids $\mathbf{F}^{-P}(\mathbf{H})$ and alpha-points of generation P, L is contained in a unique lake $\mathbf{O}$ of generation P. The map $\mathbf{F}^P$ sends $\mathbf{O}$ conformally onto an ocean $\mathbf{O}^\circ$ for some $\circ \in \{0, \infty\}$, hence $\mathbf{F}^P(X) = \mathbf{I}^\circ_{\le Q-P}$. By unboundedness, X must be attached to the alpha-point of $\mathbf{O}$. □

Definition 5.6.17. Consider two alpha-points α and α' in the same ocean $\mathbf{O}^\bullet$ with generation P and P' respectively, and suppose $P < P'$ and $\alpha \prec \alpha'$. We define the *external chain* $[\alpha, \alpha']$ to be the set of points in $\mathbf{I}^\bullet_{\le P'}$ that are inside the closure of the lakes attached to α and outside of any lake that does not contain α'.

Lemma 5.6.18. *For any triplet of alpha-points $\alpha, \alpha', \alpha''$ with $\alpha \prec \alpha' \prec \alpha''$,*

$$[\alpha, \alpha'] \cap [\alpha', \alpha''] = \{\alpha'\} \quad \textit{and} \quad [\alpha, \alpha'] \cup [\alpha', \alpha''] = [\alpha, \alpha''].$$

Proof. The first equation follows from the fact that α' is a cut point with respect to the "$\prec$" ordering. The inclusion $[\alpha, \alpha'] \cup [\alpha', \alpha''] \subset [\alpha, \alpha'']$ is obvious. Consider a point x in $[\alpha, \alpha'']\backslash[\alpha, \alpha']$. We know that x is within a lake attached to α. If x is inside of a lake that does not contain α', then this lake avoids all lakes attached to α' and in particular does not contain α'' as well, which is a contradiction. Therefore, $x \in [\alpha', \alpha'']$. □

For $P \in \mathbf{T}_{>0}$, we say that the critical point C_P on $\mathbf{H}$ is *dominant* if the interval $[0, C_P] \subset \mathbf{H}$ does not contain any critical point of generation less than P. We will enumerate dominant critical points by $\{C_{P_n}\}_{n\in\mathbb{Z}}$ where $\{P_n\}_{n\in\mathbb{Z}}$ is monotonically increasing in n.

Lemma 5.6.19. *For $\bullet \in \{0, \infty\}$, $\ldots \prec \alpha^\bullet_{P_{-2}} \prec \alpha^\bullet_{P_{-1}} \prec \alpha^\bullet_{P_0} \prec \alpha^\bullet_{P_1} \prec \alpha^\bullet_{P_2} \prec \ldots$.*

Proof. Suppose for a contradiction that $\alpha^\bullet_{P_n} \nprec \alpha^\bullet_{P_{n+1}}$ for some $\bullet \in \{0, \infty\}$ and $n \in \mathbb{Z}$. By Propositon 5.6.15, there is an alpha-point $\alpha \in \mathbf{O}^\bullet$ of some generation P less than P_n which $\prec$-separates $\alpha^\bullet_{P_n}$ and $\alpha^\bullet_{P_{n+1}}$. Then, α is contained in the closure of a lake attached to a critical point $C_Q \in \mathbf{H}$ of some generation $Q \le P$. By $\prec$-separation, C_Q is contained in the interval $(C_{P_n}, C_{P_{n+1}}) \subset \mathbf{H}$. However, this would contradict the assumption that C_{P_n} and $C_{P_{n+1}}$ are dominant. □

Consider the concatenations

$$\mathbf{R}^0 = \bigcup_{n\in\mathbb{Z}} \left[\alpha^0_{P_n}, \alpha^0_{P_{n+1}}\right] \quad \text{and} \quad \mathbf{R}^\infty = \bigcup_{n\in\mathbb{Z}} \left[\alpha^\infty_{P_n}, \alpha^\infty_{P_{n+1}}\right],$$

which we will refer to as the *inner* and *outer zero chains* respectively.

Proposition 5.6.20. *For $\bullet \in \{0, \infty\}$,*

1. $\mathbf{R}^\bullet$ *is A_*-invariant;*
2. $\mathbf{R}^\bullet$ *is an arc landing at* 0*;*
3. *alpha-points are dense on* $\mathbf{R}^\bullet$*;*
4. *points on $\mathbf{R}^\bullet$ are continuously parametrized by their escaping time ranging from 0 (near ∞) to $+\infty$ (near 0).*

Let us clarify the last statement. For $P \in \mathbb{R}_{>0} \backslash \mathbf{T}$, we can define the P^{th} escaping set to be

$$\mathbf{I}_{\leq P} := \bigcap_{Q \in \mathbf{T}, Q > P} \mathbf{I}_{\leq Q}.$$

The *escaping time* of a point x in $\mathbf{I}_{<\infty}$ is the minimum time $P \in \mathbb{R}_{>0}$ such that $x \in \mathbf{I}_{\leq P}$.

Proof. To lighten the notation, we will denote $\alpha_n^\bullet := \alpha_{P_n}^\bullet$ and $J_n^\bullet := [\alpha_n^\bullet, \alpha_{n+1}^\bullet]$ for all $\bullet \in \{0, \infty\}$ and $n \in \mathbb{Z}$.

By definition, C_P is dominant if and only if $C_{\mathbf{t}P} = A_*(C_P)$ is dominant, so there is some integer $k \geq 1$ such that $\mathbf{t}P_n = P_{n+k}$ for all $n \in \mathbb{Z}$. Therefore, A_* maps each of $[\alpha_{(n-1)k}^\bullet, \alpha_{nk}^\bullet]$ onto $[\alpha_{nk}^\bullet, \alpha_{(n+1)k}^\bullet]$. This immediately implies items (1) and (2).

Due to self-similarity, it remains for us to show that the external chain $J^\bullet := [\alpha_0^\bullet, \alpha_k^\bullet]$ is an arc that can be continuously parametrized by their escaping time, and that alpha-points are dense on $J^\bullet$. We will do so by constructing nested Markov tilings $\mathcal{P}_r$ for $r \geq 0$ on $J^\bullet$.

Firstly, we set the tiling $\mathcal{P}_0$ of level 0 to be $\{J_i^\bullet\}_{0 \leq i \leq k-1}$. The tiling $\mathcal{P}_1$ of level 1 is constructed as follows. By Lemma 2.1.17, for every chain $J_i \in \mathcal{P}_0$, there exist some $Q_i \in \mathbf{T}_{>0}$ and a pair of integers l_i and r_i such that $0 < l_i < r_i \leq i$ and $\mathbf{F}^{Q_i}$ maps $J_i^\bullet$ homeomorphically onto the chain $[\alpha_{l_i}^\bullet, \alpha_{r_i}^\bullet]$. A tile of level 1 in $\mathcal{P}_1$ is the lift of a chain $J_j^\bullet \subset [\alpha_{l_i}^\bullet, \alpha_{r_i}^\bullet]$ under the map $\mathbf{F}^{Q_i} : J_i^\bullet \to [\alpha_{l_i}^\bullet, \alpha_{r_i}^\bullet]$.

For each tile $I \in \mathcal{P}_1$ in $J_i^\bullet$, there exists some $m_I \in \mathbb{N}$ such that $A_*^{m_I}$ sends $\mathbf{F}^{Q_i}(I)$ back to a tile of level 0. Let $\mathbf{O}_i$ denote the lake of generation Q_i which contains $[\alpha_{l_i}^\bullet, \alpha_{r_i}^\bullet]$. The composition

$$\chi_I := A_*^{m_I} \circ \mathbf{F}^{Q_i} : \mathbf{O}_i \to \mathbf{O}^\bullet \tag{5.6.5}$$

expands the hyperbolic metric of $\mathbf{O}^\bullet$.

Inductively, we define tiles in $\mathcal{P}_{n+1}$ of level $n+1$ to be the preimages of tiles of level n under maps of the form (5.6.5). Since each map χ_I is expanding, the diameter of every tile of level n uniformly exponentially shrinks to zero. Since each tile in $\mathcal{P}_n$ is an external chain containing alpha-points, alpha-points are dense on J.

By Lemma 5.6.18, we can enumerate our level n tiles by $I_1^n, I_2^n, \ldots, I_{s_n}^n \in \mathcal{P}_n$ in increasing order of generation such that I_i^n and I_l^n touch if and only if $|l - i| \leq 1$. As tiles shrink, we can extend the "$\prec$" order to a total order on $J^\bullet$ by defining $x \prec y$ when $x \in I_i^n$ and $y \in I_j^n$ for sufficiently high n and some indices i, j with $i < j$.

Consider a tile I_i^n in $\mathcal{P}_n$ of some high level n, and a composition $\chi := \chi_1 \circ \chi_2 \circ \ldots \circ \chi_n$ of n maps of the form (5.6.5) sending I_i^n onto a tile in $\mathcal{P}_n$. By (5.5.2), we can write χ as $A_*^{m(n,i)} \circ \mathbf{F}^{Q(n,i)}$ for some $m(n,i) \in \mathbb{N}$ and $Q(n,i) \in \mathbf{T}_{>0}$. Therefore, the difference in the escaping time between the endpoints of I_i^n is at most

$$\mathbf{t}^{-m(r,i)}(P_k - P_0). \tag{5.6.6}$$

Since $Q_i > 0$ for all $i \in \{0, \dots, k-1\}$, there exists some integer $M \geq 1$ independent of n such that every sequence of M consecutive integers between 1 and n contains an element j_* such that χ_{j_*} has the scaling factor A_* in (5.6.5). Consequently, as $n \to \infty$, $\min_{1 \leq i \leq s_n} m(n,i) \to \infty$ and thus the quantity in (5.6.6) tends to zero. Therefore, the escaping time continuously parametrizes points on J. □

In general, for every alpha-point α, there is an infinite sequence of alpha-points $\alpha_0 = \alpha$, α_{-1}, α_{-2}, ... of generation decreasing to 0 such that $\dots \prec \alpha_{-2} \prec \alpha_{-1} \prec \alpha_0$. This allows us to generate the chain

$$(\infty, \alpha] := \bigcup_{n \leq 0} [\alpha_{n-1}, \alpha_n].$$

Corollary 5.6.21. *Consider any alpha-point α of some generation $P > 0$. The chain $(\infty, \alpha]$ is an infinite arc continuously parametrized by the escape time from $|P|$ to 0. Moreover, alpha-points are dense in $(\infty, \alpha]$.*

Proof. Suppose first that α is of the form $\alpha_P^\bullet$ for some $P \in \mathbf{T}_{>0}$ and $\bullet \in \{0, \infty\}$. Let us pick a dominant $\alpha_{P_n}^\bullet$ for some $n \in \mathbb{Z}$ such that $P_n \geq P$. There is a unique point $x \in \left(\infty, \alpha_{P_n}^\bullet\right]$ of generation $P_n - P$. Then, $\mathbf{F}^{P_n - P}$ maps the arc $\left(x, \alpha_{P_n}^\bullet\right]$ onto $(\infty, \alpha_P^\bullet]$, which implies the claim.

In general, let $\alpha = {}_J\alpha_S^{\blacksquare,\bullet}$ where $S = (P_1, P_2, \dots, P_k)$ is the corresponding itinerary. There exist alpha-points α_1, α_2, ..., $\alpha_k = \alpha$ such that $\alpha_1 \prec \alpha_2 \prec \dots \prec \alpha_k$ and for every i, α_i has itinerary $S_i := (P_1, \dots, P_i)$. Therefore, we can split $(\infty, \alpha]$ into

$$J_1 = (\infty, \alpha_1], \quad J_2 = (\alpha_1, \alpha_2], \quad \dots \quad J_k = (\alpha_{k-1}, \alpha_k].$$

When $2 \leq i \leq k$, the map $\mathbf{F}^{P_1 + \dots + P_{i-1}}$ sends J_i homeomorphically onto the chain $\left(\infty, \alpha_{P_i}^\circ\right]$ for some $\circ \in \{0, \infty\}$. By the previous paragraph, each J_i is an arc continuously parametrized by the landing time. □

As a consequence, whenever $\alpha \prec \alpha'$, the chain $[\alpha, \alpha']$ is a simple arc.

Definition 5.6.22. An *external ray* is an infinite arc of the form $\mathbf{R} = \bigcup_{n \in \mathbb{Z}} [\alpha_n, \alpha_{n+1}]$ for some sequence of alpha-points $\{\alpha_n\}_{n \in \mathbb{Z}}$ where

- ▷ $\alpha_n \prec \alpha_{n+1}$ for all n;
- ▷ the generation of α_n decreases to 0 as $n \to -\infty$;
- ▷ there is no alpha-point α such that $\alpha_n \prec \alpha$ for all $n \in \mathbb{Z}$.

The *generation* of $\mathbf{R}$ is the limit of the generation of α_n as $n \to +\infty$. For any $P \in \mathbf{T}_{>0}$, we define the image of an external ray $\mathbf{R}$ under $\mathbf{F}^P$ by

$$\mathbf{F}^P(\mathbf{R}) := \mathbf{F}^P\left(\mathbf{R} \cap \mathrm{Dom}\left(\mathbf{F}^P\right)\right).$$

We say that $\mathbf{R}$ is *periodic* if $\mathbf{F}^P(\mathbf{R}) = \mathbf{R}$ for some $P \in \mathbf{T}_{>0}$.

The zero chains $\mathbf{R}^0$ and $\mathbf{R}^\infty$ are indeed external rays, which from now on will be referred to as *zero rays.*

The following corollary is an immediate consequence of Proposition 5.6.15.

Corollary 5.6.23. *The intersection of any two external rays in the same ocean is non-empty and of the form $(\infty, \alpha]$ for some alpha-point α.*

5.6.5 Wakes

Consider a critical point ${}_JC_S^{\blacksquare}$. For every lake of the form ${}_{J,j}\mathbf{O}_S^{\blacksquare,\bullet}$ where j is either in $\{l, r\}$ or an even number, the map $\mathbf{F}^{|S|}$ sends such a lake conformally onto $\mathbf{O}^\bullet$. The zero ray $\mathbf{R}^\bullet$ lifts under $\mathbf{F}^{|S|} : {}_{J,j}\mathbf{O}_S^{\blacksquare,\bullet} \to \mathbf{O}^\bullet$ to a ray segment, which we will label as ${}_{J,k}\mathbf{R}_S^{\blacksquare,\bullet}$ where

$$k = \begin{cases} 1 & \text{if } j = r, \\ \frac{j}{2} + 1 & \text{if } j \text{ is even,} \\ d_\bullet & \text{if } j = l. \end{cases}$$

Therefore, we obtain $d_\bullet$ ray segments

$$ {}_{J,1}\mathbf{R}_S^{\blacksquare,\bullet}, \quad {}_{J,2}\mathbf{R}_S^{\blacksquare,\bullet}, \quad \ldots, {}_{J,d_\bullet}\mathbf{R}_S^{\blacksquare,\bullet} \tag{5.6.7}$$

starting from the alpha-point ${}_J\alpha_S^{\blacksquare,\bullet}$ and landing at the critical point ${}_J\mathbf{R}_S^{\blacksquare}$, labelled in an anticlockwise order about ${}_JC_S^{\blacksquare}$. See Figure 5.10. For $k \in \{1, \ldots, d_\bullet - 1\}$, the closure of ${}_{J,k}\mathbf{R}_S^{\blacksquare,\bullet} \cup {}_{J,k+1}\mathbf{R}_S^{\blacksquare,\bullet}$ bounds a Jordan domain which we denote by ${}_{J,k}\mathbf{W}_S^{\blacksquare,\bullet}$.

Definition 5.6.24. A wake $\mathbf{W}$ is a Jordan domain of the form ${}_{J,k}\mathbf{W}_S^{\blacksquare,\bullet}$. We call ${}_JC_S^{\blacksquare}$ the *root* of $\mathbf{W}$ and ${}_J\alpha_S^{\blacksquare,\bullet}$ the *alpha-point* of $\mathbf{W}$. The *generation* of $\mathbf{W}$ is $|S|$. If S is a tuple of length m, we say that m is the *level* of $\mathbf{W}$. If $m = 1$, we call $\mathbf{W}$ a *primary* wake. If $m = 2$, we call $\mathbf{W}$ a *secondary* wake.

Due to the tree structure of $\mathbf{I}_{<\infty}$, primary wakes are always pairwise disjoint.

Lemma 5.6.25. *Consider a wake ${}_{J,j}\mathbf{W}_S^{\blacksquare,\bullet}$ rooted at a critical point ${}_JC_S^{\blacksquare}$.*

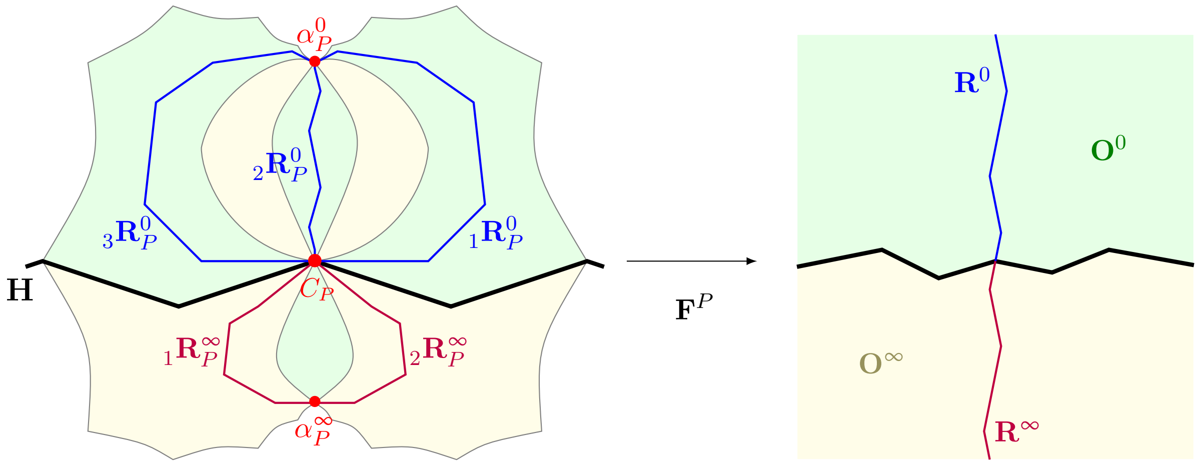

Figure 5.10: The construction of wakes rooted at C_P when $(d_0, d_\infty) = (3,2)$.

(1) *If* $\mathbf{F}^Q$ *sends* ${}_J C_S^{\blacksquare}$ *to another critical point* ${}_{J'}C_{S'}^{\square}$*, then* $\mathbf{F}^Q : \overline{{}_{J,j}\mathbf{W}_S^{\blacksquare,\bullet}} \to \overline{{}_{J',j}\mathbf{W}_{S'}^{\square,\bullet}}$ *is a homeomorphism.*

(2) *The map* $\mathbf{F}^{|S|}$ *conformally sends* ${}_{J,j}\mathbf{W}_S^{\blacksquare,\bullet}$ *onto* $\mathbb{C}\backslash\overline{\mathbf{R}^\bullet}$*.*

Proof. (1) follows from the fact that $\mathbf{F}^Q$ maps ${}_{J,j}\mathbf{R}_S^{\blacksquare,\bullet} \cup {}_{J,j+1}\mathbf{R}_S^{\blacksquare,\bullet}$ homeomorphically onto ${}_{J',j}\mathbf{R}_{S'}^{\square,\bullet} \cup {}_{J',j+1}\mathbf{R}_{S'}^{\square,\bullet}$, whereas (2) follows from the fact that $\mathbf{F}^{|S|}$ maps ${}_{J,j}\mathbf{R}_S^{\blacksquare,\bullet}$ for each $j \in \{1, \ldots, d_\bullet\}$ homeomorphically onto the zero ray $\mathbf{R}^\bullet$. $\square$

To reduce notation, let us consider the *full wake*

$$ {}_J\mathbf{W}_S^{\blacksquare,\bullet} \coloneqq \bigcup_{j=1}^{d_\bullet - 1} {}_{J,j}\mathbf{W}_S^{\blacksquare,\bullet}(j) $$

which is the union of wakes attached to the critical point ${}_J C_S^{\blacksquare,\bullet}$ on the same side.

Lemma 5.6.26 (Primary wakes shrink)**.** *For every* $n \in \mathbb{Z}$ *and every* $\varepsilon > 0$*, there are at most finitely many primary wakes of diameter at most* ε *rooted at a point on* $\mathbf{H} \cap \mathbf{S}_n^{\#}$*.*

Proof. The proof we present below is similar to [DL23, Lemma 5.29]. By self-similarity, it is sufficient to prove the lemma for $n = 0$. Let

$$ \mathbf{J}_- \coloneqq \mathbf{U}_- \cap \mathbf{H}, \qquad \mathbf{J}_+ \coloneqq \mathbf{U}_+ \cap \mathbf{H}, \quad \text{and} \quad \mathbf{J} \coloneqq \mathbf{J}_- \cup \mathbf{J}_+. $$

The maps $\mathbf{f}_- = \mathbf{F}^{(0,1,0)} : \mathbf{J}_- \to \mathbf{J}$ and $\mathbf{f}_+ = \mathbf{F}^{(0,0,1)} : \mathbf{J}_- \to \mathbf{J}$ are precisely the first return maps of $\mathbf{F}$ back to $\mathbf{J}$.

Consider the semigroup generated by $(0,1,0)$ and $(0,0,1)$ and let us label its elements by 0, Q_0, Q_1, $Q_2, \ldots$ written in increasing order. Then, every critical point on $\mathbf{J}$ is of the form C_{Q_n} for some $n \geq 0$. Let us fix $\bullet \in \{0, \infty\}$ and consider the full primary wake $\mathbf{W}_n \coloneqq \mathbf{W}_{Q_n}^{\bullet}$

attached to C_{Q_n}. For all $n > 0$, $\mathbf{W}_n$ is a preimage under $\mathbf{F}^{Q_n - Q_0}$ of the full wake $\mathbf{W}_0$ with the smallest generation.

Let us pick a curve Γ_0 in $\mathbf{W}_0$ connecting a point in $\mathbf{W}_0$ to the critical point C_{Q_0}. Consider the lift Γ_{-n} of Γ_0 under $\mathbf{F}^{Q_n - Q_0} : \mathbf{W}_n \to \mathbf{W}_0$, which connects a point in $\mathbf{W}_n$ to the critical point C_{Q_n}.

Claim. There is a sequence ε_0, ε_{-1}, ε_{-2}, ... of positive numbers decreasing to 0 such that the following holds. If the (Euclidean) diameter of Γ_0 is less than ε_0, then the diameter of Γ_{-n} is less than ε_n for all $n \geq 0$.

Proof. It is sufficient to prove the claim in the dynamical plane of the corona f_*. Let $g = f_{d_0, d_\infty, \theta}$ be the prototypical Example 4.4.5. It admits a (d_0, d_∞)-critical Herman curve $\mathbf{H}_g$ with rotation number equal to that of f_*. By Theorem D, g is quasiconformally conjugate to f_* on a neighborhood of $\mathbf{H}_g$, so it suffices to prove the claim in the dynamical plane of g. We shall do so by applying the local connectivity of the boundary of the immediate basin of attraction of $\bullet$ of g.

Recall that the critical point of g is normalized at $1 \in \mathbf{H}_g$. For $k \geq 0$, we denote $c_k := (g|_{\mathbf{H}_g})^k(1)$. Within the immediate basin of $\bullet$, let us pick two external rays R_l and R_r landing at points on $\mathbf{H}_g$ that are slightly on the left and right of c_0 respectively. Let us pick a disk D_0 of small diameter bounded by $\mathbf{H}_g$, R_l, R_r, and an equipotential within the immediate basin of $\bullet$. Let D_{-k} be the unique lift of D_0 under g^k whose boundary contains c_{-k}. The disk D_{-k} is bounded by $g^{-k}(\mathbf{H})$, a pair of external rays which are preimages of R_l and R_r, and an equipotential of an even smaller level. By local connectivity, the Euclidean diameter of D_{-k} shrinks to zero as $k \to \infty$. □

Let $\mathbf{O}_- \subset \mathbf{O}^\bullet$ be the union of all lakes of generation $(0,1,0)$ whose closure intersects $\mathbf{J}_-$, and let $\mathbf{O}_+ \subset \mathbf{O}^\bullet$ be the union of all lakes of generation $(0,0,1)$ whose closure intersects $\mathbf{J}_+$. The maps $\mathbf{f}_\pm : \mathbf{O}_\pm \to \mathbf{O}^\bullet$ expand the hyperbolic metric of $\mathbf{O}^\bullet$. Note that for all $n \geq 0$, $\mathbf{F}^{Q_{n+1} - Q_n} : \mathbf{W}_{n+1} \to \mathbf{W}_n$ is a restriction of $\mathbf{f}_\pm : \mathbf{O}_\pm \to \mathbf{O}^\bullet$. Then, due to the claim, we conclude that the Euclidean diameter of $\mathbf{W}_n$ shrinks as $n \to \infty$. □

The outer boundary of each of the full wakes attached to ${}_J C_S^{\blacksquare}$ consists of two ray segments, which we will relabel as

$$ {}^{+}_{J}\mathbf{R}_S^{\blacksquare,0} := {}_{J,1}\mathbf{R}_S^{\blacksquare,0}, \quad {}^{-}_{J}\mathbf{R}_S^{\blacksquare,0} := {}_{J,d_\bullet}\mathbf{R}_S^{\blacksquare,0}, \quad {}^{-}_{J}\mathbf{R}_S^{\blacksquare,\infty} := {}_{J,1}\mathbf{R}_S^{\blacksquare,\infty}, \quad {}^{+}_{J}\mathbf{R}_S^{\blacksquare,\infty} := {}_{J,d_\bullet}\mathbf{R}_S^{\blacksquare,\infty}. $$

For every $P \in \mathbf{T}_{>0}$, let P^- (resp. P^+) be the first entry of the left (resp. right) itinerary of the side lake ${}_l\mathbf{O}_P^0$ (resp. ${}_r\mathbf{O}_P^0$). Both P^- and P^+ are characterized by the property that $(C_{P^-}, C_{P^+}) \subset \mathbf{H}$ is the maximal open interval in which the only critical point of generation $\leq P$ is C_P.

Lemma 5.6.27 (Combinatorics of primary wakes)**.** *Given $P \in \mathbf{T}_{>0}$ and $\bullet \in \{0, \infty\}$,*

(1) *both ${}^{+}\mathbf{R}^{\bullet}_{P-}$ and ${}^{-}\mathbf{R}^{\bullet}_{P+}$ contain $\alpha^{\bullet}_{P}$;*

(2) *the closure of $\mathbf{W}^{\bullet}_{P} \cup \mathbf{W}^{\bullet}_{P-} \cup \mathbf{W}^{\bullet}_{P+}$ is a neighborhood of $\alpha^{\bullet}_{P}$;*

(3) *the ray segments ${}^{+}\mathbf{R}^{\bullet}_{P}$ and ${}^{-}\mathbf{R}^{\bullet}_{P}$ can be presented as infinite concatenations of ray segments*

$$
{}^{\pm}\mathbf{R}^{\bullet}_{P} = \left[\alpha^{\bullet}_{P}, \alpha^{\bullet}_{Q^{\pm}_{1}}\right] \cup \left[\alpha^{\bullet}_{Q^{\pm}_{1}}, \alpha^{\bullet}_{Q^{\pm}_{2}}\right] \cup \left[\alpha^{\bullet}_{Q^{\pm}_{2}}, \alpha^{\bullet}_{Q^{\pm}_{3}}\right] \cup \dots,
$$

where

$$
{}^{\pm}\mathbf{R}^{\bullet}_{P} \cap {}^{\mp}\mathbf{R}^{\bullet}_{P\pm} = \left[\alpha^{\bullet}_{P}, \alpha^{\bullet}_{Q^{\pm}_{1}}\right], \quad \text{and for } i \geq 1, \;\; {}^{\pm}\mathbf{R}^{\bullet}_{P} \cap {}^{\mp}\mathbf{R}^{\bullet}_{Q^{\pm}_{i}} = \left[\alpha^{\bullet}_{Q^{\pm}_{i}}, \alpha^{\bullet}_{Q^{\pm}_{i+1}}\right];
$$

(4) *the sequences of alpha-points $\left\{\alpha^{\bullet}_{Q^{+}_{i}}\right\}_{i\geq 1}$ and $\left\{\alpha^{\bullet}_{Q^{-}_{i}}\right\}_{i\geq 1}$ tend to C_P as $i \to \infty$.*

See Figures 5.11 and 5.12.

Proof. The left coast of ${}_{l}\mathbf{O}^{0}_{P}$ is contained in ${}_{1}\mathbf{W}^{0}_{P-}$ because it starts with a segment of the spine ${}_{1}\mathbf{H}^{0}_{P-}$ rooted at C_{P-} and is disjoint from the external rays landing at C_{P-}. Since the left coast of ${}_{l}\mathbf{O}^{0}_{P}$ lands at the alpha-point α^{0}_{P}, the boundary of the wake ${}_{1}\mathbf{W}^{0}_{P-}$ must contain α^{0}_{P}. The treatment for the other side lakes of C_P is analogous, and this implies (1).

We have established that $\alpha^{\bullet}_{P}$ is in the boundary of each of $\mathbf{W}^{\bullet}_{P}$, $\mathbf{W}^{\bullet}_{P-}$, and $\mathbf{W}^{\bullet}_{P+}$. By Corollary 5.6.23, there exist alpha-points α', α_{-}, and α_{+} such that

$$
\alpha' \prec \alpha^{\bullet}_{P}, \quad \alpha^{\bullet}_{P} \prec \alpha_{-}, \quad \alpha^{\bullet}_{P} \prec \alpha_{+}
$$

and

$$
{}^{+}\mathbf{R}^{\bullet}_{P-} \cap {}^{-}\mathbf{R}^{\bullet}_{P+} = [\alpha', \alpha^{\bullet}_{P}], \quad {}^{+}\mathbf{R}^{\bullet}_{P-} \cap {}^{-}\mathbf{R}^{\bullet}_{P} = [\alpha^{\bullet}_{P}, \alpha_{-}], \quad {}^{+}\mathbf{R}^{\bullet}_{P} \cap {}^{-}\mathbf{R}^{\bullet}_{P+} = [\alpha^{\bullet}_{P}, \alpha_{+}].
$$

Therefore, the union of $\mathbf{W}^{\bullet}_{P-}$, $\mathbf{W}^{\bullet}_{P}$, and $\mathbf{W}^{\bullet}_{P+}$ form a neighborhood of $\alpha^{\bullet}_{P}$, thus proving (2). More generally, we have just shown that every primary alpha-point is the meeting point of exactly three distinct primary full wakes.

Let us prove (3) and (4) for ${}^{-}\mathbf{R}^{\bullet}_{P}$. The treatment for ${}^{+}\mathbf{R}^{\bullet}_{P}$ is analogous. Let us define $Q^{-}_{1} \in \mathbf{T}$ to be the unique smallest moment greater than P such that $C_{Q^{-}_{1}}$ is contained on the interval $(C_{P-}, C_P) \subset \mathbf{H}$. Then, based on the previous paragraph, the alpha-point α_{-} must be equal to $\alpha^{\bullet}_{Q^{-}_{1}}$ because it is the meeting point of ${}^{-}\mathbf{R}^{\bullet}_{P}$, ${}^{+}\mathbf{R}^{\bullet}_{P-}$, and the boundary of a primary full wake, which is $\mathbf{W}^{\bullet}_{Q^{-}_{1}}$. Similarly, ${}^{+}\mathbf{R}^{\bullet}_{Q^{-}_{1}}$ and ${}^{-}\mathbf{R}^{\bullet}_{P}$ meet along a ray segment $\left[\alpha^{\bullet}_{Q^{-}_{1}}, \alpha^{\bullet}_{Q^{-}_{2}}\right]$ for some $Q^{-}_{2} > Q^{-}_{1}$. Inductively, we obtain the desired increasing sequence $\{Q^{-}_{i}\}_{i\in\mathbb{N}}$

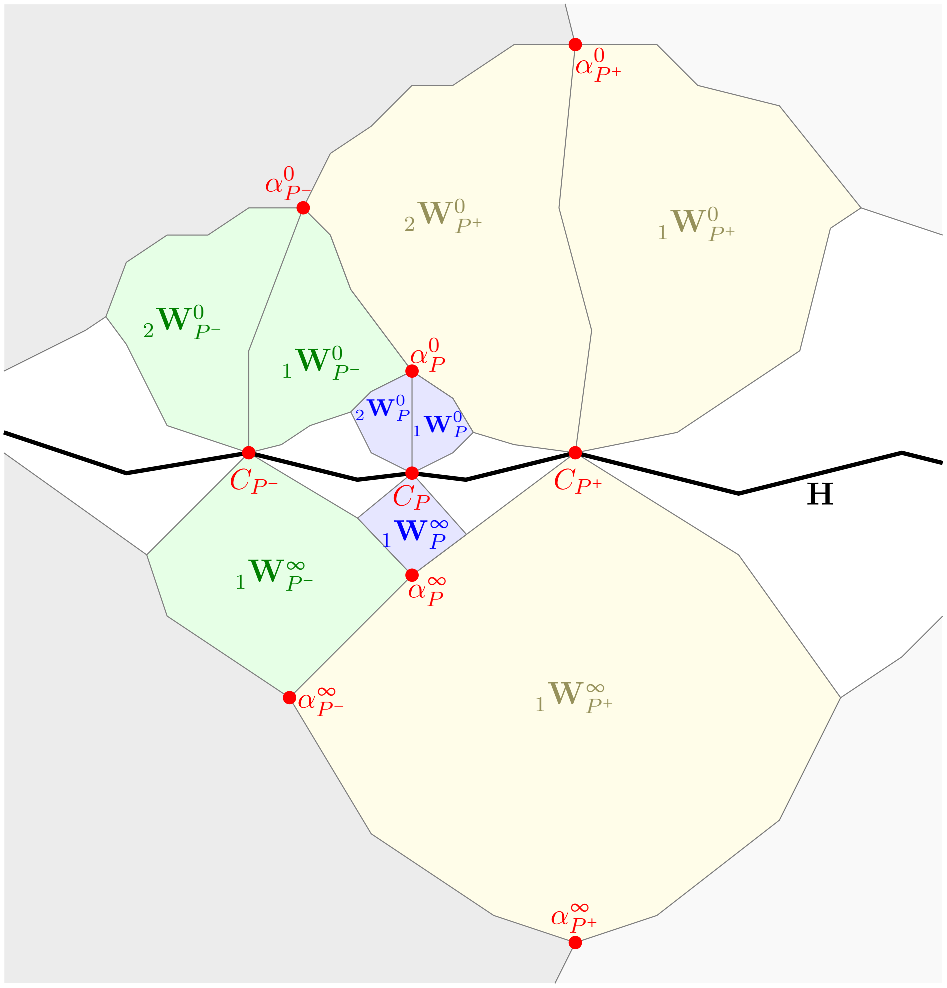

Figure 5.11: A cartoon picture of the structure of wakes when $(d_0, d_\infty) = (3, 2)$. See Figure 5.12 for a more realistic picture.

of power-triples. It remains to show that the corresponding sequence of alpha-points $\alpha^\bullet_{Q_i^-}$ indeed converges to C_P.

By Proposition 5.6.20, there exists an alpha-point α on ${}^-\mathbf{R}^\bullet_P$ close to C_P, which is the alpha-point of some primary full wake $\mathbf{W}^\bullet_Q$ where $Q > P$. Since there are at most finitely many critical points on $\mathbf{H}$ of generation less than Q between C_Q and $C_{Q_1^-}$, the ray segment $[\alpha^\bullet_{Q_1^-}, \alpha^\bullet_Q]$ intersects the boundaries of at most finitely many primary wakes. Therefore, $Q = Q_i^-$ for some $i \in \mathbb{N}$. Since α can be picked to be arbitrarily close to C_P, then $\alpha^\bullet_{Q_i^-}$ indeed converges to C_P. $\square$

Corollary 5.6.28 (Tiling of wakes)**.**

(1) *Primary wakes fill up the ocean: for* $\bullet \in \{0, \infty\}$,

$$\mathbf{O}^\bullet \subset \bigcup_{P \in \mathbf{T}_{>0}} \overline{\mathbf{W}^\bullet_P}.$$

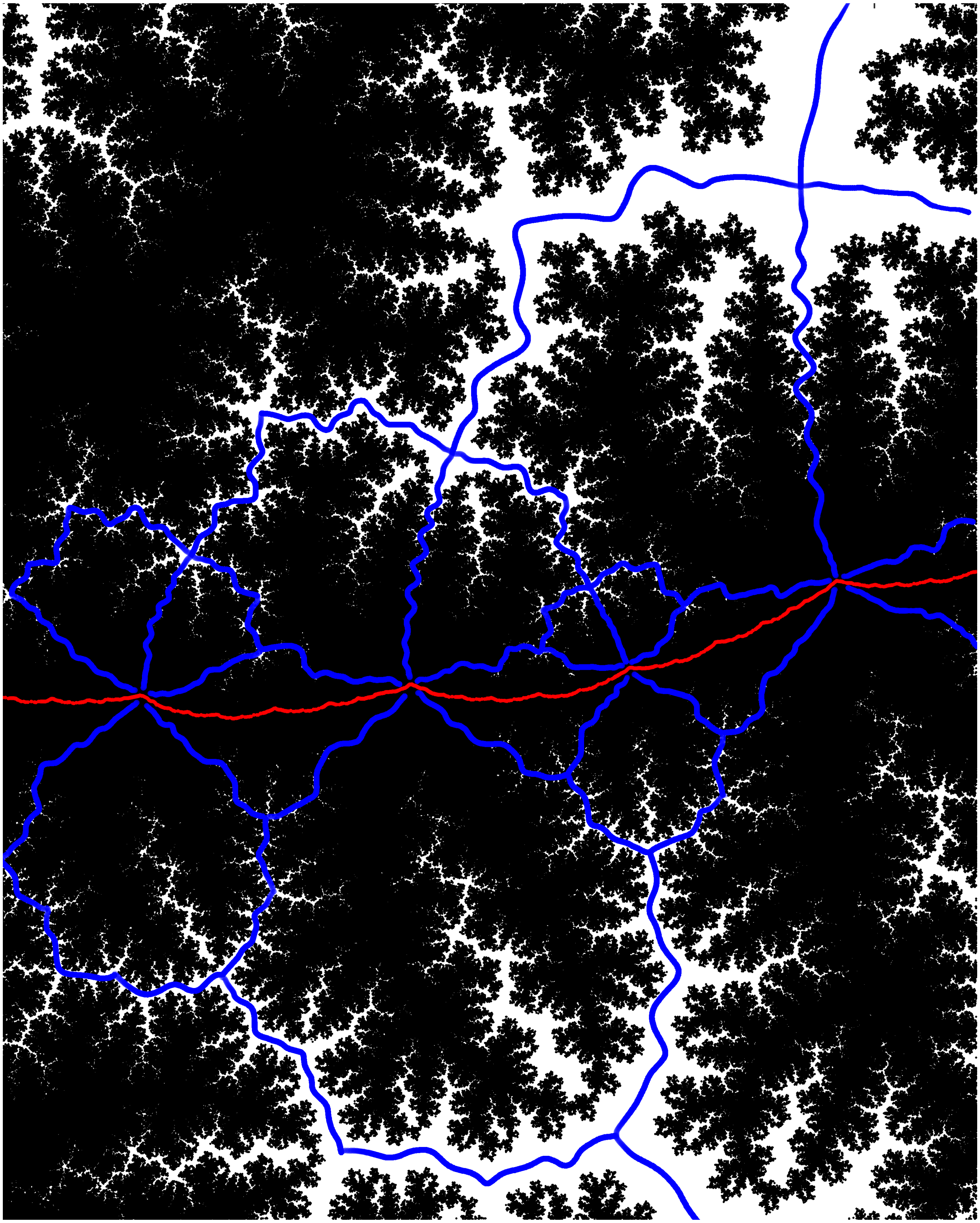

Figure 5.12: An approximate picture of the dynamical plane of $\mathbf{F}_*$ when $(d_0, d_\infty) = (3, 2)$ and θ is the golden mean irrational. This figure is obtained from the magnification of the Julia set of the rational map $f_{3,2}$ in Figure 1.1 around a point on its Herman curve. The Herman curve $\mathbf{H}$ of $\mathbf{F}_*$ is colored red and some external ray segments are displayed in blue. These external rays are the boundaries of the primary wakes attached to four critical points on $\mathbf{H}$.

(2) *The closure of a wake* ${}_{J,j}\mathbf{W}^{\blacksquare,\bullet}_S$ *is the union of spines* ${}_{J,2j-1}\mathbf{H}^{\blacksquare,\bullet}_S$ *and* ${}_{J,2j}\mathbf{H}^{\blacksquare,\bullet}_S$ *and the closure of all full wakes rooted at critical points on any of these two spines.*

(3) *For every* $z \in \mathbf{I}_{<\infty}$ *and* $m \in \mathbb{N}_{\geq 1}$*, there are at most three disjoint full wakes of level* $\geq m$ *containing* z *on their boundaries. The union of these full wakes forms a neighborhood of* z*.*

Proof. To prove (1), let us assume for a contradiction that there is a non-empty connected component X of $\mathbf{O}^\bullet \backslash \bigcup_P \overline{\mathbf{W}^\bullet_P}$. By Lemma 5.6.27, $\overline{X}$ intersects some point x on $\mathbf{H}$. There exists two sequences of power-triples $\{Q_n\}$ and $\{T_n\}$ such that for all $n \in \mathbb{N}$, the primary wakes $\mathbf{W}^\bullet_{Q_n}$ and $\mathbf{W}^\bullet_{T_n}$ touch, the union $\mathbf{H} \cup \overline{\mathbf{W}^\bullet_{Q_n}} \cup \overline{\mathbf{W}^\bullet_{T_n}}$ encloses a unique disk D_n containing X, and the corresponding roots C_{Q_n} and C_{T_n} converge to x as $n \to \infty$. By Lemma 5.6.26, the diameter of D_n tends to 0 as $n \to \infty$, which implies that such X cannot exist.

Item (2) follows from pulling back the tiling of wakes in (1) by the map $\mathbf{F}^{|S|}$ on $\overline{{}_{J,j}\mathbf{W}^{\blacksquare,\bullet}_S}$. We have thus shown that wakes of a fixed level tile each of the two oceans, and every point in the ocean is contained in the closure of at most three wakes of the same level. This implies (3). $\square$

Lemma 5.6.29. *Let us equip* $\mathbb{C}\backslash\mathbf{H}$ *with the hyperbolic metric* ρ_0*. For every* $P \in \mathbf{T}_{>0}$*,*

(1) *the map* $\mathbf{F}^P : \mathbf{W}^\bullet_P \backslash \mathbf{F}^{-P}(\mathbf{H}) \to \mathbb{C}\backslash\mathbf{H}$ *is uniformly expanding (with respect to* ρ_0*) with a factor independent of* P*;*

(2) *the hyperbolic diameter of every wake of level two is at most some uniform constant independent of* P*.*

Proof. For all $P \in \mathbf{T}$, let ρ_P be the hyperbolic metric of $\mathbb{C}\backslash\mathbf{F}^{-P}(\mathbf{H})$. To prove (1), it suffices to show that the inclusion map

$$\iota : \left(\mathbb{C}\backslash\mathbf{F}^{-P}(\mathbf{H}), \rho_P\right) \to (\mathbb{C}\backslash\mathbf{H}, \rho_0)$$

is uniformly contracting on $\mathbf{W}^\bullet_P \backslash \mathbf{F}^{-P}(\mathbf{H})$.

Clearly, ι is uniformly contracting on $\mathbf{W}^\bullet_P$ minus a small neigborhood of C_P because this region is a compact subset of $\mathbf{O}^\bullet$. The uniform contraction of ι on a neighborhood of C_P follows from the asymptotic self-similarity of $\mathbf{H}$ and $\partial\mathbf{W}^\bullet_P$ near C_P induced by pulling back A_*-invariance near 0 by $\mathbf{F}^P : C_P \mapsto 0$. One may refer to [DL23, Lemma 5.33] for further details.

Item (2) follows from essentially the same argument. By compactness, every secondary subwake of $\mathbf{W}^\bullet_P$ has uniformly bounded diameter away from a neighborhood of C_P. Near C_P, the claim again follows from the asymptotic self-similarity at C_P.

Lastly, the bounds in both claims are independent of P because every full wake can be mapped to a full wake $\mathbf{W}_Q^\bullet$ for some $\bullet \in \{0, \infty\}$ for all sufficiently small $Q \in \mathbf{T}_{>0}$. $\square$

Lemma 5.6.30. *Any infinite sequence of nested wakes shrinks to a point.*

Proof. Let us define a holomorphic map χ sending level two wakes to level one wakes as follows. Given a critical point c of $\mathbf{F}^{\geq 0}$, let $W(c)$ be the union of all wakes rooted at c. Consider a secondary critical point ${}_jC^\bullet_{P,Q}$, which is contained in $W(C_P)$. The map $\mathbf{F}^P$ sends $W({}_jC^\bullet_{P,Q})$ univalently onto $W(C_Q)$. Let $T \in \mathbf{T}$ be the smallest power-triple such that $Q - T = \mathbf{t}^n P$ for some $n \in \mathbb{Z}$. Then, $\chi := A^{-n} \circ \mathbf{F}^{P+T}$ sends $W({}_jC^\bullet_{P,Q})$ univalently back onto $W(C_P)$. By Lemma 5.6.29 (1), χ must be uniformly expanding on $W({}_jC^\bullet_{P,Q})$ with expansion factor independent of P.

Now, consider an infinite sequence of nested wakes $W_1 \supset W_2 \supset W_3 \supset \ldots$ where each W_n is of level n. By Lemma 5.6.29 (2), there is a uniform constant $C > 0$ such that for all $n \geq 3$,

$$\operatorname{diam}_{\rho_0}\left(\chi^{n-2}(W_n)\right) \leq C.$$

Since χ is uniformly expanding, the hyperbolic diameter of W_n tends to 0 exponentially fast as $n \to \infty$. $\square$

5.6.6 The structure of $\mathbf{I}_{<\infty}$ and $\mathfrak{X}$

Using wakes, we will show in this final subsection that the finite-time escaping set consists of topologically tame external rays.

Corollary 5.6.31. *Every external ray lands at a unique point.*

Proof. Let X be the accumulation set of an external ray. Since the boundary of every wake is made of ray segments, then for every wake W, either $X \subset \overline{W}$ or $X \subset \mathbb{C} \backslash W$.

If X intersects $\mathbf{H}$, then by Corollary 5.6.28, X must be contained in $\mathbf{H}$. In general, if X intersects $\mathbf{F}^{-P}(\mathbf{H})$ for some $P \in \mathbf{T}$, then $X \subset \mathbf{F}^{-P}(\mathbf{H})$. Since the roots of wakes are dense in $\mathbf{F}^{-P}(\mathbf{H})$, X must be a singleton.

Suppose X is disjoint from $\mathbf{F}^{-P}(\mathbf{H})$ for all P. Then, X is contained in an infinite sequence of nested wakes which, by Lemma 5.6.30, implies that X is a singleton. $\square$

We say that two points x and y in $\mathbf{I}_{\leq P}$ are *combinatorially equivalent* if there is no alpha-point α such that x and y belong in distinct connected components of $\mathbf{I}_{\leq P} \backslash \{\alpha\}$. Combinatorial equivalence is an equivalence relation in $\mathbf{I}_{<\infty}$.

Corollary 5.6.32.

(1) *Every combinatorial equivalence class in* $\mathbf{I}_{<\infty}$ *is a singleton.*

(2) $\mathbf{I}_{<\infty}$ *is dense in* $\mathbb{C}$ *and has empty interior.*

(3) *For every* $P \in \mathbb{R}_{>0}$,

$$\mathbf{I}_{\leq P} = \overline{\bigcup_{Q<P} \mathbf{I}_{\leq Q}}.$$

Proof. Consider a point x in $\mathbf{I}_{\leq P}$. There are two cases. If x is contained in some chain $(\infty, \alpha]$ for some alpha-point α, then the triviality of the combinatorial class follows from Corollary 5.6.21. Now, suppose x is not contained in any external chain. By Corollary 5.6.28, x is contained in an infinite sequence of nested wakes. Then, the triviality of combinatorial class of x follows from Lemma 5.6.30.

Suppose for a contradiction that the interior of $\mathbf{I}_{<\infty}$ is non-empty. Any connected component of the interior would be contained in a single combinatorial equivalence class, and this would contradict item (1). By Corollary 5.6.28 and Lemma 5.6.30, wakes of any fixed level tile the plane and any nested wakes shrink to points. Since $\mathbf{I}_{<\infty}$ intersects the closure of every wake of every level, then $\mathbf{I}_{<\infty}$ is dense in $\mathbb{C}$. This proves item (2). Lastly, item (3) follows directly from item (1). □

Since the finite-time escaping set is a subset of the Julia set, the corollary above immediately implies the following.

Corollary 5.6.33. *The Julia set of* $\mathbf{F}$ *is the whole plane:* $\mathfrak{J}(\mathbf{F}) = \mathbb{C}$.

Remark 5.6.34. Let us present an alternative proof of Corollary 5.6.33 that is independent of the entirety of this section. By Theorems 4.5.2 and 5.2.7, the critical value $c_1(f_*)$ of f_* is a deep point of the non-escaping set of the corona f_*. Magnifications of the iterated preimages of the Herman quasicircle of f_* about $c_1(f_*)$ converge to the whole plane exponentially fast in the Hausdorff metric. As we pass to the corresponding dynamical plane of the transcendental extension, 0 is a deep point of the iterated preimages of $\mathbf{H}$ under $\mathbf{F} = (\mathbf{f}_\pm)$. By self-similarity, the iterated preimages of $\mathbf{H}$ must be dense in $\mathbb{C}$, so its closure $\mathfrak{J}(\mathbf{F})$ is equal to $\mathbb{C}$.

Let us end this section with a discussion on the dynamics of $\mathbf{F}$ outside of $\mathbf{I}_{<\infty}$ and the grand orbit of $\mathbf{H}$. Consider the set

$$\mathfrak{X} := \mathbb{C} \backslash \left(\mathbf{I}_{<\infty} \cup \bigcup_{P \in \mathbf{T}} \mathbf{F}^{-P}(\mathbf{H}) \right).$$

Note that $\mathfrak{X}$ contains the infinite-time escaping set $\mathbf{I}_\infty = \mathbf{I}_\infty(\mathbf{F}_*)$ of $\mathbf{F}_*$.

Every point x in $\mathfrak{X}$ is characterized by the property that for all $P \in \mathbf{T}$, $\mathbf{F}^P(x)$ is contained in a unique primary wake. Consider the holomorphic map

$$\hat{\mathbf{F}} : \mathfrak{X} \to \mathfrak{X}, \quad \hat{\mathbf{F}}(z) = \mathbf{F}^P(z) \quad \text{if } z \in \mathbf{W}_P^0 \cup \mathbf{W}_P^\infty. \tag{5.6.8}$$

Thus, every point in $\mathfrak{X}$ is subject to infinite iteration of the map $\hat{\mathbf{F}}$.

Definition 5.6.35. For every point z in $\mathfrak{X}$, the *complete address* of z is an infinite tuple $(\mathbf{W}_0, \mathbf{W}_1, \mathbf{W}_2, \ldots)$ where for every $n \geq 0$, $\mathbf{W}_n$ is the primary wake containing the unique point $\hat{\mathbf{F}}^n(z)$. The (*incomplete*) *address* of z is the infinite tuple $(P_0, P_1, P_2, \ldots) \in \mathbf{T}_{>0}^{\mathbb{N}}$ where P_n is the generation of $\mathbf{W}_n$.

We say that an element $(P_0, P_1, \ldots)$ of $\mathbf{T}_{>0}^{\mathbb{N}}$ is *admissible* if

$$\sum_{n=0}^{\infty} P_n = \infty.$$

Moreover, we say that an infinite tuple of primary wakes is *admissible* if the corresponding tuple of generations is admissible.

Proposition 5.6.36.

(1) *An infinite tuple of primary wakes is admissible if and only if it is the complete address of a point in $\mathfrak{X}$.*

(2) *Two different points in $\mathfrak{X}$ always have distinct complete addresses.*

Proof. Given a point $z \in \mathfrak{X}$, if the sum of its incomplete address were finite, say $Q \in \mathbb{R}_{>0}$, then z would have escape time Q instead. Conversely, consider any admissible tuple of primary wakes $(\mathbf{W}_0, \mathbf{W}_1, \mathbf{W}_2, \ldots)$. Consider the sequence of nested wakes $\mathbf{W}'_0 := \mathbf{W}_0 \supset \mathbf{W}'_1 \supset \mathbf{W}'_2 \supset \ldots$ where for $n \geq 0$, $\mathbf{W}'_{n+1}$ is defined inductively by the lift of $\mathbf{W}_n$ under $\hat{\mathbf{F}}^{n+1}|_{\mathbf{W}'_n}$. The intersection of such nested wakes is precisely the set of points admitting the complete address $(\mathbf{W}_0, \mathbf{W}_1, \mathbf{W}_2, \ldots)$, and according to Lemma 5.6.30, it is a singleton. $\square$

Corollary 5.6.37. $\mathfrak{X}$ *is a dense, totally disconnected subset of* $\mathbb{C}$.

Proof. $\mathfrak{X}$ is dense because its complement has no interior. By Proposition 5.6.36 (2), two distinct points in $\mathbb{C}\backslash\mathbf{I}_{<\infty}$ have different complete itineraries and thus belong in disjoint wakes of sufficiently high generation. This implies the total disconnectivity of $\mathbb{C}\backslash\mathbf{I}_{<\infty}$. $\square$

For $R > 0$, define the large radius "non-escaping" set of $\mathbf{F}$ by

$$\mathfrak{K}_R := \left\{ z \in \mathbb{C}\backslash\mathbf{I}_{<\infty} \ : \ \left|\mathbf{F}^P(z)\right| \geq R \text{ for all } P \in \mathbf{T} \right\}.$$

Whenever $R' > R > 0$, we have

$$\mathfrak{K}_{R'} \subset \mathfrak{K}_R \subset \mathfrak{X}.$$

We say that $(P_0, P_1, P_2, \ldots) \in \mathbf{T}_{>0}^{\mathbb{N}}$ is *bounded by* $T \in \mathbf{T}$ if $P_n \le T$ for all n.

Lemma 5.6.38.

(1) *For any high $R > 0$, there exists some $Q_R \in \mathbf{T}_{>0}$ such that $Q_R \to 0$ as $R \to \infty$ and that every point z in $\mathfrak{K}_R$ has address bounded by Q_R.*

(2) *For any $Q \in \mathbf{T}_{>0}$, there is some $R_Q > 0$ such that every point in $\mathfrak{X}$ with address bounded by Q is contained in $\mathfrak{K}_{R_Q}$.*

Proof. Let us fix $R > 0$, and let $Q_R \in \mathbf{T}_{>0}$ be the smallest power-triple such that all primary wakes of generation Q_R are contained in the disk $\mathbb{D}_R := \{|z| < R\}$. (This quantity exists due to Lemma 5.5.7.) Consider a point z in $\mathfrak{X}$ and let $(P_0, P_1, P_2, \ldots)$ be its address. If $P_n \ge Q_R$ for some $n \in \mathbb{N}$, then z is eventually mapped into a wake of generation Q_R, which is contained inside of $\mathbb{D}_R$. This implies (1).

Next, let us fix $Q \in \mathbf{T}_{>0}$, and let $R_Q > 0$ be such that all primary wakes of generation $\le Q$ are disjoint from $\mathbb{D}_{R_Q}$. Suppose that $\mathbf{F}^P(z)$ is in $\mathbb{D}_{R_Q}$ for some $P \in \mathbf{T}$. Then, $\mathbf{F}^P(z)$ is contained in a wake of generation greater than Q. This implies (2). □

In the next section, we are interested in the infinite-time escaping set as well. For $\mathbf{F} = \mathbf{F}_*$, this set can be described as follows.

Corollary 5.6.39. *The infinite-time escaping set $\mathbf{I}_\infty$ of $\mathbf{F}$ is the set of points in $\mathfrak{X}$ whose address $(P_0, P_1, P_2, \ldots)$ satisfies $P_n \to 0$ as $n \to \infty$.*

Proof. Consider a point z in $\mathfrak{X}$ with some address $(P_0, P_1, P_2, \ldots)$. If $z \in \mathbf{I}_\infty$, then given any $R > 0$, $\hat{\mathbf{F}}^n(z)$ must be in $\mathfrak{K}_R$ for all sufficiently high n. By the previous lemma, $(P_n, P_{n+1}, \ldots)$ is bounded by Q_R where $Q_R \to 0$ as $R \to \infty$. Conversely, if $P_n \to 0$ as $n \to \infty$, then for all n, $(P_n, P_{n+1}, \ldots)$ is bounded by some $Q_n \in \mathbf{T}_{>0}$ where $Q_n \to 0$ as $n \to \infty$. By the previous lemma, $\hat{\mathbf{F}}^n(z)$ is contained in $\mathfrak{K}_{R_{Q_n}}$ where $R_{Q_n} \to \infty$ as $n \to \infty$, thus z is contained in $\mathbf{I}_\infty$. □

5.7 The escaping set I(F)

In this section, we will discuss the topology and rigidity of both the finite-time and the infinite-time escaping sets of a cascade in $\boldsymbol{\mathcal{W}}^u$. Our eventual goal is to prove the first half of Theorem K. In the proof, we will apply the external structure of the renormalization fixed point $\mathbf{F}_*$ addressed in Section 5.6, and adapt an argument by Rempe [Rem09] to show that the set of points in the full escaping set that remain sufficiently close to ∞ under iteration must move holomorphically with dilatation arbitrarily close to zero.

5.7.1 Invariant line field

We say that a corona $f : U \to V$ admits an *invariant line field* supported on a completely invariant set $E \subset \mathbb{C}$ if there is a measurable Beltrami differential $\mu(z)\frac{d\bar{z}}{dz}$ such that $f^*\mu = \mu$ almost everywhere on U, $|\mu| = 1$ on a positive measure subset of E, and $\mu = 0$ elsewhere.

Similarly, we say that $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$ admits an *invariant line field* supported on a completely invariant set $E \subset \mathbb{C}$ if there is a measurable Beltrami differential $\boldsymbol{\mu}(z)\frac{d\bar{z}}{dz}$ such that $(\mathbf{F}^P)^* \boldsymbol{\mu} = \boldsymbol{\mu}$ almost everywhere on $\mathrm{Dom}(\mathbf{F}^P)$ for all $P \in \mathbf{T}$, $|\boldsymbol{\mu}| = 1$ on a positive measure subset of E, and $\boldsymbol{\mu} = 0$ elsewhere.

We would like to emphasize that the latter is stronger than the former. Given a corona f in $\mathcal{W}^u_{\text{loc}}$ and its associated cascade $\mathbf{F}$, an invariant line field $\boldsymbol{\mu}$ of $\mathbf{F}$ induces a sequence of line fields μ_{-n} invariant under $\mathcal{R}^{-n}f$ for all $n \geq 0$.

In classical holomorphic dynamics, the absence of invariant line fields is equivalent to the triviality of deformation space associated to a single holomorphic map. This principle remains valid for cascades in the unstable manifold.

Proposition 5.7.1. *Suppose $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$ admits an invariant line field $\boldsymbol{\mu}$.*

(1) *There exist a holomorphic family $\{\mathbf{G}_t\}_{t\in\mathbb{D}}$ in $\boldsymbol{\mathcal{W}}^u$ and a holomorphic family of normalized quasiconformal maps $\{\phi_t : \mathbb{C} \to \mathbb{C}\}_{t\in\mathbb{D}}$ such that $\mathbf{G}_0 = \mathbf{F}$ and $\mathbf{F}^{\geq 0}$ is quasiconformally conjugate to $\mathbf{G}_t^{\geq 0}$ via ϕ_t.*

(2) *The support of $\boldsymbol{\mu}$ is equal to the set of points at which the conjugacy ϕ_t is not conformal.*

(3) *Suppose $\mathbf{F}$ is in $\boldsymbol{\mathcal{W}}^u_{\text{loc}}$, let $f \in \mathcal{W}^u_{\text{loc}}$ be the associated corona, and let $f_{-n} := \mathcal{R}^{-n}(f)$ for $n \geq 0$. For each $n \geq 0$,*

   (a) *there exist a holomorphic family $\{g_{-n,t}\}_{t\in\mathbb{D}}$ in $\mathcal{W}^u_{\text{loc}}$ and a holomorphic family of quasiconformal maps $\{\phi_{-n,t} : V \to V\}_{t\in\mathbb{D}}$ such that $g_{-n,0} = f_{-n}$ and f_{-n} is quasiconformally conjugate to $g_{-n,t}$ via $\phi_{-n,t}$;*

   (b) *$\mathcal{R}g_{-n-1,t} = g_{-n,t}$ for all $t \in \mathbb{D}$,*

   (c) *$\mathbf{G}_t \equiv \mathbf{F}$ for all t if and only if $g_{-n,t} \equiv f_{-n}$ for all t.*

Proof. A standard application of the measurable Riemann mapping theorem gives us the desired holomorphic family $\{\mathbf{G}_t\}_{t\in\mathbb{D}}$, but a priori we do not know whether this family lives in $\boldsymbol{\mathcal{W}}^u$. To fix this issue, we shall descend to the realm of coronas.

By anti-renormalizing, let us assume without loss of generality that $\mathbf{F}$ is in $\boldsymbol{\mathcal{W}}^u_{\text{loc}}$. Pick a pair of integers $m, n \geq 0$. Let us project $\boldsymbol{\mu}$ to the dynamical plane of f_{-m-n} and obtain an invariant line field μ_{-m-n} of f_{-m-n}. Then, we integrate $t\mu_{-m-n}$ for every $t \in \mathbb{D}$ to obtain a Beltrami path $\{f_{-m-n,t}\}_{t\in\mathbb{D}}$ of coronas in a neighborhood of f_*. Let us renormalize m times to obtain a new path $f^{(m)}_{-n,t} \coloneqq \mathcal{R}^m f_{-m-n,t}$ about $f^{(m)}_{-n,0} \equiv f_{-n}$. When $|t| < \frac{1}{2}$, $f^{(m)}_{-n,t}$ is quasiconformally conjugate to f_{-n} with uniformly bounded dilatation. Therefore, we can take a limit as $m \to \infty$ and obtain a holomorphic path $g_{-n,t}$ of infinitely anti-renormalizable coronas.

For sufficiently small $\varepsilon > 0$, the limiting path $\{g_{-n,t}\}_{|t|<\varepsilon}$ lies in the local unstable manifold $\mathcal{W}^u_{\text{loc}}$, and it satisfies the relation $\mathcal{R}g_{-n-1,t} = g_{-n,t}$ for all $n \geq 0$ and t. In particular, $\{g_{0,t}\}$ corresponds to the desired holomorphic path $\{\mathbf{G}_t\}$ in $\boldsymbol{\mathcal{W}}^u_{\text{loc}}$. Let us elaborate on the last property (3) (c). Recall from Section 5.4.1 that $\mathbf{G}_t$ is constructed as the "union" of analytic extensions of rescalings of $g_{-n,t}$ across all $n \leq 0$. Thus, $\mathbf{G}_t$ is a trivial family if and only if $g_{-n,t}$ is trivial for all n, but by (3) (b), this occurs if and only if $g_{-n,t}$ is trivial for some n. □

Lemma 5.7.2. *The renormalization fixed point* $\mathbf{F}_*$ *admits no invariant line field.*

Proof. Suppose for a contradiction that $\mathbf{F}_*$ admits an invariant line field $\boldsymbol{\mu}$. By Proposition 5.7.1, the invariant line field induces a family $\{\mathbf{G}_t\}_{t\in\mathbb{D}}$ in $\boldsymbol{\mathcal{W}}^u$ where $\mathbf{G}_0 \equiv \mathbf{F}_*$, together with quasiconformal maps $h_t : \mathbb{C} \to \mathbb{C}$ conjugating $\mathbf{F}_*$ with $\mathbf{G}_t$ for all $t \in \mathbb{D}$. Each of $\mathbf{G}_t$ induces a rotational corona g_t with rotation number θ, which, by Theorem 5.3.9, implies that g_t must also be on the local stable manifold. Therefore, $g_t \equiv f_*$ and the family $\mathbf{G}_t$ is trivial. For every t, h_t commutes with $\mathbf{F}_*$ along the Herman curve $\mathbf{H}$ of $\mathbf{F}_*$. As such, h_t is the identity on $\mathbf{H}$, and so on the iterated preimages $\bigcup_P \mathbf{F}^{-P}(\mathbf{H})$ of $\mathbf{H}$ as well. By Corollary 5.6.33, the closure of iterated preimages of $\mathbf{H}$ is $\mathbb{C}$, so h_t is the identity map on the whole plane. This contradicts the assumption that the support of $\boldsymbol{\mu}$ has positive measure. □

5.7.2 The finite-time escaping set

Let us fix

$$T \coloneqq \min\{(0,1,0),(0,0,1)\}.$$

Lemma 5.7.3. *There is a unique equivariant holomorphic motion of* $\mathbf{I}_{\leq T}(\mathbf{F})$ *over some neighborhood* $\boldsymbol{\mathcal{U}}$ *of* $\mathbf{F}_*$.

Proof. By Lemma 5.5.6, the set of critical values $\text{CV}(\mathbf{F}^T)$ of $\mathbf{F}^T$ moves holomorphically within a small neighborhood of $\mathbf{F}_*$. By Lemma 5.4.8, there is a small neighborhood $\boldsymbol{\mathcal{U}}$ of $\mathbf{F}_*$ and some point $x \in \mathbb{C}$ such that x belongs in the interior of $\mathbf{U}_-(\mathbf{F})$ and does not collide with $\text{CV}(\mathbf{F}^T)$ for all $\mathbf{F} \in \boldsymbol{\mathcal{U}}$. Moreover, $\mathbf{F}^{-S}(x)$ moves holomorphically with $\mathbf{F} \in \boldsymbol{\mathcal{U}}$ for all $S \leq T$.

Given $Q, S \in \mathbf{T}_{>0}$, if $Q < S \leq T$, then $\mathbf{F}^{-S}(x)$ is disjoint from $\mathbf{F}^{-Q}(x)$ because every point is mapped by $\mathbf{F}^{S}$ and $\mathbf{F}^{Q}$ to different tiles of the zeroth renormalization tiling of $\mathbf{F}$. Hence, $\bigcup_{S \leq T} \mathbf{F}^{-S}(x)$ moves holomorphically and equivariantly with $\mathbf{F} \in \boldsymbol{\mathcal{U}}$. By the λ-lemma, this holomorphic motion extends to the closure. Then, by Corollaries 5.6.32 (2) and 5.5.14, $\mathbf{I}_{\leq T}(\mathbf{F})$ has no interior and moves holomorphically and equivariantly over $\boldsymbol{\mathcal{U}}$.

Let us show that the motion τ of $\mathbf{I}_{\leq T}(\mathbf{F})$ obtained above is independent of x. Let us pick another point $y = y(\mathbf{F}) \in \mathbb{C} \backslash \mathrm{CV}(\mathbf{F})$ which depends holomorphically on $\mathbf{F} \in \boldsymbol{\mathcal{U}}$. By shrinking $\boldsymbol{\mathcal{U}}$, we can connect x and y by a simple arc $l = l(\mathbf{F})$ which is surrounded by an annulus $A = A(\mathbf{F}) \subset \mathbb{C} \backslash \mathrm{CV}(\mathbf{F})$. Every preimage of l under $\mathbf{F}^{T}$ is separated from $\mathbf{I}_{\leq T}(\mathbf{F})$ by a conformal preimage of A. Therefore, any sequence of preimages of l under $\mathbf{F}^{T}$ which accumulates at a point in $\mathbf{I}_{\leq T}(\mathbf{F})$ necessarily shrinks in diameter. As a result, the holomorphic motion τ coincides with the motion of $\overline{\mathbf{F}^{-T}(y(\mathbf{F}))}$.

Finally, let us show that the equivariant holomorphic motion τ of $\mathbf{I}_{\leq T}(\mathbf{F})$ over $\boldsymbol{\mathcal{U}}$ is unique. Suppose there is another equivariant holomorphic motion τ' of $\mathbf{I}_{\leq T}(\mathbf{F})$. Pick any $S \in \mathbf{T}_{>0}$ where $S < T$ and consider the motion $y(\mathbf{F})$ of a point in $\mathbf{I}_{\leq S}(\mathbf{F})$ induced by τ'. By equivariance, $\mathbf{F}^{-(T-S)}(y(\mathbf{F}))$ moves holomorphically by τ'. However, since $\mathbf{I}_{\leq T-S}(\mathbf{F})$ is contained in the closure of $\mathbf{F}^{-(T-S)}(y(\mathbf{F}))$, then τ and τ' coincide on $\mathbf{I}_{\leq T-S}(\mathbf{F})$ for all $S \in \mathbf{T}_{>0}$. By Corollary 5.6.32 (3), $\tau \equiv \tau'$. □

Definition 5.7.4. We say that a holomorphic motion of a set $E \subset \mathbb{C}$ is a *conformal motion* if its dilatation on E is zero.

Theorem 5.7.5. *For every $\mathbf{F} \in \boldsymbol{\mathcal{W}}^{u}$, $\mathbf{I}_{<\infty}(\mathbf{F})$ has empty interior and supports no invariant line field. For every $P \in \mathbf{T}_{>0}$, on every connected component of the open set $\{\mathbf{F} \in \boldsymbol{\mathcal{W}}^{u} : 0 \notin \mathbf{I}_{\leq P}(\mathbf{F})\}$, there is a unique equivariant holomorphic motion of $\mathbf{I}_{\leq P}$, and this motion is conformal.*

Proof. Let us fix $P \in \mathbf{T}_{>0}$ and consider the set $\boldsymbol{\Omega}_P := \{\mathbf{F} \in \boldsymbol{\mathcal{W}}^{u} : 0 \notin \mathbf{I}_{\leq P}(\mathbf{F})\}$. If $P < T$, then clearly the neighborhood $\boldsymbol{\mathcal{U}}$ of $\mathbf{F}_*$ from Lemma 5.7.3 is contained in $\boldsymbol{\Omega}_P$. Else, if $P \geq T$, then $\mathbf{F} \in \boldsymbol{\Omega}_P \cap \boldsymbol{\mathcal{U}}$ if and only if $\mathbf{F}^{P-T}(0) \notin \mathbf{I}_{\leq T}(\mathbf{F})$, which is an open condition because $\mathbf{I}_{\leq T}$ moves holomorphically over $\boldsymbol{\mathcal{U}}$. Therefore, $\boldsymbol{\Omega}_P \cap \boldsymbol{\mathcal{U}}$ is open for all P.

If $\mathbf{F} \in \boldsymbol{\Omega}_P \cap \boldsymbol{\mathcal{U}}$, we can obtain the unique equivariant holomorphic motion of $\mathbf{I}_{\leq P}$ by pulling back the holomorphic motion of $\mathbf{I}_{\leq T}$ via $\mathbf{F}^{P-T}$. Otherwise, we can pick a sufficiently large $n \in \mathbb{N}$ such that $\mathbf{F}_{-n}$ is in $\boldsymbol{\mathcal{U}}$. Clearly, $\mathbf{F} \in \boldsymbol{\Omega}_P$ if and only if $\mathbf{F}_{-n} \in \boldsymbol{\Omega}_{\mathbf{t}^n P}$, so $\boldsymbol{\Omega}_P$ is always an open subset of $\boldsymbol{\mathcal{W}}^{u}$ on which $\mathbf{I}_{\leq P}(\mathbf{F})$ moves holomorphically and equivariantly. The dilatation of the motion of $\mathbf{I}_{\leq T}(\mathbf{F})$ over $\boldsymbol{\mathcal{U}}$ goes to zero as $\mathbf{F} \to \mathbf{F}_*$. Therefore, for every $\mathbf{F} \in \boldsymbol{\Omega}_P$, we can take an arbitrarily high n to ensure that the dilatation of the motion of $\mathbf{I}_{\leq T}$ at $\mathbf{F}_{-n}$, and hence that of the motion of $\mathbf{I}_{\leq \mathbf{t}^{-n} T}$ at $\mathbf{F}$ as well, are arbitrarily small. Pulling back via $\mathbf{F}^{P-\mathbf{t}^{-n}T}$ does not affect the dilatation, so the motion of $\mathbf{I}_{\leq P}$ is indeed conformal over $\boldsymbol{\Omega}_P$.

By Corollary 5.6.32 (2) and Lemma 5.7.2, $\mathbf{I}_{\leq \mathbf{t}^n P}(\mathbf{F}_{-n})$ has empty interior and supports no invariant line field of $\mathbf{F}_{-n}$. Therefore, $\mathbf{I}_{\leq P}(\mathbf{F})$ also has empty interior and supports no invariant line field of $\mathbf{F}$. $\square$

In the study of dynamics of transcendental entire functions $g : \mathbb{C} \to \mathbb{C}$, a fundamental yet highly non-trivial result is the fact that the escaping set $I(g)$ of g is non-empty [Ere89; Dom98], and as a consequence, the Julia set of g is the closure of the boundary of $I(g)$. Similarly, we have the following.

Corollary 5.7.6. *For all $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, the finite-time escaping set $\mathbf{I}_{<\infty}(\mathbf{F})$ is non-empty and $\mathfrak{J}(\mathbf{F}) = \overline{\mathbf{I}_{<\infty}(\mathbf{F})}$.*

Proof. Pick any $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$. From the previous theorem, there exist some small $P \in \mathbf{T}_{>0}$ and some open neighborhood $\boldsymbol{\mathcal{U}} \subset \boldsymbol{\mathcal{W}}^u$ of $\mathbf{F}_*$ containing $\mathbf{F}$ in which the P^{th} escaping set moves holomorphically. Therefore, $\mathbf{I}_{\leq P}(\mathbf{F})$ is clearly non-empty.

Consider any open disk $D \subset \mathbb{C}$ disjoint from $\mathbf{I}_{<\infty}(\mathbf{F})$. By Montel's theorem, since $\mathbf{I}_{<\infty}(\mathbf{F})$ contains more than two points, then $\{\mathbf{F}^P : D \to \mathbb{C} \backslash \mathbf{I}_{<\infty}(\mathbf{F})\}_P$ forms a normal family. Thus, such a disk D is necessarily contained in $\mathfrak{F}(\mathbf{F})$. In particular, any open disk centered at a point in $\mathfrak{J}(\mathbf{F})$ must intersect $\mathbf{I}_{<\infty}(\mathbf{F})$. $\square$

5.7.3 The infinite-time escaping set

For $R > 0$ and $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, define

$$\mathfrak{J}_R(\mathbf{F}) := \left\{ z \in \mathbb{C} \backslash \mathbf{I}_{<\infty}(\mathbf{F}) \ : \ \left| \mathbf{F}^P(z) \right| \geq R \text{ for all } P \in \mathbf{T} \right\}.$$

The forward orbit of every point in $\mathfrak{J}(\mathbf{F}) \cap \mathbf{I}_\infty(\mathbf{F})$ is eventually contained in $\mathfrak{J}_R(\mathbf{F})$. The following lemma is inspired by [Rem09].

Lemma 5.7.7. *For every $\mathbf{F}$ on a neighborhood $\boldsymbol{\mathcal{U}} \subset \boldsymbol{\mathcal{W}}^u_{\text{loc}}$ of $\mathbf{F}_*$, there exists a totally disconnected subset $\Lambda(\mathbf{F})$ of $\mathbb{C} \backslash \mathbf{I}_{<\infty}(\mathbf{F})$ with the following properties.*

(1) *$\Lambda(\mathbf{F})$ is forward invariant under $\mathbf{F}^{\geq 0}$.*

(2) *There is a unique equivariant holomorphic motion of Λ over $\boldsymbol{\mathcal{U}}$.*

(3) *There exists some $R > 1$ such that $\Lambda(\mathbf{F})$ contains $\mathfrak{J}_R(\mathbf{F})$.*

Proof. In the dynamical plane of $\mathbf{F}_*$, every point in the forward orbit of a point in $\mathfrak{J}_R(\mathbf{F}_*)$ must be contained in a wake of sufficiently low generation in order to avoid the disk $\mathbb{D}_R := \{|z| < R\}$. We consider all such points and define $\Lambda(\mathbf{F}_*)$. In the proof below, we apply the motion of the finite-time escaping set from the previous subsection to show that $\Lambda(\mathbf{F})$ can be defined naturally via a unique holomorphic motion. The proof will be broken down into four steps.

Step 1: Construct truncated wakes which move holomorphically.

Let us pick $r > 0$ such that all primary wakes of $\mathbf{F}_*$ of generation at most $T := \min\{(0,1,0),(0,0,1)\}$ are compactly contained in the domain $\mathbf{V} := \mathbb{C}\backslash\overline{\mathbb{D}_r}$. Let us enumerate primary wakes of $\mathbf{F}_*$ of generation at most T by $\{\mathbf{W}_i\}_{i\in I}$ for some countable index set I. Denote the generation of each wake $\mathbf{W}_i$ by P_i. For every $i \in I$, consider the truncated wake

$$\hat{\mathbf{W}}_i := \mathbf{W}_i \cap \mathbf{F}_*^{-P_i}(\mathbf{V})$$

obtained by removing from $\mathbf{W}_i$ a small neighborhood of the critical point C_{P_i} that gets mapped to $\overline{\mathbb{D}_r}$.

For each $\bullet \in \{0,\infty\}$, there exists a unique point $z^\bullet$ on the intersection of $\partial\mathbf{V}$ and the zero ray $\mathbf{R}^\bullet$ such that the external ray segment

$$\hat{\mathbf{R}}^\bullet := (\infty, z^\bullet) \subset \mathbf{R}^\bullet$$

is contained in $\mathbf{V}$. The ray segments $\hat{\mathbf{R}}^0$ and $\hat{\mathbf{R}}^\infty$ are contained in $\mathbf{I}_{\leq Q}(\mathbf{F}_*)$ where Q is the maximum of the escaping times of z^0 and z^∞. By Lemma 5.7.7, the Q^{th} escaping set $\mathbf{I}_{\leq Q}$ moves holomorphically and equivariantly on a small neighborhood $\boldsymbol{\mathcal{U}}$ of $\mathbf{F}_*$. By the λ-lemma, such a motion induces a holomorphic motion of $\hat{\mathbf{R}}^0(\mathbf{F}) \cup \hat{\mathbf{R}}^\infty(\mathbf{F}) \cup \partial\mathbf{V}(\mathbf{F})$, which, by shrinking $\boldsymbol{\mathcal{U}}$ if necessary, can be assumed to not collide with $\mathrm{CV}(\mathbf{F}^T)$. This allows us to pull back via $\mathbf{F}^P$ for all $P \leq T$ and further extend this motion to a holomorphic motion of

$$\hat{\mathbf{R}}^0(\mathbf{F}) \cup \hat{\mathbf{R}}^\infty(\mathbf{F}) \cup \partial\mathbf{V}(\mathbf{F}) \cup \bigcup_{i\in I} \partial\hat{\mathbf{W}}_i(\mathbf{F})$$

that is equivariant on $\partial\hat{\mathbf{W}}_i(\mathbf{F})$ with respect to $\mathbf{F}^{P_i}$ for every $i \in I$. By λ-lemma, this motion can again be extended to a holomorphic motion Φ_0 on the whole plane that is equivariant with respect to $\mathbf{F}^{P_i}$ on $\partial\hat{\mathbf{W}}_i(\mathbf{F})$ for every $i \in I$.

Step 2: Construct Λ which moves holomorphically and equivariantly.

Consider $\mathbf{V}_0(\mathbf{F}) := \bigcup_{i\in I} \hat{\mathbf{W}}_i(\mathbf{F})$ and define the holomorphic map

$$\hat{\mathbf{F}} : \mathbf{V}_0(\mathbf{F}) \to \mathbf{V}(\mathbf{F}), \quad \hat{\mathbf{F}}(z) = \mathbf{F}^{P_i}(z) \text{ for } z \in \hat{\mathbf{W}}_i(\mathbf{F}).$$

This map satisfies a Markov-like property that $\mathbf{V}_0(\mathbf{F}) \subset \mathbf{V}(\mathbf{F})$ and $\hat{\mathbf{F}}$ sends every connected component of $\mathbf{V}_0(\mathbf{F})$ univalently onto a dense subset of $\mathbf{V}(\mathbf{F})$. Note that $\hat{\mathbf{F}}_*$ coincides with the map defined in (5.6.8).

Consider the non-escaping set $\Lambda(\mathbf{F})$ of $\hat{\mathbf{F}}$ which is defined by

$$\Lambda(\mathbf{F}) := \bigcap_{n\geq 0} \mathbf{V}_{-n}(\mathbf{F}) \quad \text{where} \quad \mathbf{V}_{-n}(\mathbf{F}) := \hat{\mathbf{F}}^{-n}(\mathbf{V}_0(\mathbf{F})).$$

Clearly, $\Lambda(\mathbf{F})$ is non-empty and forward invariant under $\mathbf{F}^{\geq 0}$. For $\mathbf{F} = \mathbf{F}_*$, the set $\Lambda(\mathbf{F}_*)$ is a subset of the set $\mathfrak{X}$ defined in Section §5.6.6 and is totally disconnected.

Let us treat the holomorphic motion $\Phi_0 = \Phi_0(\mathbf{F})$ discussed in Step 1 as a map from the dynamical plane of $\mathbf{F}_*$ to the dynamical plane of $\mathbf{F}$. We will apply the pullback argument to Φ_0 as follows. For $n \geq 0$, let us inductively define the lift of Φ_n to be

$$\Phi_{n+1} := \begin{cases} \Phi_n & \text{on } \mathbb{C}\backslash \mathbf{V}_{-n}(\mathbf{F}_*), \\ \left(\hat{\mathbf{F}}|_{\hat{\mathbf{W}}_i(\mathbf{F})}\right)^{-1} \circ \Phi_n \circ \hat{\mathbf{F}}_* & \text{on } \mathbf{V}_{-n}(\mathbf{F}_*) \cap \hat{\mathbf{W}}_i(\mathbf{F}_*) \text{ for each } i \in I. \end{cases}$$

By equivariance, for all n, Φ_n is quasiconformal on $\mathbb{C}$ with uniformly bounded dilatation and it eventually stabilizes at every point outside of $\Lambda(\mathbf{F}_*)$. Since $\Lambda(\mathbf{F}_*)$ has no interior, Φ_n converges in subsequence to a limiting holomorphic motion Φ which is equivariant on $\Lambda(\mathbf{F})$.

Step 3: Show that the equivariant holomorphic motion of Λ is unique.

Suppose Ψ is another holomorphic motion of $\Lambda(\mathbf{F})$ on some small neighborhood $\boldsymbol{\mathcal{U}} \subset \boldsymbol{\mathcal{W}}^u_{\text{loc}}$ of $\mathbf{F}_*$. We will use the notation $\Psi_{\mathbf{F}}(x)$ to highlight the dependence of $\mathbf{F}$. Let us pick any point $x \in \Lambda(\mathbf{F}_*)$. By Proposition 5.6.36, there is some $(i_0, i_1, \ldots) \in I^{\mathbb{N}}$ such that x is the unique point with address $(i_0, i_1, \ldots)$, that is, $\hat{\mathbf{F}}^n_*(x)$ lies in the truncated wake $\hat{\mathbf{W}}_{i_n}(\mathbf{F}_*)$ for all n.

Suppose for a contradiction that $\Psi_{\mathbf{F}}(x)$ and $\Phi_{\mathbf{F}}(x)$ are distinct. Then, the address of $\Psi_{\mathbf{F}}(x)$ is not equal to $(i_0, i_1, \ldots)$ and, in particular, there is some $n \in \mathbb{N}$ such that $\hat{\mathbf{F}}^n(\Psi_{\mathbf{F}}(x))$ lies in a truncated wake other than $\hat{\mathbf{W}}_{i_n}(\mathbf{F})$. Since the boundary of $\hat{\mathbf{W}}_{i_n}(\mathbf{F})$ moves holomorphically and equivariantly, there exists some $\mathbf{G} \in \boldsymbol{\mathcal{W}}^u_{\text{loc}}$ sufficiently close to $\mathbf{F}_*$ such that $x'_n := \hat{\mathbf{G}}^n(\Psi_{\mathbf{G}}(x))$ is on the boundary of $\hat{\mathbf{W}}_{i_n}(\mathbf{G})$. Then, the image $y'_n := \mathbf{G}^{P_{i_n}}(x'_n)$ would lie on $\hat{\mathbf{R}}^0(\mathbf{G}) \cup \hat{\mathbf{R}}^\infty(\mathbf{G}) \cup \partial \mathbf{V}(\mathbf{G})$, which is disjoint from $\Lambda(\mathbf{G})$. However, due to forward invariance, y'_n must be contained in $\Lambda(\mathbf{G})$, hence a contradiction.

Step 4: Show that $\Lambda(\mathbf{F})$ contains $\mathfrak{J}_R(\mathbf{F})$ for some $R > 0$ independent of $\mathbf{F} \in \boldsymbol{\mathcal{U}}$.

It suffices to find R such that for all $\mathbf{F} \in \boldsymbol{\mathcal{U}}$, every point outside of $\mathbf{I}_{<\infty}(\mathbf{F}) \cup \Lambda(\mathbf{F})$ will be sent into the disk $\mathbb{D}_R$ by $\mathbf{F}^P$ for some $P \in \mathbf{T}$.

Let us recall the renormalization tiling $\boldsymbol{\Delta}_n(\mathbf{F})$ defined in §5.5.2. In the dynamical plane of $\mathbf{F}_*$, there exists some sufficiently large $N \in \mathbb{N}$ such that all primary wakes rooted at critical points located in $\Delta_0(0, \mathbf{F}_*) \cup \Delta_0(1, \mathbf{F}_*)$ are contained in the tile $\Delta_{-N}(i, \mathbf{F}_*)$ for some $i \in \{0, 1\}$. Then, every wake of generation greater than T is contained in the tiling $\boldsymbol{\Delta}_{-N}(\mathbf{F}_*)$. In particular, $\overline{\mathbf{V}_0(\mathbf{F}_*)}$ is disjoint from $\boldsymbol{\Delta}_{-N}(\mathbf{F}_*)$.

By shrinking $\boldsymbol{\mathcal{U}}$ if needed, the tiling $\boldsymbol{\Delta}_{-N}(\mathbf{F})$ moves holomorphically and equivariantly over $\boldsymbol{\mathcal{U}}$ and always contains $\mathbb{C}\backslash\overline{\mathbf{V}_0(\mathbf{F})}$. Therefore, for all $\mathbf{F} \in \boldsymbol{\mathcal{U}}$, every point outside of $\mathbf{I}_{<\infty}(\mathbf{F}) \cup \Lambda(\mathbf{F})$ is eventually mapped to a point in $\mathbb{C}\backslash\overline{\mathbf{V}_0(\mathbf{F})}$, which is eventually mapped to another point in $\mathbf{F}^{(-N,0,1)}(\Delta_{-N}(0,\mathbf{F})) \cup \mathbf{F}^{(-N,1,0)}(\Delta_{-N}(1,\mathbf{F}))$, which is contained in the disk $\mathbb{D}_R$ for some large $R > 0$ independent of $\mathbf{F}$. $\square$

Theorem 5.7.8. *For every $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, $\mathbf{I}_\infty(\mathbf{F})$ is a totally disconnected subset of $\mathfrak{J}(\mathbf{F})$ and supports no invariant line field. Moreover, on every connected component of the interior of $\{\mathbf{F} \in \boldsymbol{\mathcal{W}}^u : 0 \notin \mathbf{I}_\infty(\mathbf{F})\}$, there is a unique equivariant holomorphic motion of $\mathbf{I}_\infty(\mathbf{F})$, and this motion is conformal.*

Proof. Let $\boldsymbol{\mathcal{U}}$, Λ, and R be from the previous lemma. For every $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$, there is some sufficiently large $n \in \mathbb{N}$ such that the n^{th} anti-renormalization $\mathbf{F}_{-n}$ lies in $\boldsymbol{\mathcal{U}}$. Since $\mathbf{F}^P = A_*^{-n} \circ \mathbf{F}_{-n}^{P/\mathbf{t}^{-n}} \circ A_*^n$ for all $P \in \mathbf{T}$, the set

$$\Lambda_{-n}(\mathbf{F}) := A_*^{-n}(\Lambda(\mathbf{F}_{-n}))$$

is forward invariant, contains $\mathfrak{J}_{|\mu_*|^{-n}R}(\mathbf{F})$, and admits a unique equivariant holomorphic motion Φ_{-n} over $\mathcal{R}^n(\boldsymbol{\mathcal{U}})$. The dilatation of Φ_{-n} near $\mathbf{F}$ can be made arbitrarily small by choosing $\mathbf{F}_{-n}$ arbitrarily close to $\mathbf{F}_*$, or equivalently, n to be an arbitraily large. In particular, there is a unique equivariant holomorphic motion of $\mathbf{I}_\infty(\mathbf{F}) \cap \Lambda_{-n}(\mathbf{F})$ and its dilatation near $\mathbf{F}$ shrinks to zero as $n \to \infty$.

Every point in $\mathbf{I}_\infty(\mathbf{F})$ is eventually mapped to $\mathbf{I}_\infty(\mathbf{F}) \cap \Lambda_{-n}(\mathbf{F})$. Since $\Lambda_{-n}(\mathbf{F})$ is totally disconnected, then so is $\mathbf{I}_\infty(\mathbf{F})$. To show that $\mathbf{I}_\infty(\mathbf{F})$ is in the Julia set, suppose for a contradiction that $\mathbf{I}_\infty(\mathbf{F})$ contains a point x in the Fatou set. By normality, points sufficiently close to x are also attracted to ∞, which contradicts the total disconnectivity of $\mathbf{I}_\infty(\mathbf{F})$.

On a component $\boldsymbol{\Omega}$ of the interior of $\{\mathbf{F} \in \boldsymbol{\mathcal{W}}^u : 0 \notin \mathbf{I}_\infty(\mathbf{F})\}$, for $\mathbf{F} \in \boldsymbol{\Omega}$, we can extend the motion Φ_{-n} by iteratively pulling back the holomorphic motion of $\mathbf{I}_\infty(\mathbf{F}) \cap \Lambda_{-n}(\mathbf{F})$, yielding a unique equivariant holomorphic motion $\widetilde{\Phi}_{-n}$ of $\mathbf{I}_\infty(\mathbf{F})$. Since we are pulling back by a holomorphic map, the dilatation of $\widetilde{\Phi}_{-n}$ is equal to that of Φ_{-n}. By the uniqueness of the motion, $\widetilde{\Phi} = \widetilde{\Phi}_{-n}$ is independent of n. Moreover, since the dilatation shrinks to zero as $n \to \infty$, then $\widetilde{\Phi}$ is a conformal motion of $\mathbf{I}_\infty$.

Lastly, suppose for a contradiction that $\mathbf{I}_\infty(\mathbf{G})$ supports an invariant line field $\boldsymbol{\mu}$ of some $\mathbf{G} \in \boldsymbol{\mathcal{W}}^u$. Since $\mathbf{I}_\infty(\mathbf{F}) \cap \Lambda_{-n}(\mathbf{F})$ moves holomorphically over a neighborhood of $\mathbf{F}_*$ containing $\mathbf{G}$ for some sufficiently high n, then there is a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ which has zero dilatation on $\mathbf{I}_\infty(\mathbf{F}_*) \cap \Lambda_{-n}(\mathbf{F})$ and conjugates $\mathbf{F}_*|_{\mathbf{I}_\infty(\mathbf{F}_*) \cap \Lambda_{-n}(\mathbf{F}_*)}$ to $\mathbf{G}|_{\mathbf{I}_\infty(\mathbf{G}) \cap \Lambda_{-n}(\mathbf{G})}$. Consider $\boldsymbol{\mu}' = \phi^* \boldsymbol{\mu}$ on $\mathbf{I}_\infty(\mathbf{F}_*) \cap \Lambda_{-n}(\mathbf{F})$ and pull it back via $\mathbf{F}_*$ to obtain a $\mathbf{F}_*$-invariant Beltrami differential $\boldsymbol{\mu}'$ supported on $\mathbf{I}_\infty(\mathbf{F}_*)$. Then, $\boldsymbol{\mu}'$ would be an invariant line field of $\mathbf{F}_*$ supported on $\mathbf{I}_\infty(\mathbf{F}_*)$, which is impossible due to Lemma 5.7.2. $\square$

5.8 Hyperbolic cascades

Definition 5.8.1. We say that a cascade $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$ is *hyperbolic* if $\mathbf{F}$ admits an attracting cycle of periodic points.

If $\mathbf{F}$ is hyperbolic, the critical orbit $\mathbf{F}^P(0)$ automatically converges to an attracting periodic cycle (Proposition 5.5.19) and so $\mathbf{F}$ has a unique attracting periodic cycle.

In this section, we will provide a proof the second half of Theorem K as well as a proof of Theorem J (4). Roughly speaking, we will show that hyperbolic cascades exist and that any hyperbolic component in $\boldsymbol{\mathcal{W}}^u$ is one-dimensional.

5.8.1 Expansion

Before we discuss the properties of hyperbolic cascades, let us state a number of classical properties of the Julia set which we can now deduce from the equation $\mathfrak{J}(\mathbf{F}) = \overline{\mathbf{I}_{<\infty}(\mathbf{F})}$ that we have established in Corollary 5.7.6.

Proposition 5.8.2. *Either* $\mathfrak{J}(\mathbf{F}) = \mathbb{C}$ *or* $\mathfrak{J}(\mathbf{F})$ *has no interior.*

Proof. Suppose $\mathfrak{J}(\mathbf{F})$ contains an open disk B. Corollary 5.7.6 tells us that $\mathbf{I}_{<\infty}(\mathbf{F}) \cap B$ is dense in B. By Lemma 5.5.13, there is some $P \in \mathbf{T}_{>0}$ such that $\mathbf{F}^P(B \backslash \mathbf{I}_{\leq P}(\mathbf{F}))$ is dense in $\mathbb{C}$. Thus, $\mathfrak{J}(\mathbf{F})$ is the whole plane. $\square$

For any tangent vector v at a point z in $\mathbb{C} \backslash \mathfrak{P}(\mathbf{F})$, denote by $\|v\|$ the norm of v with respect to the hyperbolic metric of $\mathbb{C} \backslash \mathfrak{P}(\mathbf{F})$. If $z \in \mathfrak{P}$, we set $\|v\| = \infty$.

Lemma 5.8.3 (Julia expansion). *For every point z in $\mathfrak{J}(\mathbf{F}) \backslash \mathbf{I}_{<\infty}(\mathbf{F})$,*

$$\left\| \left(\mathbf{F}^P\right)'(z) \right\| \to \infty \quad \text{as} \quad P \to \infty.$$

Proof. Let us fix a point $z \in \mathfrak{J}(\mathbf{F}) \backslash \mathbf{I}_{<\infty}(\mathbf{F})$. Without loss of generality, assume that z does not eventually land on $\mathfrak{P}(\mathbf{F})$.

For any $P \in \mathbf{T}_{>0}$, let

$$\mathfrak{P}_P := \mathbf{I}_{\leq P}(\mathbf{F}) \cup \mathbf{F}^{-P}(\mathfrak{P}(\mathbf{F})).$$

The map $\mathbf{F}^P : \mathbb{C} \backslash \mathfrak{P}_P \to \mathbb{C} \backslash \mathfrak{P}$ is a local isometry with respect to their hyperbolic metrics. Since the closure of the union $\bigcup_{P \in \mathbf{T}} \mathfrak{P}_P$ contains the Julia set (thanks to Corollaries 5.5.14 and 5.7.6), the distance between $\mathfrak{P}_P$ and z shrinks to 0 as $P \to \infty$. Consequently, the distance r_P between z and $\mathfrak{P}_P$ with respect to the hyperbolic metric of $\mathbb{C} \backslash \mathfrak{P}$ tends to 0 as $P \to \infty$. The inclusion map $\iota : \mathbb{C} \backslash \mathfrak{P}_P \to \mathbb{C} \backslash \mathfrak{P}$ is contracting by some factor $C(r_P)$ where $C(r) \to 0$ as $r \to 0$. Therefore, as $P \to \infty$, $\left\| (\mathbf{F}^P)'(z) \right\| \geq C(r_P)^{-1} \to \infty$. $\square$

Denote by $\operatorname{dist}_{\hat{\mathbb{C}}}(\cdot,\cdot)$ the spherical distance between two subsets of $\hat{\mathbb{C}}$.

Theorem 5.8.4 (Measure-theoretic attractor)**.** *If $\mathfrak{J}(\mathbf{F})$ has no interior, then for almost every point z in $\mathfrak{J}(\mathbf{F})\backslash\mathbf{I}_{<\infty}(\mathbf{F})$,*

$$\operatorname{dist}_{\hat{\mathbb{C}}}\left(\mathbf{F}^P(z),\mathfrak{P}(\mathbf{F})\cup\{\infty\}\right)\to 0 \qquad \textit{as } P\to\infty.$$

In other words, almost every non-escaping point in the Julia set is attracted to the postcritical set.

Proof. Suppose for a contradiction that there exist a positive number $\varepsilon>0$ and a positive area subset E of $\mathfrak{J}(\mathbf{F})\backslash\mathbf{I}_{<\infty}(\mathbf{F})$ such that for all $z\in E$,

$$\limsup_{P\to\infty}\operatorname{dist}_{\hat{\mathbb{C}}}\left(\mathbf{F}^P(z),\mathfrak{P}(\mathbf{F})\cup\{\infty\}\right)\geq\varepsilon.$$

Let z be a Lebesgue density point of E. There is a sequence of power-triples P_n such that $P_n\to\infty$ and $y_n:=\mathbf{F}^{P_n}(z)$ lies in the compact subset

$$K:=\{z\in\mathbb{C}\,:\,\operatorname{dist}_{\hat{\mathbb{C}}}(z,\mathfrak{P}(\mathbf{F})\cup\{\infty\})\geq\varepsilon\}.$$

For each $n\in\mathbb{N}$, consider the spherical ball B_n of radius $\varepsilon/2$ centered at y_n, and let B_n' be the lift of B_n under $\mathbf{F}^{P_n}$ containing z.

By Lemma 5.8.3, $\left\|(\mathbf{F}^{P_n})'(z)\right\|\to\infty$. Since K is compact and $\mathbf{F}^{P_n}|_{B_n'}$ has bounded distortion, the disks B_n' must shrink to a point. Since z is a density point of E,

$$\lim_{n\to\infty}\frac{\operatorname{area}(B_n'\cap E)}{\operatorname{area}(B_n')}=1.$$

Therefore, we also have

$$\lim_{n\to\infty}\frac{\operatorname{area}(B_n\cap\mathfrak{J}(\mathbf{F})\backslash\mathbf{I}_{<\infty}(\mathbf{F}))}{\operatorname{area}(B_n)}=1.$$

Since K is compact, y_n converges in subsequence to some point $y\in K$. Then, the ball B of radius $\varepsilon/2$ centered at y must have the same area as $B\cap\mathfrak{J}(\mathbf{F})\backslash\mathbf{I}_{<\infty}(\mathbf{F})$. Since $\mathfrak{J}(\mathbf{F})$ is closed, then the ball B has to be contained in $\mathfrak{J}(\mathbf{F})$. This contradicts the assumption that $\mathfrak{J}(\mathbf{F})$ has no interior. □

Corollary 5.8.5. *If $\mathbf{F}\in\boldsymbol{\mathcal{W}}^u$ is hyperbolic, then $\mathfrak{J}(\mathbf{F})\backslash\mathbf{I}(\mathbf{F})$ has zero Lebesgue measure.*

Proof. Suppose $\mathbf{F}$ is hyperbolic. By Proposition 5.5.19, the postcritical set $\mathfrak{P}(\mathbf{F})$ is contained in the Fatou set. The assertion immediately follows from Proposition 5.8.2 and Theorem 5.8.4. □

Hyperbolicity is clearly an open condition. An open subset $\mathbf{\Omega}$ of $\boldsymbol{\mathcal{W}}^u$ is called a *hyperbolic component* if it is a connected component of the set of hyperbolic cascades in $\boldsymbol{\mathcal{W}}^u$.

Corollary 5.8.6. *Consider a hyperbolic component $\mathbf{\Omega}$ of $\boldsymbol{\mathcal{W}}^u$. There is a unique equivariant holomorphic motion of $\mathfrak{J}(\mathbf{F})$ over $\mathbf{F} \in \mathbf{\Omega}$, and such a motion is a conformal motion. For $\mathbf{F} \in \mathbf{\Omega}$, $\mathfrak{J}(\mathbf{F})$ supports no invariant line field of $\mathbf{F}$.*

Proof. For $\mathbf{F} \in \mathbf{\Omega}$, the critical value 0 is not contained in $\mathbf{I}(\mathbf{F})$, and so the assertion follows from Theorems 5.7.5 and 5.7.8, Corollary 5.8.5, and the λ-lemma. □

This completes the proof of Theorem K. To prove Theorem J, we first need to unravel further properties of hyperbolic cascades.

One feature of transcendental dynamics that distinguishes itself from polynomial dynamics is the emergence of wandering domains and Baker domains. If $\mathbf{F}$ is hyperbolic, such domains do not exist.

Proposition 5.8.7. *If $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$ is hyperbolic, the Fatou set of $\mathbf{F}$ is equal to the basin of the unique attracting periodic cycle of $\mathbf{F}$.*

Proof. Let $\mathcal{A}$ denote the basin of attraction of the unique attracting cycle of $\mathbf{F}$, and suppose for a contradiction that $\mathfrak{F}(\mathbf{F})\backslash\mathcal{A}$ is non-empty. Let us pick a connected component Ω of $\mathfrak{F}(\mathbf{F})\backslash\mathcal{A}$.

Let us pick any point x in Ω. By Theorem 5.7.8, $\mathbf{I}_\infty(\mathbf{F})$ is contained in the Julia set and so it is disjoint from Ω. Hence, there exist some $R > 1$ and some increasing sequence of times $P_0 := 0$, P_1, P_2, ... in $\mathbf{T}$ such that $P_n \to \infty$ and that each of $x_n := \mathbf{F}^{P_n}(x)$ is contained in

$$K := \{z \in \mathbb{C} \, : \, |z| \le R \text{ and } z \notin \mathcal{A}\}.$$

Let P denote the period of the attracting cycle of $\mathbf{F}$. Since K is a compact subset of $\mathbb{C}\backslash\mathfrak{P}(\mathbf{F})$, the hyperbolic metric $\rho(z)dz$ of $\mathbb{C}\backslash\mathfrak{P}(\mathbf{F})$ satisfies

$$\rho(z) \asymp 1 \qquad \text{for all } z \in K, \tag{5.8.1}$$

and the hyperbolic distance between any point in K and $\mathbf{F}^{-P}(\mathfrak{P}(\mathbf{F})) \cup \mathbf{I}_{\le P}(\mathbf{F})$ is uniformly bounded from above. As such, since each of x_n is in K, then there is some constant $C > 1$ such that for all $n \ge 1$, $\left\|(\mathbf{F}^P)'(x_n)\right\| \ge C$. Let us pass to a subsequence and assume that $P_{n+1} - P_n \ge P$ for all $n \ge 1$. By chain rule,

$$\left\|\left(\mathbf{F}^{P_n}\right)'(x)\right\| \ge \prod_{k=0}^{n-1} \left\|\left(\mathbf{F}^{P_{k+1}-P_k}\right)'(x_k)\right\| \ge C^n \to \infty \quad \text{as } n \to \infty. \tag{5.8.2}$$

Since Ω is simply connected (Proposition 5.5.16) and does not contain any critical point of $\mathbf{F}^{\geq 0}$, then $\mathbf{F}^{P_n}$ is univalent on Ω for all n. Suppose Ω contains the Euclidean disk $D := \mathbb{D}(x, \varepsilon)$ for some $\varepsilon > 0$. By Koebe quarter, $\mathbf{F}^{P_n}(D)$ contains the Euclidean disk $\mathbb{D}(x_n, r_n)$ where

$$r_n := \frac{\varepsilon}{4} \left| \left(\mathbf{F}^{P_n}\right)' (x) \right|.$$

By (5.8.1) and (5.8.2), we have

$$r_n \asymp \left| \left(\mathbf{F}^{P_n}\right)' (x) \right| = \frac{\rho(x)}{\rho(x_n)} \left\| \left(\mathbf{F}^{P_n}\right)' (x) \right\| \asymp \left\| \left(\mathbf{F}^{P_n}\right)' (x) \right\| \to \infty.$$

We have just established that $\mathbf{F}^{P_n}(\Omega)$ contains the disk $\mathbb{D}(x_n, r_n)$ where $|x_n| \leq R$ and $r_n \to \infty$ as $n \to \infty$. Therefore, $\mathbf{F}^{P_n}(\Omega)$ converges to the whole plane in the Hausdorff metric, which is impossible because the Fatou set $\mathfrak{F}(\mathbf{F})$ is not the whole plane. $\square$

5.8.2 Superattracting cascades

We say that $\mathbf{F} \in \boldsymbol{\mathcal{W}}^u$ is *superattracting* if 0 is a periodic point of $\mathbf{F}^{\geq 0}$. Superattracting cascades are clearly hyperbolic.

Lemma 5.8.8 (Density of hyperbolicity at $\mathbf{F}_*$). *Every neighborhood $\boldsymbol{\mathcal{U}} \subset \boldsymbol{\mathcal{W}}^u_{\text{loc}}$ of the renormalization fixed point $\mathbf{F}_*$ contains a superattracting cascade.*

Proof. Suppose for a contradiction that there is a small neighborhood $\boldsymbol{\mathcal{U}}$ of $\mathbf{F}_*$ in which for all $\mathbf{F} \in \boldsymbol{\mathcal{U}}$, we have $\mathbf{F}^{P+Q}(0) \neq \mathbf{F}^{Q}(0)$ for all $P \in \mathbf{T}_{>0}$, $Q \in \mathbf{T}$. By λ-lemma, this implies that the postcritical set of $\mathbf{F}$ moves holomorphically over $\boldsymbol{\mathcal{U}}$. In the realm of coronas, the corresponding neighborhood $\mathcal{V} \subset \mathcal{W}^u_{\text{loc}}$ of f_* consists of rotational coronas. By Theorem 5.3.9, $\mathcal{V}$ must lie in the stable manifold, which is impossible.

Therefore, every neighborhood $\boldsymbol{\mathcal{U}}$ of $\mathbf{F}_*$ contains some $\mathbf{G}$ such that $\mathbf{G}^{P+Q}(0) = \mathbf{G}^{Q}(0)$ for some $P \in \mathbf{T}_{>0}$ and $Q \in \mathbf{T}$. If $Q = 0$, then $\mathbf{G}$ is superattracting and we are done. Hence, let us assume that $Q > 0$. In this case, $\mathbf{G}^{Q}(0)$ is a periodic point of period P, and by Proposition 5.5.19, it must be repelling in nature.

Consider any sufficiently small one-dimensional disk $\boldsymbol{\mathcal{U}}'$ about $\mathbf{G}$ embedded in $\boldsymbol{\mathcal{U}}$. By implicit function theorem, every $\mathbf{F} \in \boldsymbol{\mathcal{U}}'$ admits a repelling periodic point $x_{\mathbf{F}}$ of period P such that $x_{\mathbf{G}} = \mathbf{G}^{Q}(0)$ and $x_{\mathbf{F}}$ depends holomorphically on $\mathbf{F}$. By Corollaries 5.5.14 and 5.7.6, there exists a sequence of critical points $x_{\mathbf{F},n}$ of some generation P_n depending holomorphically on $\mathbf{F} \in \boldsymbol{\mathcal{U}}'$ such that $P_n \to \infty$ and $x_{\mathbf{F},n} \to x_{\mathbf{F}}$ as $n \to \infty$. By Rouché's theorem, for sufficiently large n, the number of zeros of $\mathbf{F}^{Q+P_n}(x_{\mathbf{F},n}) - x_{\mathbf{F},n}$ as a function of $\mathbf{F} \in \boldsymbol{\mathcal{U}}'$ is equal to that of $\mathbf{F}^{Q+P_n}(x_{\mathbf{F},n}) - x_{\mathbf{F}}$, which is at least one (e.g. $\mathbf{G}$). Therefore, there exist some large $n \in \mathbb{N}$ and some $\mathbf{F} \in \boldsymbol{\mathcal{U}}'$ such that $\mathbf{F}^{Q+P_n}(x_{\mathbf{F},n}) = x_{\mathbf{F},n}$ and so $\mathbf{F}^{Q+P_n}(0) = 0$. $\square$

Lemma 5.8.9. *For $P \in \mathbf{T}$, the set $\{\mathbf{F} \in \boldsymbol{\mathcal{W}}^u \,:\, \mathbf{F}^P(0) = 0\}$ of superattracting cascades of period P is a zero-dimensional analytic variety.*

Proof. The equation "$\mathbf{F}^P(0) = 0$" surely cuts out an analytic variety in $\boldsymbol{\mathcal{W}}^u$. Suppose for a contradiction that it has a component of dimension at least one. Then, there exists an embedded holomorphic curve $\mathbb{D} \to \boldsymbol{\mathcal{W}}^u, t \mapsto \mathbf{F}_t$ such that each $\mathbf{F}_t$ is superattracting of period P. Below, we will run the pullback argument to obtain a contradiction.

Let D_t be the immediate basin of attraction of 0 for the cascade $\mathbf{F}_t$. The only critical point of $\mathbf{F}_t^P$ in D_t is 0 itself, so by Riemann-Hurwitz formula, D_t is simply connected. Let $b_t : (D_t, 0) \to (\mathbb{D}, 0)$ be a Böttcher conjugacy, i.e. a Riemann mapping which conjugates $\mathbf{F}_t^P$ with the power map $z \mapsto z^d$ where $d = d_0 + d_\infty - 1$. Observe that

$$B_t := b_t^{-1} \circ b_0 : (D_0, 0) \to (D_t, 0)$$

conjugates $\mathbf{F}_0^P$ with $\mathbf{F}_t^P$. The Böttcher conjugacy is unique up to multiplication by some roots of unity. We can select them such that b_t depends holomorphically on t and so B_0 is the identity map on D_0.

By Corollary 5.8.6, the Julia set $\mathfrak{J}(\mathbf{F}_t)$ moves conformally and equivariantly in t. More precisely, there exists a holomorphic family of quasiconformal maps $\phi_t : \mathbb{C} \to \mathbb{C}$ that have zero dilatation on $\mathfrak{J}(\mathbf{F}_0)$ and conjugates $\mathbf{F}_0|_{\mathfrak{J}(\mathbf{F}_0)}$ and $\mathbf{F}_t|_{\mathfrak{J}(\mathbf{F}_t)}$.

We shall modify the map ϕ_t on the attracting basin as follows. For $r \in (0,1)$, let $E_t(r) := b_t^{-1}(\mathbb{D}_r)$ be a disk neighborhood of 0 cut out by an equipotential. Let $\varepsilon = \frac{1}{2}$ and $\varepsilon' = \varepsilon^d$. Define the global quasiconformal map

$$\psi_{t,0}(z) := \begin{cases} \phi_t(z) & \text{if } z \in \mathbb{C} \backslash \bigcup_{0 \le T < P} \mathbf{F}_0^T(E_t(\varepsilon)) \\ \mathbf{F}_t^T \circ B_t \circ \left(\mathbf{F}_0^T|_{E_0(\varepsilon')}\right)^{-1} & \text{if } z \in \mathbf{F}_0^T(E_0(\varepsilon')) \text{ for some } T < P \\ \text{quasiconformal interpolation} & \text{if otherwise.} \end{cases}$$

On $\mathfrak{J}(\mathbf{F}_0)$ and a neighborhood of the periodic cycle $\{\mathbf{F}_0^T(0)\}_T$, $\psi_{t,0}$ conjugates $\mathbf{F}_0^P$ and $\mathbf{F}_t^P$. Inductively, we define for all $n \ge 1$ the quasiconformal map $\psi_{t,n} : \mathbb{C} \to \mathbb{C}$ by lifting $\psi_{t,n-1}$ such that

$$\mathbf{F}_t^P \circ \psi_{t,n} = \psi_{t,n-1} \circ \mathbf{F}_0^P.$$

The map $\psi_{t,n}$ has dilatation equal to that of $\psi_{t,0}$ and it agrees with $\psi_{t,n-1}$ on a neighborhood of $\mathfrak{J}(\mathbf{F}_0)$ and on increasingly large part of the Fatou set $\mathfrak{F}(\mathbf{F}_0)$. Moreover $\psi_{t,n}$ is a conformal conjugacy between $\mathbf{F}_0^P$ and $\mathbf{F}_t^P$ on $\mathbf{F}^{-nP}\left(\mathbf{F}^T(E_0(\varepsilon))\right)$ for all $0 \le T < P$.

As $n \to \infty$, $\psi_{t,n}$ stabilizes and converges to a quasiconformal map ψ_t conjugating $\mathbf{F}_0^P$ to $\mathbf{F}_t^P$ everywhere. By Proposition 5.8.7, ψ_t is conformal on the whole Fatou set and has zero dilatation almost everywhere on the Julia set. By Weyl's lemma, ψ_t is a linear conjugacy between $\mathbf{F}_0$ and $\mathbf{F}_t$.

Without loss of generality, we can reparametrize $\mathbf{F}_t$ and assume that $\psi_t(z) = (1+t)z$ where $|t|$ is sufficiently small. Then, within the global parameter space $\boldsymbol{\mathcal{W}}^u$, we have a one-dimensional slice $\mathbf{F}_t = \{\psi_t \circ \mathbf{F}_0 \circ \psi_t^{-1}\}_t$. For all $n \geq 1$, denote the n^{th} anti-renormalization of $\mathbf{F}_t$ by $\mathbf{F}_{t,-n}$. As $n \to \infty$, we have

$$\mathbf{F}_* = \lim_{n\to\infty} \mathbf{F}_{t,-n} = \lim_{n\to\infty} \psi_t \circ \mathbf{F}_{0,-n} \circ \psi_t^{-1} = \psi_t \circ \mathbf{F}_* \circ \psi_t^{-1}.$$

However, the only holomorphic map which commutes with the linear map ψ_t for all t is a linear map, and clearly $\mathbf{F}_*^P$ is not a linear map for every $P \in \mathbf{T}_{>0}$. □

5.8.3 Dimension of $\boldsymbol{\mathcal{W}}^u$

We are now ready to prove Theorem J (4).

Theorem 5.8.10. *The global unstable manifold $\boldsymbol{\mathcal{W}}^u$ is biholomorphic to $\mathbb{C}$.*

Proof. By Lemma 5.8.8, there exists a superattracting cascade in $\boldsymbol{\mathcal{W}}^u$ of some period $P > 0$. The equation "$\mathbf{F}^P(0) = 0$" defines a non-empty analytic hypersurface in $\boldsymbol{\mathcal{W}}^u$. By Lemma 5.8.9, the dimension of $\boldsymbol{\mathcal{W}}^u$ must be equal to one. Since $\mathcal{R}$ is an automorphism of $\boldsymbol{\mathcal{W}}^u$ admitting a unique repelling fixed point $\mathbf{F}_*$, then the claim will imply that $\mathcal{R} : \boldsymbol{\mathcal{W}}^u \to \boldsymbol{\mathcal{W}}^u$ is conformally conjugate to the linear map $\mathbb{C} \to \mathbb{C}, z \mapsto \lambda z$ where λ is the repelling eigenvalue of $\mathcal{R}$. □

Let us conclude with a proof of Corollary L.

Corollary 5.8.11. *Consider a small Banach neighborhood $N(f)$ of a (d_0, d_∞)-critical quasicircle map f of preperiodic type rotation number τ. The space $S(f)$ of maps in $N(f)$ that admit a (d_0, d_∞)-critical Herman quasicircle of rotation number τ forms an analytic submanifold of $N(f)$ of codimension at most one. The Herman quasicircles of maps in $S(f)$ move holomorphically.*

Proof. Let G be the Gauss map. There exists some $k \in \mathbb{N}$ such that $\theta := G^k(\tau)$ is a periodic type irrational. Consider the corona renormalization operator $\mathcal{R} : (\mathcal{U}, f_*) \to (\mathcal{B}, f_*)$ from Corollary 5.3.2 associated to the data (d_0, d_∞, θ).

By Lemma 5.3.3, there is a compact analytic corona renormalization operator $\mathcal{R}_1$ on a neighborhood of f such that $\mathcal{R}_1 f$ is sufficiently close to the fixed point f_* of $\mathcal{R}$, and thus it lies in the stable manifold of f_*. Then, the preimage $S := \mathcal{R}_1^{-1}(\mathcal{W}^s_{\text{loc}})$ is an analytic submanifold of the Banach neighborhood of f consisting of perturbations of f which admit a (d_0, d_∞)-critical Herman quasicircle of rotation number τ. By Theorem J, the codimension of $\mathcal{W}^s_{\text{loc}}$ is one, so there is an analytic function $\phi : \mathcal{U}' \to \mathbb{C}$ on a Banach neighborhood $\mathcal{U}'$ of f_* such that $\mathcal{W}^s_{\text{loc}} = \phi^{-1}(0)$. Therefore, S is the zero set of $\phi \circ \mathcal{R}_1$ and so the codimension of S is at most one.

The Herman quasicircle of a corona in $\mathcal{W}^s_{\text{loc}}$ moves holomorphically over $\mathcal{W}^s_{\text{loc}}$ due to λ-lemma. Since $\mathcal{R}_1$ is analytic, the Herman quasicircles of maps in S also move holomorphically over S. $\square$

Chapter 6

Questions and Conjectures

We conclude this dissertation with a couple of questions and conjectures.

In Chapter 3, we construct Herman curves as a limit of degenerating Herman rings. It is natural to ask the following questions:

Question I. *When is a limit of degenerating Herman rings a Herman curve?*

Question II. *When is a Herman curve a limit of degenerating Herman rings?*

Given a degree $d \geq 2$ rational map f containing an invariant bounded type Herman quasicircle $\mathbf{H}$, one can perform Douady-Ghys surgery [Ghy84; Dou87] to both sides of $\mathbf{H}$ and obtain a pair of rational maps g_+ and g_- having invariant Siegel disks Z_+ and Z_- of complementary rotation numbers. Applying Shishikura's surgery [Shi87] to g_+ and g_-, we obtain a family of degree d rational maps F_t admitting an invariant Herman ring $\mathbb{H}_t$ of modulus $t > 0$ and of the same rotation number and combinatorics as $f|_{\mathbf{H}}$. The dynamics of F_t on each component of $\hat{\mathbb{C}} \backslash \overline{\mathbb{H}_t}$ is quasiconformally conjugate to the dynamics of f on a component of $\hat{\mathbb{C}} \backslash \mathbf{H}$. We believe in the following conjecture.

Conjecture III. *Every Herman quasicircle with bounded type rotation number arises as a limit of degenerating Herman rings. More precisely, given f and F_t above, $[F_t] \to [f]$ as $t \to 0$ in the moduli space* $\mathrm{Rat}_d/\mathrm{PSL}_2(\mathbb{C})$.

Note that the bounded type assumption is essential in the realization and rigidity of maps in $\mathcal{X}_{d_0,d_\infty,\theta}$. Recently, Yang [Yan22] proved the existence of a cubic rational map whose Julia set has positive Lebesgue measure and contains a smooth Herman curve of high type Brjuno rotation number. Such a rational map is also constructed as a limit of degenerating Herman rings, but the problem of realization and rigidity for general irrational rotation number θ and degrees d_0, d_∞ remains open.

To the best of our knowledge, all examples of Herman curves that are known have quasiconformal regularity. It would be interesting to know of any examples of Herman curves that have cusps.

Question IV. *Does there exist a holomorphic map with a Herman curve that is a not a quasicircle? If so, what are the possible rotation numbers?*

Consider the unicritical family $\{F_c\}_{c \in \mathbb{C}}$ from Proposition 4.3.8. Numerical experiments (or simply lack of counterexamples) suggest the following.

Conjecture V (Uniform a priori bounds + full combinatorial rigidity)**.** *For every irrational $\theta \in \mathbb{R}/\mathbb{Z}$, there exists a unique parameter $c_\theta \in \mathbb{C}$ such that F_{c_θ} admits a Herman curve $\mathbf{H}_\theta$ with rotation number θ passing through its free critical point 1. Moreover, $\mathbf{H}_\theta$ is a uniform quasicircle.*

In proving Theorem C, we show that the Julia set of any rational map in $\mathcal{X}$ supports no invariant line field. It is also reasonable to ask the following related question.

Question VI. *Given $f \in \mathcal{X}$, does $J(f)$ have zero Lebesgue measure? Is the Hausdorff dimension of $J(f)$ less than 2?*

We believe that Question VI is a much more difficult problem. Points along the Herman quasicircle of f are deep points of its Julia set (cf. Theorem 4.5.2). This hints at the similarity in complexity to Feigenbaum Julia sets (see [McM96]). One possible direction towards this problem is an adaptation of the methods developed by Avila and Lyubich [AL08] in studying the Lebesgue measure of Feigenbaum Julia sets. In particular, it may be possible to formulate a criterion for zero or positive area in terms of escape probabilities and, similar to [DS20; AL22; DL23], apply either rigorous computer estimates or various renormalization schemes to obtain a conclusive answer.

In Theorem C, we constructed rational maps admitting multicritical Herman curves of arbitrary combinatorics and proved a rigidity property for such maps. It is natural to expect for Theorem D to hold in the multicritical setting too.

Conjecture VII. *$C^{1+\alpha}$ rigidity holds for multicritical quasicircle maps with bounded type rotation number.*

We have all the ingredients available to prove this conjecture except for complex bounds for multi-quasicritical circle maps, which can be applied to show quasiconformal rigidity. One may ask whether or not it is possible to obtain complex bounds directly on the level of holomorphic maps, i.e. multicritical quasicircle maps.

Many of the tools in the proof of Theorem J, in particular the proof of $\dim \mathcal{W}^u_{\text{loc}} \leq 1$, are fairly soft. We conjecture that the philosophy that we apply here should hold in a more general setting.

Conjecture VIII. *Consider a compact analytic renormalization operator with a hyperbolic fixed point such that every map on the unstable manifold $\mathcal{W}^u_{\text{loc}}$ admits a global transcendental extension. Then,*

$$\dim(\mathcal{W}^u_{\text{loc}}) \leq \text{number of critical orbits.}$$

In Corollary L, we deduce that within its natural Banach neighborhood, the conjugacy class of a critical quasicircle map with pre-periodic type rotation number forms an analytic submanifold of codimension at most one. We firmly believe that the codimension should be equal to one, and more generally:

Conjecture IX. *Consider a Banach neighborhood $N(f)$ of a (d_0, d_∞)-critical quasicircle map f with irrational rotation number θ. The space S of maps in $N(f)$ which restrict to a (d_0, d_∞)-critical quasicircle map with rotation number θ forms a codimension one analytic submanifold of $N(f)$. In particular, critical quasicircle maps are structurrally unstable.*

So far, this conjecture is known to be true for

- ▷ periodic type critical quasicircle maps that are close to the associated renormalization fixed point f_* (due to Theorem J), and
- ▷ critical circle maps with arbitrary irrational rotation number (due to standard monotonicity properties of the rotation number).

We suspect that it can be solved via an infinitesimal argument similar to unimodal maps [ALM03], although the lack of real symmetry and nice external structure presents a great challenge.

In [Yam03], the hyperbolicity of renormalization explains the golden mean parameter universality of critical circle maps. This should be the case for critical quasicircle maps too.

Consider a one-dimensional holomorphic family of unicritical holomorphic maps $\{f_\lambda\}_{\lambda\in\Lambda}$, and suppose there is a unique parameter $\lambda_* \in \Lambda$ such that f_{λ_*} has a unicritical Herman quasicircle of periodic type rotation number θ. By Lemma 5.3.3, if Λ is a small disk around λ_*, the family $\{f_\lambda\}_{\lambda\in\Lambda}$ can be corona renormalized to a one-parameter family $\{g_\lambda\}_{\lambda\in\Lambda}$ near the renormalization fixed point f_* described in Theorem J. This family intersects $\mathcal{W}^s_{\text{loc}}$ at a single point g_{λ_*} with some intersection multiplicity r. If $r = 1$, then the intersection is transversal.

Conjecture X (Parameter self-similarity)**.** *Consider $\{f_\lambda\}_{\lambda\in\Lambda}$ discussed above. The union of hyperbolic components within Λ is asymptotically self-similar at λ_* with a universal self-similarity factor depending only on θ, the criticality, and the intersection multiplicity r.*

For example, based on numerical experiments (Figure 6.1), the unicritical family of rational maps $\{F_c\}_{c\in\mathbb{C}^*}$ in Proposition 4.3.8 supports this conjecture with $r = 1$. Our hyperbolicity result provides a step forward towards this conjecture. However, we suspect that attaining a complete solution would require hyperbolicity of the renormalization horseshoe for bounded type rotation numbers, as well as a thorough study of parameter rays and hyperbolic components of the unstable manifold as a parameter space of transcendental σ-proper maps.

We also propose the following conjecture on the global structure of the bifurcation locus of $\{F_c\}_{c\in\mathbb{C}^*}$.

Conjecture XI (Necklace structure)**.** *There exists a set $Q \subset \mathbb{C}^*$ and a continuous surjection $\rho : Q \to \mathbb{R}/\mathbb{Z}$ with the following properties.*

(1) *Q is a quasicircle separating 0 and ∞.*

(2) *For every $c \in Q$, F_c contains an invariant quasicircle H_c on which it is a (d_0, d_∞)-critical quasicircle map with rotation number $\rho(c)$.*

(3) *If θ is irrational, the fiber $\rho^{-1}(\theta)$ is the singleton $\{c_\theta\}$ from Conjecture V.*

(4) *If $\theta = p/q$ is rational,*

  (a) *the fiber $\rho^{-1}(p/q)$ is a non-degenerate closed interval which is a proper arc in a hyperbolic component, and*

  (b) *for every $c \in \rho^{-1}(p/q)$, the free critical orbit converges to a periodic cycle of period q and the corresponding multiplier is real and in $[0, 1]$.*

When $d_0 = d_\infty$, there exists a unique curve Q satisfying the conjecture above, and it is precisely the unit circle $\mathbb{T}$. Indeed, under combinatorial symmetry, when $|c| = 1$, each F_c is a Blaschke product and $F_c : \mathbb{T} \to \mathbb{T}$ forms a real analytic family of critical circle maps. The conjecture easily follows from standard monotonicity properties of the rotation number.

When $d_0 \neq d_\infty$, we have candidates of parameters in $\rho^{-1}(\theta)$ when θ is rational or bounded type irrational. The closure of all such parameters should give the set Q, but during the time this dissertation is written, we do not even know whether Q is connected. This conjecture is related to the conjectural non-existence of irrational *ghost limbs*, as well as the conjectural local connectivity of the bifurcation locus. We believe the answer should be reachable once the hyperbolicity of the full renormalization horseshoe is established.

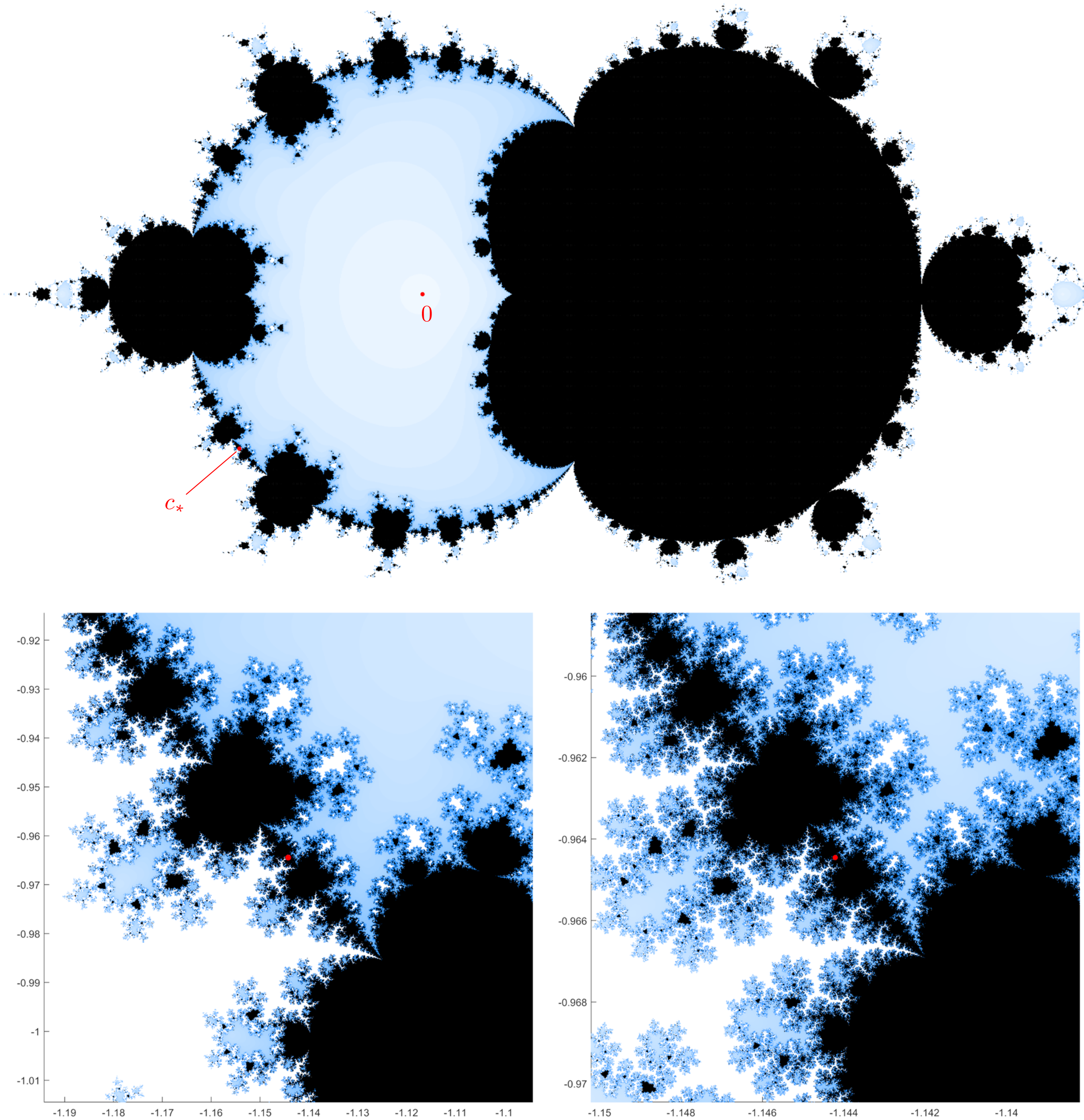

Figure 6.1: The parameter space of the family

$$\left\{ F_c(z) = cz^3 \frac{4-z}{1-4z+6z^2} \right\}_{c \in \mathbb{C}^*}$$

characterized by critical points 0, ∞, and 1 of local degrees 2, 3, and 4 respectively, where both 0 and ∞ are fixed and $F_c(1) = c$. There are two types of escape loci: in white the critical orbit escapes to ∞, and in blue the critical orbit escapes to 0. The non-escaping locus $\mathcal{M}$ is colored black, and the figure above indicates that $\mathcal{M}$ contains a necklace of Mandelbrot copies separating 0 and ∞. There is a unique parameter $c_* \approx -1.144208 - 0.964454i$ such that F_{c_*} has a golden mean Herman quasicircle. (The Julia set of F_{c_*} is shown in Figure 1.1.) The bottom figure shows magnifications by different scales about c_* marked in red.

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