THE COMBINATORICS OF SECTOR RENORMALIZATION

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ABSTRACT. The goal of this note is provide a reference on the combinatorial properties of sector renormalization operation on rigid rotations. We demonstrate the form of modified continued fractions that is most appropriate for sector renormalization and carefully discuss their properties. This leads to a natural dynamical compactification upon taking into account infinite first return times. We also discuss the corresponding natural extension; for example, we describe how a bi-infinite tower of sector renormalizations of irrational rotations can be packaged in a single dynamical plane as a cascade of translations. This note will be applied in the study of sector renormalizations of holomorphic maps with an irrationally indifferent fixed point, in particular neutral quadratic polynomials.

CONTENTS

1. INTRODUCTION

Let

$$\Theta := \left(-\frac{1}{2}, \frac{1}{2}\right) \setminus \mathbb{Q}.$$

For any $\theta \in \Theta$, denote by

$$\mathtt{r}_\theta : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$$

the rigid rotation by angle θ on the closed unit disk $\bar{\mathbb{D}} \subset \mathbb{C}$.

1.1. **Sector renormalization.** The sector renormalization of $\mathtt{r}_\theta$ is constructed as follows. Let S be a sector on $\bar{\mathbb{D}}$ bounded by two arcs γ and $\mathtt{r}_\theta(\gamma)$ where γ is a radial arc from 0 to a point on the unit circle. The power map $\psi(z) = z^{1/|\theta|}$ glues the two sides of the sector S together, sending S onto $\bar{\mathbb{D}}$. The first return map of $\mathtt{r}_\theta$ back to S is a piecewise continuous map consisting of a pair different iterates $(\mathtt{r}_\theta^{\bar{a}}, \mathtt{r}_\theta^{\bar{a}+1})$. Under ψ, this pair projects onto a new rotation $\mathtt{r}_{\mathfrak{g}(\theta)} : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$, called the sector renormalization of $\mathtt{r}_\theta$. The new rotation number $\mathfrak{g}(\theta)$ is an element of Θ that is independent of the choice of γ.

What we have just described above is a combinatorial version to an operation, again called sector renormalization, that can be done to arbitrary holomorphic germs with linear term $e^{2\pi i\theta}z$, pioneered by Douady [Dou87] and Yoccoz [Yoc95]. In this note, we aim to work with just linear maps $\mathbb{r}_\theta$ and describe the combinatorics of sector renormalization in detail. Our goal is to describe:

(I) the dynamical and arithmetic aspects of $\mathfrak{g} : \Theta \to \Theta$;
(II) the appropriate compactification $\overline{\Theta}$ of Θ so that the map $\mathfrak{g}$ and the embedding $\Theta \to [0,1]$, $\theta \mapsto \theta \pmod 1$ extends continuously to $\overline{\Theta}$;
(III) the properties of the natural extension of $\mathfrak{g}$.

1.2. **Modified continued fractions.** Goal (I) leads to a variant of the nearest integer continued fraction algorithm. It first appeared in the work of Hurwitz [Hur89] in 1889. We take a more dynamical point of view and emphasize that the expansion is parameterized by the infinite sequence space

$$\Sigma^{\mathbb{N}} := \left\{ \langle(\varepsilon_n, \bar{a}_n)\rangle_{n\geq 1} \ : \ \varepsilon_n \in \{-1,+1\}, \bar{a}_n \in \mathbb{N}_{\geq 2} \right\}.$$

For any sequence of non-zero integers $x_1, x_2, x_3, \ldots$, denote

$$[x_1, x_2, x_3, \ldots]_- := \cfrac{1}{x_1 - \cfrac{1}{x_2 - \cfrac{1}{x_3 - \cdots}}}. \tag{1.1}$$

When $|x_n| \geq 2$ for cofinitely many $n \geq 1$, the expression above is a well-defined irrational number. One of the theorems in this note is the following.

Theorem A. *The function $\mathfrak{X} : \Sigma^{\mathbb{N}} \to \Theta$ given by*

$$\mathfrak{X}(\langle(\varepsilon_n, \bar{a}_n)\rangle_{n\geq 1}) = [\varepsilon_1 b_1, \varepsilon_2 b_2, \varepsilon_3, b_3, \ldots]_- \quad \textit{where} \quad b_n = \bar{a}_n + \frac{1+\varepsilon_n\varepsilon_{n+1}}{2}$$

is a homeomorphism conjugating the standard shift map $\mathfrak{s} : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ with the map $\mathfrak{g} : \Theta \to \Theta$.

The proof comes down to the study of Diophantine approximations related to the modified continued fraction above. For $\sigma = \langle(\varepsilon_n, \bar{a}_n)\rangle_{n\geq 1} \in \Sigma^{\mathbb{N}}$ and b_n's being as introduced in the theorem, we can write

$$[\varepsilon_1 b_1, \varepsilon_2 b_2, \ldots, \varepsilon_n b_n]_- = \frac{p_{[n]}}{q_{[n]}}$$

where $p_{[n]}$ and $q_{[n]}$ are co-prime integers and $q_{[n]} \geq 1$. The continuants $q_{[n]}$ can be characterized in many other ways including as the first return times of an iterated pre-renormalization (Proposition 2.3). Theorem A essentially follows from studying the arithmetic properties of these convergents.

We are aware that the study of convergents was already done by Hurwitz. Nonetheless, we aim to provide a modern reference in the style of renormalization. In Section 3, we will also make a comparison with:

- the regular continued fraction algorithm, and
- the nearest integer continued fraction algorithm.

The conversion to the regular one will come in handy in applications. The nearest integer variant is similar to our expansion and has been used several times in complex dynamics, e.g. by Yoccoz himself in [Yoc95].

1.3. **Compactification and natural extension.** Let us identify the unit circle $\mathbb{T}$ as the quotient of $[-\frac{1}{2}, \frac{1}{2}]$ by identifying its endpoints. The modified Gauss map $\mathfrak{g}$ naturally extends to a map on $\mathbb{T}$ that is continuous everywhere except at the origin. See Figure 2 for a reference. In fact, wild oscillations are produced on both the left and the right side of the origin. To distinguish these two sides, we instead consider the embedding

$$\eta : \Theta \to [0,1]\,, \qquad \eta(\theta) \equiv \theta \pmod 1.$$

Goal (II) essentially asks for a compactification that respects both $\mathfrak{g}$ and η. Theorem A leads to a natural candidate, which is the compact sequence space

$$\overline{\Theta} = \left\{ \langle(\varepsilon_n, \bar{a}_n)\rangle_{n\geq 1} \,:\, \varepsilon_n \in \{-1,+1\}, \bar{a}_n \in \{2,3,\dots,\infty\} \right\}.$$

Via $\mathfrak{X}$, the modified Gauss map $\mathfrak{g}$ naturally extends to the shift map $\mathfrak{s}$ on $\overline{\Theta}$. In Theorem 4.3, we show that $\overline{\Theta}$ is indeed the smallest compactification of Θ that allows for $\mathfrak{g}$ and η to extend continuously to $\overline{\Theta}$.

For each element $\boldsymbol{\theta}$ of $\overline{\Theta}$, we can assign a totally ordered additive abelian group called the *time group* $\boldsymbol{T}^{\mathrm{gp}}_{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ whose generators are a generalization of the continuants $q_{[n]}$'s discussed above. The positive cone $\boldsymbol{T}_{\boldsymbol{\theta}} = \{p \in \boldsymbol{T}^{\mathrm{gp}}_{\boldsymbol{\theta}} \,:\, p > 0\}$, which we call the *time semigroup*, will be important in applications.

To realize goal (III), we take the inverse limit of $\mathfrak{s} : \overline{\Theta} \to \overline{\Theta}$, thereby giving us a bi-infinite shift space $\mathfrak{s} : \underleftarrow{\overline{\Theta}} \to \underleftarrow{\overline{\Theta}}$. Each element $\underline{\boldsymbol{\theta}}$ of $\underleftarrow{\overline{\Theta}}$ induces a totally ordered group $\mathbf{T}^{\mathrm{gp}}_{\underline{\boldsymbol{\theta}}}$ with a countably infinite number of generators $Q_{[n]}$, $n \in \mathbb{Z}$ mimicking the properties of continuants. Again, the *time semigroup* $\mathbf{T}_{\underline{\boldsymbol{\theta}}} = \{P \in \mathbf{T}^{\mathrm{gp}}_{\underline{\boldsymbol{\theta}}} \,:\, P > 0\}$ is useful in applications.

Throwing out ∞ gives us the shift-invariant subspace $\underleftarrow{\Theta}$ of $\underleftarrow{\overline{\Theta}}$. Every element of $\underleftarrow{\Theta}$ induces a unique bi-infinite tower of sector renormalizations. In Section 5.2, we describe how this tower can be fit into a single dynamical plane as a group of real translations, called *cascades*.

1.4. **Motivation.** This note provides the background on the combinatorics of sector renormalization of quadratic polynomials with an irrationally indifferent neutral fixed point. Following [DL26], sector renormalization of neutral quadratic polynomials $f_\theta(z) = e^{2\pi i\theta}z + z^2$ can be defined in such a way that the resulting maps $\mathcal{R}^n_{\mathrm{sec}} f_\theta$ are pre-compact in the sense that their critical orbit is well-contained in its domain of definition. Every element in the closure of the space of renormalization orbits $\{(\mathcal{R}^{n+m}_{\mathrm{sec}} f_\theta)_{n\geq 0} \,:\, m \geq 0, \theta \in \Theta\}$ has combinatorics that can be uniquely encoded by an element $\boldsymbol{\theta}$ of the compactification $\overline{\Theta}$, and it also comes with an associated full Lavaurs-Epstein renormalization tower that can be parametrized by the time semigroup $\boldsymbol{T}_{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$.

In [DLL26], we study the full attractor of neutral renormalization. Elements of the attractor are bi-infinite towers $\langle f_n \rangle_{n\in\mathbb{Z}}$ of renormalization and their combinatorics are encoded as elements of $\underleftarrow{\overline{\Theta}}$. We show:

Theorem 1.1 ([DLL26])**.** *The full renormalization attractor for neutral quadratic polynomials is combinatorially rigid. In particular, modulo conformal conjugacy, the renormalization attractor is conjugate to the shift map* $\mathfrak{s} : \underleftarrow{\overline{\Theta}} \to \underleftarrow{\overline{\Theta}}$*.*

In the proof, we bring a bi-infinite tower of renormalizations $\langle f_n \rangle_{n\in\mathbb{Z}}$ to a single dynamical plane of a semigroup $\mathbf{F}$ of σ-proper holomorphic maps onto $\mathbb{C}$ called a

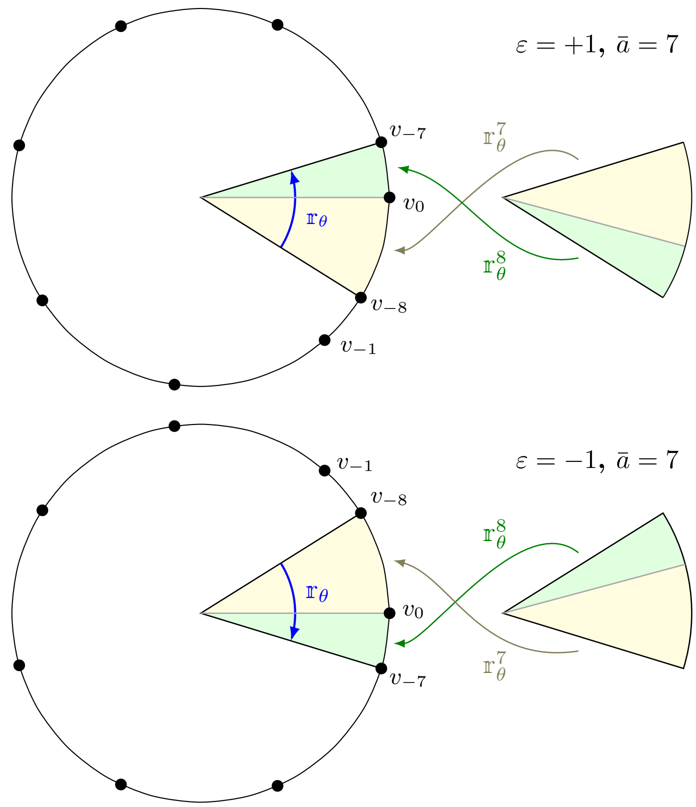

FIGURE 1. Two examples of the first return map on the shaded sector S_θ.

neutral cascade which can parametrized by the time semigroup $\mathbf{T}_{\underline{\boldsymbol{\theta}}}$ of $\underline{\boldsymbol{\theta}}$. We then prove uniform complex bounds for neutral cascades and apply them to construct an affine conjugacy between any two combinatorially equivalent neutral cascades.

2. Renormalization of irrational rotations

2.1. **Sector renormalization.** For any irrational $\theta \in \Theta$, denote

$$\varepsilon(\theta) := \operatorname{sgn}(\theta) \in \{-1, +1\} \qquad \text{and} \qquad \bar{a}(\theta) := \left\lfloor \frac{1}{|\theta|} \right\rfloor \in \mathbb{N}_{\geq 2}.$$

The sign $\varepsilon = \varepsilon(\theta)$ tells us whether or not $\mathbb{r}_\theta$ rotates clockwise or anticlockwise. The number $\bar{a} = \bar{a}(\theta)$ is the unique positive integer such that

$$\bar{a}|\theta| < 1 < (\bar{a}+1)|\theta|.$$

Let us first fix $\theta \in \Theta$. For $k \in \mathbb{Z}$, denote

$$v_k = v_k(\theta) := \mathbb{r}_\theta^k(1).$$

For any two distinct integers $k, l \in \mathbb{Z}$, we denote by $\triangle_\theta(k, l)$ the closed radial sector bounded by the radial line segment from 0 to v_k, the radial line segment from 0 to v_l, and the shortest circular arc in $\partial\mathbb{D}$ joining v_k and v_l. We will in particular consider the sector

$$S_\theta := \triangle_\theta(-\bar{a}-1, -\bar{a}).$$

It contains the point $v_0 = 1$. The first return map of $\mathbb{r}_\theta$ back to S_θ is the piecewise linear map

$$\mathrm{FRM}_\theta(z) = \begin{cases} \mathbb{r}_\theta^{\bar{a}+1}(z) & \text{if } z \in \triangle_\theta(-\bar{a}-1, -2\bar{a}-1), \\ \mathbb{r}_\theta^{\bar{a}}(z) & \text{if } z \in \triangle_\theta(-2\bar{a}-1, -\bar{a}). \end{cases}$$

See Figure 1 for an illustration. The power map

$$\psi_\theta : S_\theta \to \bar{\mathbb{D}}, \quad \psi_\theta(z) = z^{1/|\theta|}$$

sends the interior of S_θ conformally onto the unit disk minus the radial slit $\gamma_\theta = \{\arg z = |\theta|^{-1} \arg v_{-\bar{a}}\}$, and it glues the two radial edges of S_θ together onto the slit. Under ψ_θ, FRM_θ projects to a new rigid rotation $\mathbb{r}_{\mathfrak{g}(\theta)} : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ for some irrational $\mathfrak{g}(\theta) \in \Theta$. The slit γ_θ is actually equal to $\{\arg z = -2\pi\, \mathfrak{g}(\theta)\}$. The map

$$\mathfrak{g} : \Theta \to \Theta$$

should be thought as a modified Gauss map. We call $\mathbb{r}_{\mathfrak{g}(\theta)}$ the *sector renormalization* of $\mathbb{r}_\theta$.

Lemma 2.1. *For $\theta \in \Theta$, $\mathfrak{g}(\theta)$ is the unique irrational in Θ such that*

$$\mathfrak{g}(\theta) \equiv -\frac{1}{\theta} \quad (\textit{mod } 1).$$

More precisely, we have

$$\mathfrak{g}(\theta) = -\frac{1}{\theta} + \varepsilon(\theta)\, b(\theta) \qquad \textit{where } b(\theta) = \bar{a}(\theta) + \frac{1 + \varepsilon(\theta)\, \varepsilon(\mathfrak{g}(\theta))}{2}.$$

The map $\mathfrak{g} : \Theta \to \Theta$ is an infinite-to-one surjective continuous function.

The graph of $\mathfrak{g}$ is illustrated in Figure 2.

Proof. Denote $\varepsilon_1 = \varepsilon(\theta)$, $\varepsilon_2 = \varepsilon(\mathfrak{g}(\theta))$, and $\bar{a} = \bar{a}(\theta)$. The first return map back to S_θ can be written as the pair of maps $\mathbb{r}_{\bar{a}\theta-\varepsilon_1}$ and $\mathbb{r}_{(\bar{a}+1)\theta-\varepsilon_1}$ where both $\bar{a}\theta - \varepsilon_1$ and $(\bar{a}+1)\theta - \varepsilon_1$ are contained in $(-\frac{1}{2}, \frac{1}{2})$. Therefore, for $z \in \bar{\mathbb{D}}$, there is some $k = k(z) \in \{\bar{a}, \bar{a}+1\}$ such that

$$\mathbb{r}_{\mathfrak{g}(\theta)}(z) = \psi_\theta \circ \mathbb{r}_{k\theta-\varepsilon_1} \circ \psi_\theta^{-1}(z) = e^{2\pi i (k\theta - \varepsilon_1)/|\theta|} z = e^{2\pi i (\varepsilon_1 k - \frac{1}{\theta})} z,$$

so then $\mathfrak{g}(\theta) \equiv -\frac{1}{\theta}$ mod 1.

Let us write $\mathfrak{g}(\theta) = -\frac{1}{\theta} + x$ where $x = x(\theta)$ is the unique integer closest to $\frac{1}{\theta}$. Since $\bar{a} < \frac{\varepsilon_1}{\theta} < \bar{a}+1$, then either $x = \varepsilon_1 \bar{a}$ or $x = \varepsilon_1(\bar{a}+1)$. If $x = \varepsilon_1 \bar{a}$, then

$$\varepsilon_1 \mathfrak{g}(\theta) = -\frac{\varepsilon_1}{\theta} + \varepsilon_1 x = -\frac{\varepsilon_1}{\theta} + \bar{a} < 0.$$

which implies that $\varepsilon_1 \varepsilon_2 = -1$. Else, if $x = \varepsilon_1(\bar{a}+1)$, then

$$\varepsilon_1 \mathfrak{g}(\theta) = -\frac{\varepsilon_1}{\theta} + \varepsilon_1 x = -\frac{\varepsilon_1}{\theta} + \bar{a} + 1 > 0$$

and so $\varepsilon_1 \varepsilon_2 = +1$. In both cases, we do have $x = \varepsilon_1 b(\theta)$.

The function $\mathfrak{g}$ is continuous since $-\frac{1}{\theta}$ and $x(\theta)$ are continuous in $\theta \in \Theta$. It is clear that $\mathfrak{g}$ is infinite-to-one and surjective. ☐

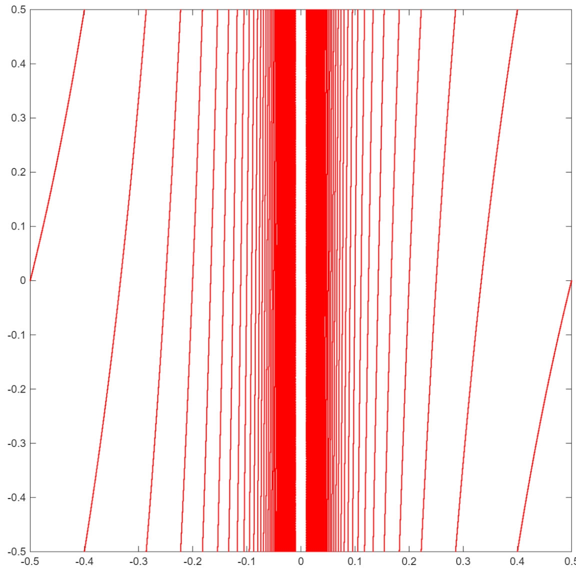

FIGURE 2. The graph of $\mathfrak{g}$

2.2. **More notation.** Denote $\mathbb{N} = \{1, 2, 3 \ldots\}$ and $\Sigma := \{-1, +1\} \times \mathbb{N}_{\geq 2}$. Consider the infinite sequence space

$$\Sigma^{\mathbb{N}} = \{\langle(\varepsilon_n, \bar{a}_n)\rangle_{n\geq 1} \ : \ (\varepsilon_n, \bar{a}_n) \in \Sigma \text{ for all } n \geq 1\},$$

equipped with the infinite product topology. For every $\sigma = \langle(\varepsilon_n, \bar{a}_n)\rangle_{n\geq 1} \in \Sigma^{\mathbb{N}}$, we denote

$$b_n = b_n(\sigma) := \bar{a}_n + \frac{1 + \varepsilon_n \varepsilon_{n+1}}{2} \qquad \text{for } n \geq 1.$$

Associated to σ are the pair of sequences $\{p_{[n]} = p_{[n]}(\sigma)\}_{n\geq 0}$ and $\{q_{[n]} = q_{[n]}(\sigma)\}_{n\geq 0}$ where

$$p_{[0]} = 0, \quad q_{[0]} = 1, \qquad p_{[1]} = \varepsilon_1, \quad q_{[1]} = b_1,$$

and recursively for $n \geq 2$,

$$p_{[n]} = b_n p_{[n-1]} - \varepsilon_{n-1}\varepsilon_n p_{[n-2]} \qquad \text{and} \qquad q_{[n]} = b_n q_{[n-1]} - \varepsilon_{n-1}\varepsilon_n q_{[n-2]}. \tag{2.1}$$

We have a well-defined function

$$\mathfrak{Y} : \Theta \to \Sigma^{\mathbb{N}}, \quad \theta \mapsto \left\langle (\varepsilon(\mathfrak{g}^{n-1}(\theta), \bar{a}(\mathfrak{g}^{n-1}(\theta)) \right\rangle_{n\geq 1}.$$

Throughout the rest of this section, we will fix an irrational number $\theta \in \Theta$. We denote $\theta_n := \mathfrak{g}^n(\theta)$ for $n \geq 0$ and let $\langle(\varepsilon_n, \bar{a}_n)\rangle_{n\geq 1} = \mathfrak{Y}(\theta)$, $b_n = b_n(\mathfrak{Y}(\theta))$ for all $n \geq 1$, and $p_{[n]} = p_{[n]}(\mathfrak{Y}(\theta))$ and $q_{[n]} = q_{[n]}(\mathfrak{Y}(\theta))$ for all $n \geq 0$. By Lemma 2.1, we have

$$\theta_n = -\frac{1}{\theta_{n-1}} + \varepsilon_n b_n \qquad \text{for all } n \geq 1. \tag{2.2}$$

2.3. **Iterated sector renormalizations.** For any $m \geq 0$ and $n \in \mathbb{Z}$, denote

$$S^1_m = S_{\theta_m}, \quad \psi_{\theta_m} = \psi_m, \quad \text{and} \quad I^n_m := \{z \in \bar{\mathbb{D}} \ : \ \arg z = \arg \mathfrak{r}^n_{\theta_m}(1)\}.$$

Observe that the radial slit γ_{θ_m} discussed previously is equal to I^{-1}_{m+1} and is always disjoint from the sector $S_{m+1,1}$. Hence, the sector S^1_{m+1} can be lifted under the gluing map ψ_m to a sector S_m^2 contained in S_m^1. By repeating this process, we obtain the nest of sectors

$$S_m^1 \supset S_m^2 \supset S_m^3 \supset S_m^4 \supset \dots$$

where

$$S_m^n := \psi_{\theta_m}^{-1}(S_{m+1}^{n-1}) \quad \text{for all } m \geq 0 \text{ and } n \geq 2.$$

For every $n \geq 1$, the angular width of S_0^n is equal to

$$l_{[n-1]} := |\theta_0\theta_1 \dots \theta_{n-1}|.$$

For convenience, we will also set $l_{[-1]} = 1$.

Proposition 2.2. *For all $n \geq 1$,*

- (1_n) $l_{[n-2]} = b_n l_{[n-1]} - \varepsilon_n \varepsilon_{n+1} l_{[n]}$,
- (2_n) $1 = q_{[n]} l_{[n-1]} - \varepsilon_n \varepsilon_{n+1} q_{[n-1]} l_{[n]}$,
- (3_n) $\varepsilon_{n+1} l_{[n]} = q_{[n]}\theta - p_{[n]}$.

Proof. We can rewrite (2.2) as

$$\varepsilon_n = b_n \theta_{n-1} - \varepsilon_{n+1} |\theta_{n-1}\theta_n|. \tag{2.3}$$

Multiplying this equation by $\varepsilon_n l_{[n-2]}$ gives us (1_n). Observe that when $n = 1$, the equation above gives us (2_1) and (3_1) as well. For $n \geq 2$, (2_n) follows directly from (2_{n-1}) and the recurrence relation (2.1). Observe also that (3_0) holds when the left hand side is to be taken to be θ. For $n \geq 2$, (3_n) follows from (1_n), (3_{n-1}), (3_{n-2}), and the recurrence relations (2.1) as follows:

$$\begin{aligned} \varepsilon_{n+1} l_{[n]} &= b_n \varepsilon_n l_{[n-1]} - \varepsilon_n l_{[n-2]} \\ &= b_n (q_{[n-1]}\theta - p_{[n-1]}) - \varepsilon_{n-1}\varepsilon_n (q_{[n-2]}\theta - p_{[n-2]}) \\ &= q_{[n]}\theta - p_{[n]}. \end{aligned}$$

□

These $q_{[n]}$'s are known as the first return times of θ due to the following proposition.

Proposition 2.3. *For all $n \geq 1$,*

1. *$\mathtt{r}_\theta^{q_{[n]}}$ is the rigid rotation by angle $\varepsilon_{n+1} l_{[n]} = |\theta_0 \dots \theta_{n-1}|\theta_n$;*
2. *the sector S_0^n is bounded by radial segments $I_0^{-q_{[n]}+\varepsilon_n\varepsilon_{n+1}q_{[n-1]}}$ and $I_0^{-q_{[n]}}$;*
3. *$q_{[n]}$ is the unique smallest positive integer that satisfies*

$$|[1, \mathtt{r}_\theta^{q_{[n]}}(1)]| \leq \frac{1}{2} |[1, \mathtt{r}_\theta^{q_{[n-1]}}(1)]|$$

where $|\cdot|$ refers to the normalized angular measure on the unit circle.

We will take a dynamical approach to prove the proposition. We will make use of a nest of dynamical triangulations $\{\Delta_{m,n}\}_{n\geq 0}$ of the closed unit disk for all $m \geq 0$. Each $\Delta_{m,n}$ is a collection of closed sectors in $\bar{\mathbb{D}}$ with vertex at 0 such that their union is $\bar{\mathbb{D}}$ and that they have pairwise disjoint interiors. By a nest, we mean that for every n, every sector in $\Delta_{m,n}$ is a union of finitely many sectors in $\Delta_{m,n+1}$. Here is how these triangulations are constructed.

The level 0 triangulation $\Delta_{m,0}$ is defined by the union of the two closed sectors formed by cutting $\bar{\mathbb{D}}$ along I_m^0 and I_m^{-1}. We will denote by A_m^0 the sector in $\Delta_{m,0}$ with angular width $1 - |\theta_m|$ and by A_m^1 the sector in $\Delta_{m,0}$ with angular width $|\theta_m|$.

For all $m \geq 0$ and $n \geq 1$, the sector S_m^n is the union of two triangles $A_{m,n}^0$ and $A_{m,n}^1$ which are the lifts of A_m^0 and A_m^1 under the map $\psi_{m+n-1} \circ \ldots \circ \psi_m$. The angular widths of $A_{m,n}^0$ and $A_{m,n}^1$ are $l_{[n-1]}(\theta_m) - l_{[n]}(\theta_m)$ and $l_{[n]}(\theta_m)$ respectively.

For $m \geq 0$ and $j \in \{0,1\}$, the first time $k \geq 1$ for which $\mathbb{r}_{\theta_m}^{-k}(A_{m,1}^j)$ intersects the interior of S_m^1 is $k = b_m + j\varepsilon_m$. In fact, $\mathbb{r}_{\theta_m}^{-b_m - j\varepsilon_m}(A_{m,1}^j)$ is contained in S_m^1. In general, for $n \geq 1$, the level n triangulation $\Delta_{m,n}$ consists of sectors with angular widths $l_{[n]}(\theta_m)$ and $l_{[n-1]}(\theta_m) - l_{[n]}(\theta_m)$. It is defined inductively in increasing n as follows. Suppose $\Delta_{m,n-1}$ has been defined for all $m \geq 0$. Pull back $\Delta_{m+1,n-1}$ under ψ_{θ_m} to obtain a triangulation $\Delta'_{m,n}$ of S_m^1. For every sector $B \in \Delta'_{m,n}$, denote $j_B \in \{0,1\}$ such that B is contained in $A_{m,1}^{j_B}$. Then,

$$\Delta_{m,n} := \left\{ \mathbb{r}_{\theta_m}^{-k} B \right\}_{B \in \in \Delta'_{m,n},\ k = 0,1,\ldots,b_m + j_B\varepsilon_m - 1}.$$

Proof. Item (1) follows directly from Proposition 2.2 (3_n). Let us prove (2).

Observe that the triangulation $\Delta_{0,n}$ is obtained out of cutting $\overline{\mathbb{D}}$ along the radial arcs $I_0^0, I_0^{-1}, \ldots I_0^{-N+1}$ where N is the first positive integer such that I_0^{-N} intersects the interior of the sector S_0^n. Since S_0^n contains I_0^0, it is clear that $A_{0,n}^0 \cap A_{0,n}^1 = I_0^0$. For $j \in \{0,1\}$, we just need to find the value of $k_j \in \{1, 2, \ldots, N-1\}$ such that $A_{0,n}^j$ is bounded by I_0^0 and $I_0^{-k_j}$. Since $A_{0,n}^1$ has angular width equal to $l_{[n]} = l_{[n]}(\theta)$, then by (1), k_1 has to be equal to $q_{[n]}$. By (1) again, $\mathbb{r}_\theta^{\varepsilon_n \varepsilon_{n+1} q_{[n-1]}}$ is the rotation by angle $\varepsilon_{n+1} l_{[n-1]}$ which has the same sign as $\mathbb{r}_\theta^{q_{[n]}}$. As such, $\mathbb{r}_\theta^{q_{[n]} - \varepsilon_n \varepsilon_{n+1} q_{[n-1]}}$ rotates by $l_{[n-1]} - l_{[n]}$ in the direction opposite to $\mathbb{r}_\theta^{q_{[n]}}$. This implies that $I_0^{k_1}$ has to be the image of I_0^0 under $\mathbb{r}_\theta^{-q_{[n]} + \varepsilon_n \varepsilon_{n+1} q_{[n-1]}}$ and so $k_1 = -q_{[n]} + \varepsilon_n \varepsilon_{n+1} q_{[n-1]}$.

The inequality in (3) follows from (1) and the fact that $|\theta_n| < \frac{1}{2}$. We will use (2) and the triangulation $\Delta_{0,n}$ to prove the rest of (3). From the previous paragraph, we have $N \geq q_{[n]}$. Let $A_{0,n}^2 \in \Delta_{0,n}$ be the sector next to $A_{0,n}^1$ that is not equal to A_n^0, and let $A_{0,n}^3 \in \Delta_{0,n}$ be the sector next to $A_{0,n}^2$ that is not equal to $A_{0,n}^1$. The argument is split into two cases.

- Suppose $\varepsilon_n \varepsilon_{n+1} = +1$. Then, $A_{0,n}^2$ is bounded by $I_0^{-q_{[n]}}$ and $I_0^{-q_{[n-1]}}$ and its angular length is $l_{[n-1]} - l_{[n]}$. For any integer k with $1 \leq k < q_{[n]}$, the radial segment I_0^{-k} is disjoint from the interior of $S_0^n \cup A_{0,n}^2$ which implies that the angular distance between I_0^{-k} and I_0^0 is bounded from below by $l_{[n-1]} - l_{[n]} > \frac{1}{2} l_{[n]}$.
- Suppose $\varepsilon_n \varepsilon_{n+1} = -1$. Then, $A_{0,n}^2$ is bounded by $I_0^{-q_{[n]}}$ and $I_0^{-2q_{[n]}}$ and $A_{0,n}^3$ is bounded by $I_0^{-q_{[n]} + q_{[n-1]}}$. In this case, for any integer k with $1 \leq k < q_{[n]}$, the radial segment I_0^{-k} is disjoint from the interior of $S_0^n \cup A_{0,n}^2 \cup A_{0,n}^3$ which implies that the angular distance between I_0^{-k} and I_0^0 is again at least $l_{[n-1]} - l_{[n]} > \frac{1}{2} l_{[n]}$.

In both cases, we have proven the optimality of $q_{[n]}$ described in item (3). □

From the proposition above, for any $n \geq 1$, the first return time of $\mathbb{r}_\theta$ at any point in the interior of S_0^n is either $q_{[n]}$ or $q_{[n]} - \varepsilon_n \varepsilon_{n+1} q_{[n-1]}$. In other words, $\mathbb{r}_\theta^{q_{[n]}}|_{S_0^n}$ is the n^{th} pre-renormalization of $\mathbb{r}_\theta$:

$$\mathbb{r}_{\theta_n} = \mathbb{r}_\theta^{q_{[n]}}|_{S_0^n} / \mathbb{r}_\theta^{q_{[n-1]}}.$$

The gluing map $\psi_{n-1} \circ \ldots \circ \psi_0$ projects the map $\mathbb{r}_\theta^{q_{[n]}}|_{S_0^n}$, modulo $\mathbb{r}_\theta^{q_{[n-1]}}$, to the rotation $\mathbb{r}_{\theta_n}$.

3. Continued fractions

In this section, we explore the arithmetic properties of irrationals relative to the modified Gauss map $\mathfrak{g} : \Theta \to \Theta$ defined in the previous section. Most of the results in this section are elementary and similar to those in the study of regular continued fractions. (E.g. compare with [Khi97].) Experts in Diophantine approximation should be familiar with most of the content in this section. We provide details for the sake of completeness.

3.1. **The convergents.** Given a finite sequence of non-zero integers $x_1, x_2, \ldots, x_n$, we will denote

$$[x_1, x_2, \ldots, x_n]_- := 1/(x_1 - 1/(x_2 - \ldots - 1/(x_{n-1} - 1/x_n) \ldots)).$$

Example 3.1. The golden mean irrational

$$\theta_{\text{gm}} = \frac{3 - \sqrt{5}}{2} = 0.381966\ldots$$

is a fixed point of the map $x \mapsto \frac{1}{3-x}$ and so it can be written as

$$\theta_{\text{gm}} = [3, 3, 3, 3, 3, \ldots]_-.$$

Lemma 3.2. *Consider $\sigma = \langle(\varepsilon_n, \bar{a}_n)\rangle_{n\geq 1} \in \Sigma^{\mathbb{N}}$ and the associated sequences of integers $b_n = b_n(\sigma)$, $p_{[n]} = p_{[n]}(\sigma)$ and $q_{[n]} = q_{[n]}(\sigma)$ described in §2.2. For all $n \geq 1$, we have*

(1_n) $p_{[n]}q_{[n-1]} - p_{[n-1]}q_{[n]} = \varepsilon_n$,

(2_n) $\dfrac{p_{[n]}}{q_{[n]}} = \displaystyle\sum_{k=1}^{n} \frac{\varepsilon_k}{q_{[k-1]}q_{[k]}}$,

(3_n) $[\varepsilon_1 b_1, \varepsilon_2 b_2, \ldots, \varepsilon_n b_n]_- = \dfrac{p_{[n]}}{q_{[n]}}$,

(4_n) $-\dfrac{1}{2} \leq \dfrac{p_{[n]}}{q_{[n]}} \leq \dfrac{1}{2}$,

(5_n) $q_{[n-1]} < (1 - \theta_{\text{gm}})q_{[n]}$.

Proof. For $n = 1$, (1_1) holds since $p_{[0]} = 0$, $q_{[0]} = 1$, and $p_{[1]} = \varepsilon_1$. In general, the recurrence relations (2.1) and (1_{n-1}) imply (1_n) as shown below.

$$\begin{aligned}
&p_{[n]}q_{[n-1]} - p_{[n-1]}q_{[n]} \\
=&(b_n p_{[n-1]} - \varepsilon_n \varepsilon_{n-1} p_{[n-2]})q_{[n-1]} - p_{[n-1]}(b_n q_{[n-1]} - \varepsilon_n \varepsilon_{n-1} q_{[n-2]}) \\
=&\varepsilon_n \varepsilon_{n-1}(p_{[n-1]}q_{[n-2]} - p_{[n-2]}q_{[n-1]}) \\
=&\varepsilon_n \varepsilon_{n-1} \varepsilon_{n-1} = \varepsilon_n.
\end{aligned}$$

Let us prove (2_n). The base case (2_n) is clear. For $n \geq 2$, (2_n) follows from (4_{n-1}) and (1_n) since

$$\frac{p_{[n]}}{q_{[n]}} - \frac{p_{[n-1]}}{q_{[n-1]}} = \frac{p_{[n]}q_{[n-1]} - p_{[n]}q_{[n]}}{q_{[n-1]}q_{[n]}} = \frac{\varepsilon_n}{q_{[n-1]}q_{[n]}}.$$

By elementary calculation, we have (3_1) and (3_2):

$$[\varepsilon_1 b_1]_- = \frac{\varepsilon_1}{b_1} = \frac{p_{[1]}}{q_{[1]}}, \qquad [\varepsilon_1 b_1, \varepsilon_2 b_2]_- = \frac{\varepsilon_1 b_2}{b_2 b_1 - \varepsilon_2 \varepsilon_1} = \frac{p_{[2]}}{q_{[2]}}.$$

Suppose we have (3_n) for all $n \in \{1, \dots, k\}$ for some integer $k \geq 2$. Let $x = b_k - \frac{\varepsilon_k \varepsilon_{k+1}}{b_{k+1}}$. Then, by the recurrence relations,

$$\begin{aligned}[\varepsilon_1 b_1, \dots, \varepsilon_{k-1} b_{k-1}, \varepsilon_k b_k, \varepsilon_{k+1} b_{k+1}]_- &= [\varepsilon_1 b_1, \dots, \varepsilon_{k-1} b_{k-1}, \varepsilon_k x]_- \\ &= \frac{x p_{[k-1]} - \varepsilon_{k-1}\varepsilon_k p_{[k-2]}}{x q_{[k-1]} - \varepsilon_{k-1}\varepsilon_k q_{[k-2]}} \\ &= \frac{p_{[k]} - \varepsilon_k \varepsilon_{k+1} p_{[k-1]}/b_{k+1}}{q_{[k]} - \varepsilon_k \varepsilon_{k+1} q_{[k-1]}/b_{k+1}} = \frac{p_{[k+1]}}{q_{[k+1]}}.\end{aligned}$$

Hence, (3_n) holds for all n.

Next, we will use (3_n) to prove (4_n). The base case (4_1) is clear. Suppose (4_{n-1}) holds for some $n \geq 2$. In particular, $y := \varepsilon_2[\varepsilon_2 b_2, \dots, \varepsilon_n b_n]_-$ is contained in the interval $[0, \frac{1}{2}]$. Then, we can write

$$\left|\frac{p_{[n]}}{q_{[n]}}\right| = \left|\frac{1}{\varepsilon_1 b_1 - \varepsilon_2 y}\right| = \frac{1}{b_1 - \varepsilon_1\varepsilon_2 y} = \begin{cases} \frac{1}{\bar{a}_1 + 1 - y} & \text{if } \varepsilon_1\varepsilon_2 = +1, \\ \frac{1}{\bar{a}_1 + y} & \text{if } \varepsilon_1\varepsilon_2 = -1. \end{cases}$$

Since $\bar{a}_1 \geq 2$ and $0 \leq y \leq 2$, the equation above gives us (4_n).

To prove (5_n), we will show that for all $n \geq 1$, we have

$$(\spadesuit_n) \qquad \theta_{\text{gm}} - 1 < \varepsilon_n \varepsilon_{n+1} \frac{q_{[n-1]}}{q_{[n]}} < \theta_{\text{gm}}.$$

The base case $(\spadesuit_1)$ follows from $\frac{1}{3} < \theta_{\text{gm}} < \frac{1}{2} < 1 - \theta_{\text{gm}}$ and the following observation:

$$\varepsilon_1\varepsilon_2 \frac{q_{[0]}}{q_{[1]}} = \frac{\varepsilon_1\varepsilon_2}{b_1} = \begin{cases} \frac{1}{\bar{a}_1+1} \in (0, \frac{1}{3}] & \text{if } \varepsilon_1\varepsilon_2 = +1, \\ -\frac{1}{\bar{a}_1} \in [-\frac{1}{2}, 0) & \text{if } \varepsilon_1\varepsilon_2 = -1. \end{cases}$$

In general, for $n \geq 2$, the recurrence relation 2.1 gives us:

$$\varepsilon_n\varepsilon_{n+1}\frac{q_{[n-1]}}{q_{[n]}} = \frac{\varepsilon_n\varepsilon_{n+1}}{b_n - \varepsilon_n\varepsilon_{n-1}\frac{q_{[n-2]}}{q_{n-1}}} = \begin{cases} \dfrac{1}{\bar{a}_n + 1 - \varepsilon_n\varepsilon_{n-1}\frac{q_{[n-2]}}{q_{n-1}}} & \text{if } \varepsilon_n\varepsilon_{n+1} = +1, \\ -\dfrac{1}{\bar{a}_n - \varepsilon_n\varepsilon_{n-1}\frac{q_{[n-2]}}{q_{n-1}}} & \text{if } \varepsilon_1\varepsilon_{n+1} = -1. \end{cases}$$

Then, $(\spadesuit_n)$ follows from $(\spadesuit_{n-1})$:

$$\varepsilon_n\varepsilon_{n+1}\frac{q_{[n-1]}}{q_{[n]}} \in \begin{cases} \left(0, \frac{1}{3-\theta_{\text{gm}}}\right) = (0, \theta_{\text{gm}}) & \text{if } \varepsilon_n\varepsilon_{n+1} = +1, \\ \left(-\frac{1}{2-\theta_{\text{gm}}}, 0\right) = (\theta_{\text{gm}} - 1, 0) & \text{if } \varepsilon_1\varepsilon_{n+1} = -1. \end{cases}$$

□

Proposition 3.3. *Consider $\sigma = \langle(\varepsilon_n, \bar{a}_n)\rangle_{n\geq 1} \in \Sigma^{\mathbb{N}}$ and the associated sequences $b_n = b_n(\sigma)$, $p_{[n]} = p_{[n]}(\sigma)$, and $q_{[n]} = q_{[n]}(\sigma)$. The limit $\mathfrak{X}(\sigma) = \lim_{n\to\infty} \frac{p_{[n]}}{q_{[n]}}$ converges to an irrational in Θ equal to*

$$\mathfrak{X}(\sigma) = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{q_{[k-1]} q_{[k]}}$$

Moreover, for every $n \geq 1$,

$$\frac{\theta_{\text{gm}}}{q_{[n]} q_{[n+1]}} \leq \left|\mathfrak{X}(\sigma) - \frac{p_{[n]}}{q_{[n]}}\right| \leq \frac{2 - \theta_{\text{gm}}}{q_{[n]} q_{[n+1]}}.$$

Proof. For $k \geq 1$, denote $x_k = \dfrac{\varepsilon_k}{q_{[k-1]}q_{[k]}}$. By Lemma 3.2 (5), we have

$$\left|\frac{x_k}{x_{k+1}}\right| = \frac{q_{[k-1]}}{q_{[k+1]}} < (1-\theta_{\mathrm{gm}})^2 = \theta_{\mathrm{gm}} < 1. \tag{3.1}$$

Therefore, the series $\sum_{k\geq 1} x_k$ converges to an irrational number θ. By items (2), (3), and (4) of Lemma 3.2, θ is in Θ, is equal to $\mathfrak{X}(\sigma)$, and has an infinite modified continued fraction expansion $\theta = [\varepsilon_1 b_1, \varepsilon_2 b_2, \varepsilon_3 b_3, \ldots]_-$. Moreover, for $n \geq 1$, the inequality (3.1) gives us

$$\left|\theta - \frac{p_{[n]}}{q_{[n]}}\right| = \left|\sum_{k=n+1}^{\infty} x_n\right| \leq \frac{|x_{n+1}|}{1-\theta_{\mathrm{gm}}} = \frac{1}{(1-\theta_{\mathrm{gm}})q_{[n]}q_{[n+1]}} = \frac{2-\theta_{\mathrm{gm}}}{q_{[n]}q_{[n+1]}}$$

and

$$\begin{aligned}\left|\theta - \frac{p_{[n]}}{q_{[n]}}\right| \geq |x_{n+1}| - \left|\sum_{k=n+2}^{\infty} x_n\right| &\geq \frac{1}{q_{[n]}q_{[n+1]}} - \frac{2-\theta_{\mathrm{gm}}}{q_{[n+1]}q_{[n+2]}} \\ &\geq \frac{1-(2-\theta_{\mathrm{gm}})(1-\theta_{\mathrm{gm}})^2}{q_{[n]}q_{[n+1]}} = \frac{\theta_{\mathrm{gm}}}{q_{[n]}q_{[n+1]}}.\end{aligned}$$

Note that we have used Lemma 3.2 (5) on the last line above. □

The proposition above gives us a well-defined function

$$\mathfrak{X} : \Sigma^{\mathbb{N}} \to \Theta.$$

Recall the functions $\mathfrak{g} : \Theta \to \Theta$ and $\mathfrak{Y} : \Theta \to \Sigma^{\mathbb{N}}$ defined in the previous section.

Theorem 3.4. *The function $\mathfrak{X} : \Sigma^{\mathbb{N}} \to \Theta$ is a homeomorphism with inverse $\mathfrak{Y} : \Theta \to \Sigma^{\mathbb{N}}$. Moreover, $\mathfrak{X}$ is a conjugacy between the standard shift map $\mathfrak{s} : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ and the map $\mathfrak{g} : \Theta \to \Theta$.*

This theorem settles Theorem A.

Proof. Let $\sigma = \langle(\varepsilon_n, \bar{a}_n)\rangle_{n\geq 1} \in \Sigma^{\mathbb{N}}$ and let $b_n = b_n(\sigma)$ for $n \geq 1$. We have

$$\mathfrak{X}(\mathfrak{s}(\sigma)) = [\varepsilon_2 b_2, \varepsilon_3 b_3, \ldots]_- = -\frac{1}{\mathfrak{X}(\sigma)} + \varepsilon_1 b_1 = \mathfrak{g}(\mathfrak{X}(\sigma)).$$

Let $\theta = \mathfrak{X}(\sigma)$. Then, $\varepsilon(\theta) = \vartheta_1$ and $\bar{a}(\theta) = \bar{a}_1$. Since $\mathfrak{g}^{n-1}(\theta) = \mathfrak{X}(\mathfrak{s}^{n-1}(\theta))$, we also have that $\varepsilon(\mathfrak{g}^{n-1}(\theta)) = \vartheta_n$ and $\bar{a}(\mathfrak{g}^{n-1}(\theta)) = \bar{a}_n$. Therefore, $\mathfrak{Y} \circ \mathfrak{X}$ is equal to the identity map on $\Sigma^{\mathbb{N}}$.

We have established that $\mathfrak{Y}$ is surjective. Now, we will show the injectivity of $\mathfrak{Y}$. Fix $\sigma = \langle(\varepsilon_n, \bar{a}_n)\rangle_{n\geq 1}$. For $k \geq 1$, denote

$$E_k(\sigma) = \left\{\theta \in \Theta \;:\; \varepsilon(\mathfrak{g}^{n-1}(\theta)) = \varepsilon_n \text{ and } \bar{a}(\mathfrak{g}^{n-1}(\theta)) = \bar{a}_n \text{ for } 1 \leq n \leq k\right\}.$$

The preimage $\mathfrak{Y}^{-1}(\sigma)$ is non-empty and it is contained in $\cap_{n\geq 1} E_k(\sigma)$. The set $E_1(\sigma)$ is the open subset of Θ consisting of irrationals between $\frac{1}{\varepsilon_1 \bar{a}_1}$ and $\frac{1}{\varepsilon_1(\bar{a}_1+1)}$; since $\bar{a} \geq 2$, this open set has Euclidean length bounded above by 6^{-1}. In general, $\mathfrak{g}^{k-1}$ sends $E_k(\sigma)$ homeomorphically onto $E_1(\mathfrak{s}^{k-1}(\sigma))$. Observe that $\mathfrak{g}$ is uniformly expanding: $\mathfrak{g}' \geq 4$. This implies that $E_k(\sigma)$ is equal to $\Theta \cap J$ for some open interval J of Euclidean length at most $6^{-1} \cdot 4^{1-k}$. In particular, $\mathfrak{Y}^{-1}(\sigma)$ is a singleton and thus $\mathfrak{Y}$ is injective.

Pick $\sigma \in \Sigma^{\mathbb{N}}$ and let $\theta = \mathfrak{X}(\sigma)$. The sets $E_k(\sigma)$, $k \geq 1$ described above form a basis of open neighborhoods of θ. The images $\mathfrak{Y}(E_k(\sigma))$, $k \geq 1$ are precisely the cylinder sets about σ which form a basis of open neighborhoods of σ with respect to the topology of $\Sigma^{\mathbb{N}}$. Therefore, both $\mathfrak{X}$ and $\mathfrak{Y}$ are continuous. □

Example 3.5. For the golden mean irrationals in Θ are $\theta_{\mathrm{gm}} = \frac{3-\sqrt{5}}{2}$ and $-\theta_{\mathrm{gm}} = \frac{\sqrt{5}-3}{2}$, we have

$$\mathfrak{Y}(\theta_{\mathrm{gm}}) = \langle (+,2),(+,2),(+,2),(+,2),\ldots \rangle,$$
$$\mathfrak{Y}(-\theta_{\mathrm{gm}}) = \langle (-,2),(-,2),(-,2),(-,2),\ldots \rangle.$$

The corresponding modified continued fraction expansions are

$$\theta_{\mathrm{gm}} = [3,3,3,\ldots]_- = \cfrac{1}{3-\cfrac{1}{3-\cfrac{1}{3-\ldots}}}, \qquad -\theta_{\mathrm{gm}} = [-3,-3,-3,\ldots]_- = \cfrac{1}{-3-\cfrac{1}{-3-\cfrac{1}{-3-\ldots}}}.$$

Both θ_{gm} and $-\theta_{\mathrm{gm}}$ have the same first return times, namely

$$q_{[0]} = 1, \quad q_{[1]} = 3, \quad q_{[2]} = 8, \quad q_{[3]} = 21, \quad q_{[4]} = 55, \quad q_{[5]} = 144, \quad \ldots.$$

Each $q_{[n]}$ is the $(2n+2)^{\mathrm{th}}$ Fibonacci number.

3.2. **The nearest integer continued fraction algorithm.** Consider the nearest integer continued fraction map

$$f_{1/2} : \Theta \to \Theta, \qquad f_{1/2}(\theta) = \frac{1}{|\theta|} - \left\lfloor \frac{1}{|\theta|} + \frac{1}{2} \right\rfloor.$$

For $n \geq 0$, we define the sequences $\{\tilde{a}_n\}_{n\geq 1}$ in $\mathbb{N}_{\geq 2}$, $\{\tilde{\theta}_n\}_{n\geq 0}$ in $(0,\frac{1}{2})\backslash\mathbb{Q}$, and $\{\tilde{\varepsilon}_n\}_{n\geq 1}$ in $\{-1,+1\}$ inductively as follows:

- $\tilde{\varepsilon}_{n+1}$ is the sign of $\tilde{\theta}_n$;
- $\tilde{a}_{n+1} = \left\lfloor |\tilde{\theta}_n|^{-1} + \frac{1}{2} \right\rfloor$ is the integer closest to $|\tilde{\theta}_n|^{-1}$;
- $\tilde{\theta}_{n+1} = f_{1/2}(\tilde{\theta}_n) = |\tilde{\theta}_n|^{-1} - \tilde{a}_{n+1}$.

Then, we can write

$$(\ddagger) \qquad \theta = \cfrac{\tilde{\varepsilon}_1}{\tilde{a}_1 + \cfrac{\tilde{\varepsilon}_2}{\tilde{a}_2 + \cfrac{\tilde{\varepsilon}_3}{\tilde{a}_3 + \ldots}}}.$$

The map $f_{1/2}$ is an even function and it is related to $\mathfrak{g}$ by

$$f_{1/2}(\theta) = \begin{cases} \mathfrak{g}(\theta) & \text{if } \theta < 0, \\ -\mathfrak{g}(\theta) & \text{if } \theta > 0. \end{cases}$$

It produces the same convergents $\tilde{p}_{[n]}/\tilde{q}_{[n]}$ as those $p_{[n]}/q_{[n]}$ of $\mathfrak{g}$ up to a sign change. The terms $\tilde{a}_n$'s coincide with the b_n's described in Section 2.2. Our modified Gauss map $\mathfrak{g}$ has two primary advantages over $f_{1/2}$.

- The map $\mathfrak{g}$ locally preserves orientation everywhere, whereas $f_{1/2}$ reverses orientation the positive irrationals.
- While $f_{1/2}$ still acts as a shift on $(\tilde{\varepsilon}_n, \tilde{a}_n)$'s, these $\tilde{a}_n$'s satisfy the awkward rule that $\tilde{a}_n \geq 3$ whenever $\tilde{\varepsilon}_n\tilde{\varepsilon}_{n+1} = -1$.

Nearest integer continued fraction was initially introduced by Bernhard Minnigerode [Min73] in 1873. The notation $f_{1/2}$ was adapted from Nakada [Nak81] who introduced the one-parameter family $f_\alpha : [\alpha - 1, \alpha) \to [\alpha - 1, \alpha)$, $\alpha \in [\frac{1}{2}, 1]$ of continued fractions map, where f_1 is the classical Gauss map.

In the field of complex dynamics, the expansion (‡) first appeared in the seminal work of Yoccoz [Yoc95] in which he introduced sector renormalization. The adaptation of (‡) was also evident in the work of Inou-Shishikura [IS06] on near-parabolic renormalization as well as its numerous applications, for example [CC15, AC17, Che25]. In those works, the authors are primarily working with the sequence of positive irrationals

$$\alpha_n = |\tilde{\theta}_n|, \qquad n \geq 0.$$

3.3. **The regular continued fraction algorithm.** Every irrational $\theta \in \Theta$ admits a unique continued fraction expansion

$$\theta = \varepsilon\,[a_1, a_2, a_3, \ldots]_+ := \cfrac{\varepsilon}{a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \ldots}}}$$

where $\varepsilon \in \{-1, +1\}$, $a_1 \in \mathbb{N}_{\geq 2}$, and $a_n \in \mathbb{N}$ for all $n \geq 2$. For $n \geq 1$, the n^{th} rational approximation of θ is

$$\frac{p_n}{q_n} = \varepsilon\,[a_1, a_2, \ldots, a_n]_+$$

where p_n and q_n are co-prime integers and $q_n > 0$. These p_n's and q_n's start with

$$p_n = 0,\ q_0 = 1, \qquad p_1 = \varepsilon,\ q_1 = a_1,$$

and the rest follow the recurrence relation

$$p_n = a_n p_{n-1} + p_{n-2}, \qquad q_n = a_n q_{n-1} + q_{n-2} \quad \text{ for } n \geq 2.$$

Note that the continuants q_n are independent of the sign ε. The numbers $q_n\theta - p_n$ alternate in sign, that is

$$(q_n\theta - p_n)(q_{n+1}\theta - p_{n+1}) < 0 \qquad \text{for all } n \geq 0.$$

There are two main advantages of the regular continued fraction expansion, namely the simpler recursive formula for the rational approximations and the alternating nature of points of closest return. The main disadvantage is that the expansion does not get transformed as nicely under sector renormalization. Indeed,

$$\mathfrak{g}\left(\varepsilon[a_1, a_2, \ldots]_+\right) = \begin{cases} -\varepsilon\,[a_2, a_3, \ldots]_+ & \text{if } a_2 \geq 2, \\ \varepsilon\,[a_3 + 1, a_4, \ldots]_+ & \text{if } a_2 = 1. \end{cases}$$

There is a singularization procedure to convert the regular continued fraction expansion $\theta = \varepsilon\,[a_1, a_2, \ldots]_+$ to the modified expansion with terms governed by $\langle(\varepsilon_n, \bar{a}_n)\rangle_{n \geq 1} \in \Sigma^{\mathbb{N}}$. First, set $\varepsilon_1 = \varepsilon$ and $\bar{a}_1 = a_1$. To get $(\varepsilon_2, \bar{a}_2)$, there are two cases.

- If $a_2 \geq 2$, then $\varepsilon_2 = -\varepsilon_1$, and $\bar{a}_2 = a_2$.
- Else if $a_2 = 1$, then $\varepsilon_2 = \varepsilon_1$, and $\bar{a}_2 = a_3 + 1$.

In general, it is convenient to use the increasing sequence of integers

$$0 = t_0 < t_1 < t_2 < t_3 < \ldots$$

defined as follows. For $n \geq 0$,

- if $a_{t_n+2} \geq 2$, then $t_{n+1} = t_n + 1$, $\varepsilon_{n+2} = -\varepsilon_{n+1}$ and $\bar{a}_{n+2} = a_{t_n+2}$;
- if $a_{t_n+2} = 1$, then $t_{n+1} = t_n + 2$, $\varepsilon_{n+2} = \varepsilon_{n+1}$ and $\bar{a}_{n+2} = a_{t_n+3} + 1$.

With this convention, we have that

$$q_{[n]} = q_{t_n} \qquad \text{for all } n \geq 0.$$

Geometrically, here is what happens. For $n \geq 1$, consider the lengths

$$l_n := |q_n\theta - p_n|.$$

The sequence $\{t_n\}_{n\geq 0}$ is the maximal increasing sequence of non-negative integers starting with $t_0 = 0$ such that

$$l_{t_{n+1}} < \frac{l_{t_n}}{2}.$$

4. The compactification

In this section, we will discuss a natural compactification of Θ that respects the behavior of $\mathfrak{g}$.

4.1. **The smallest compactification.**

Let us identify the unit circle $\mathbb{T}$ with the closed interval $[-\frac{1}{2}, \frac{1}{2}]$ with the endpoints identified. The modified Gauss map $\mathfrak{g} : \Theta \to \Theta$ extends to a self-map of $\mathbb{T}$ that is continuous (in fact, C^1 smooth) everywhere except at the origin. See Figure 2 for reference. Notice that the origin is singular from both the left and right. For this reason, it makes more sense to distinguish between the left side of 0 and the right side of 0. This motivates us to consider the embedding

$$\eta : \Theta \to [0,1], \qquad \eta(\theta) = \begin{cases} \theta & \text{if } \theta > 0, \\ \theta + 1 & \text{if } \theta < 0, \end{cases}$$

which identifies the ends $-\frac{1}{2}$ and $\frac{1}{2}$ and splits the left and right sides of the origin to two separate ends 0 and 1. The composition $\eta \circ \mathfrak{X}$ can be written symbolically as

$$\eta \circ \mathfrak{X}(\langle (\varepsilon_n, \bar{a}_n) \rangle_{n\geq 1}) = \frac{1-\varepsilon_1}{2} + [\varepsilon_1 b_1, \varepsilon_2 b_2, \ldots]_-.$$

Definition 4.1. A *compactification* of the triplet $(\Theta, \mathfrak{g}, \eta)$ is a tuple $(\tilde{\Theta}, \tilde{\iota}, \tilde{\mathfrak{g}}, \tilde{\eta})$ where

- $\tilde{\Theta}$ is a compact space,
- $\tilde{\iota} : \Theta \to \tilde{\Theta}$ is an embedding with dense image,
- $\tilde{\mathfrak{g}} : \tilde{\Theta} \to \tilde{\Theta}$ is a surjective continuous map such that $\tilde{\mathfrak{g}} \circ \tilde{\iota} = \tilde{\iota} \circ \mathfrak{g}$, and
- $\tilde{\eta} : \tilde{\Theta} \to [0,1]$ is a continuous map such that $\tilde{\eta} \circ \tilde{\iota} = \eta$.

Theorem A naturally leads to the following example.

Example 4.2. Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, where the addition of the point $\infty = \lim_{n\to\infty} n$ makes it a compactification of the discrete space $\mathbb{N}$. Let

$$\overline{\Sigma} := \{-1, +1\} \times \overline{\mathbb{N}}_{\geq 2} \qquad \text{and} \qquad \overline{\Theta} := \overline{\Sigma}^{\mathbb{N}}.$$

The infinite sequence space $\overline{\Theta}$ is compact, perfect, totally disconnected, and metrizable (hence a Cantor set). The inclusion map $\iota : \Sigma^{\mathbb{N}} \to \overline{\Theta}$ induces a dense embedding

$$\bar{\iota} := \iota \circ \mathfrak{X} : \Theta \to \overline{\Theta}.$$

We also have a continuous map $\overline{\eta} : \overline{\Theta} \to [0,1]$ given by

$$\overline{\eta}\big(\langle (\varepsilon_n, \bar{a}_n) \rangle_{n \geq 1} \big) = \frac{1-\varepsilon_1}{2} + [\varepsilon_1 b_1, \varepsilon_2 b_2, \ldots]_-,$$

where we make the convention that if there exists a first time $N \geq 0$ such that $\bar{a}_{N+1} = \infty$, then

$$[\varepsilon_1 b_1, \varepsilon_2 b_2, \ldots]_- = \begin{cases} 0 & \text{if } N = 0, \\ [\varepsilon_1 b_1, \varepsilon_2 b_2, \ldots, \varepsilon_N b_N]_- & \text{if } N \geq 1. \end{cases}$$

We let $\overline{\mathfrak{g}}$ be equal to the standard shift map $\mathfrak{s}$ on $\overline{\Theta}$. It is clear that $(\overline{\Theta}, \overline{\iota}, \overline{\mathfrak{g}}, \overline{\eta})$ is a compactification of $(\Theta, \mathfrak{g}, \eta)$.

Theorem 4.3 (Universal property of $\overline{\Theta}$)**.** *There exists a unique smallest compactification of $(\Theta, \mathfrak{g}, \eta)$ and, up to homeomorphism, it is equal to the triple $(\overline{\Theta}, \overline{\iota}, \overline{\mathfrak{g}}, \overline{\eta})$. More precisely, it satisfies the universal property that for any compactification $(\tilde{\Theta}, \tilde{\iota}, \tilde{\mathfrak{g}}, \tilde{\eta})$, there exists a continuous surjective map $\phi : \tilde{\Theta} \to \overline{\Theta}$ such that*

$$(\dagger) \qquad \phi \circ \tilde{\iota} = \overline{\iota}, \qquad \phi \circ \tilde{\mathfrak{g}} = \overline{\mathfrak{g}} \circ \phi, \qquad \tilde{\eta} = \overline{\eta} \circ \phi.$$

Proof. Consider the compact sequence space $[0,1]^{\mathbb{N}}$, equipped with the infinite product topology. Let us embed Θ into $[0,1]^{\mathbb{N}}$ via the map

$$\Phi : \Theta \to [0,1]^{\mathbb{N}}, \qquad H(\theta) = \langle \eta \circ \mathfrak{g}^n(\theta) \rangle_{n \geq 0}.$$

It induces a conjugacy between $\mathfrak{g} : \Theta \to \Theta$ and the shift map on the image $\Phi(\Theta)$. Let us set

- $\hat{\Theta}$ to be the closure of $\Phi(\Theta)$ as a subset of $[0,1]^{\mathbb{N}}$,
- $\hat{\iota} : \Theta \to \hat{\Theta}$ to be equal to $\hat{\iota}(\theta) = \Phi(\theta)$,
- $\hat{\mathfrak{g}} : \hat{\Theta} \to \hat{\Theta}$ to be the shift map, and
- $\hat{\eta} : \hat{\Theta} \to [0,1]$ to be the projection to the first coordinate: $\hat{\eta}(\langle x_n \rangle_{n \geq 0}) = x_0$.

By design, it is clear that $(\hat{\Theta}, \hat{\iota}, \hat{\mathfrak{g}}, \hat{\eta})$ is a compactification of $(\Theta, \mathfrak{g}, \eta)$.

Next, let us show that $(\hat{\Theta}, \hat{\iota}, \hat{\mathfrak{g}}, \hat{\eta})$ is the smallest compactification. Pick an arbitrary compactification $(\tilde{\Theta}, \tilde{\iota}, \tilde{\mathfrak{g}}, \tilde{\eta})$. Consider the map

$$\tilde{\phi} : \tilde{\Theta} \to [0,1]^{\mathbb{N}}, \qquad \tilde{\phi}(z) = \langle \tilde{\eta} \circ \tilde{\mathfrak{g}}^n(z) \rangle_{n \geq 0}.$$

Observe that $\tilde{\phi} \circ \tilde{\iota} = \Phi$ and so the image $\tilde{\phi}(\tilde{\iota}(\Theta))$ is dense in $\hat{\Theta}$. Since $\tilde{\iota}(\Theta)$ is dense in $\tilde{\Theta}$ and $\tilde{\phi}$ is continuous, then the image of $\tilde{\phi}$ has to be equal to $\hat{\Theta}$. Therefore, $\tilde{\phi}$ induces a continuous surjective map $\tilde{\phi} : \tilde{\Theta} \to \hat{\Theta}$. It is straightforward to check that the three equations in $(\dagger)$.

Lastly, let us show that $(\hat{\Theta}, \hat{\iota}, \hat{\mathfrak{g}}, \hat{\eta})$ is equivalent to the compactification $(\overline{\Theta}, \overline{\iota}, \overline{\mathfrak{g}}, \overline{\eta})$. From the second paragraph, the projection map $\overline{\phi} : \overline{\Theta} \to [0,1]^{\mathbb{N}}$ associated to $\overline{\Theta}$ can be written as follows. For every element $\boldsymbol{\theta} = \langle (\varepsilon_n, \bar{a}_n) \rangle_{n \geq 1}$ of $\overline{\Theta}$, we have $\overline{\phi}(\boldsymbol{\theta}) = \langle x_n \rangle_{n \geq 0}$ where for all $n \geq 1$,

$$x_{n-1} = \frac{1-\varepsilon_n}{2} + [\varepsilon_n b_n, \varepsilon_n b_n, \ldots]_- \quad \text{where} \quad b_n = b_n(\boldsymbol{\theta}) = a_n + \frac{1+\varepsilon_n \varepsilon_{n+1}}{2}$$

We need to show that $\overline{\phi}$ is a homeomorphism. Since $\overline{\Theta}$ is Hausdorff and $\overline{\phi}$ is surjective, it remains to show that $\overline{\phi}$ is injective.

Suppose for a contradiction that $\overline{\phi}$ is not injective. There exist two distinct elements $\boldsymbol{\theta} = \langle (\varepsilon_n, \bar{a}_n) \rangle_{n \geq 1}$ and $\boldsymbol{\theta}' = \langle (\varepsilon_n', \bar{a}_n') \rangle_{n \geq 1}$ of $\overline{\Theta}$ such that $\langle x_n \rangle_{n \geq 0} = \overline{\phi}(\boldsymbol{\theta})$

is equal to $\langle x'_n \rangle_{n\geq 0} = \overline{\phi}(\boldsymbol{\theta}')$. After shifting, we will assume without loss of generality that $(\varepsilon_1, \bar{a}_1) \neq (\varepsilon_2, \bar{a}_2)$. There are two cases.

- Suppose $\varepsilon_1 \neq \varepsilon'_1$. Without loss of generality, suppose $\varepsilon_1 = +$ and $\varepsilon_2 = -$. Since $x_0 = x'_0$, then both x_0 and x'_0 must be equal to $\frac{1}{2}$. Therefore, $\bar{a}_1 = \bar{a}'_1 = 2$, $(\varepsilon_2, \bar{a}_2) = (-, \infty)$, and $(\varepsilon'_2, \bar{a}'_2) = (+, \infty)$. But this would imply that $x_1 = 1$ and $x'_1 = 0$, which are not equal.
- Suppose $\varepsilon_1 = \varepsilon'_1$ and $\bar{a}_1 \neq \bar{a}'_1$. Without loss of generality, suppose $\varepsilon_1 = +$ and $\bar{a}_1 < \bar{a}'_1$. Since $x_0 = x'_0$, then both x_0 and x'_0 must be equal to $\frac{1}{k}$ for some integer $k \geq 3$. Therefore, $\bar{a}_1 = k-1$, $\bar{a}'_1 = k$, $(\varepsilon_2, \bar{a}_2) = (+, \infty)$, and $(\varepsilon_1, \bar{a}_1) = (-, \infty)$. But this would again imply that $x_1 = 0$ and $x'_1 = 1$, which are not equal. □

We will state one more property of $\overline{\Theta}$. Let $\iota_{\mathbb{T}} : \Theta \to \mathbb{T}$ be the inclusion map.

Proposition 4.4. *The embedding $\iota_{\mathbb{T}} \circ \mathfrak{X} : \Sigma^{\mathbb{N}} \to \mathbb{T}$ uniquely extends to a surjective continuous map*

$$\overline{\mathfrak{X}} : \overline{\Theta} \to \mathbb{T}.$$

For every rational number θ in $\mathbb{T}$, $\mathfrak{X}^{-1}(\theta)$ is homeomorphic to $\overline{\Theta}$ itself.

Hence, with respect to $\overline{\mathfrak{X}}$, the space $\overline{\Theta}$ can be viewed as a way of blowing up the circle $\mathbb{T}$ at its rational points.

Proof. For $\sigma = \langle (\varepsilon_n, \bar{a}_n) \rangle_{n\geq 1} \in \overline{\Theta} \backslash \Sigma^{\mathbb{N}}$, if $k \geq 1$ is the first integer such that $\bar{a}_k = \infty$, then we set $\overline{\mathfrak{X}}(\sigma) = 0$ if $k = 1$ and $\overline{\mathfrak{X}}(\sigma) = [\varepsilon_1 b_1, \ldots, \varepsilon_{k-1} b_{k-1}]_-$ if $k \geq 2$. The continuity of $\overline{\mathfrak{X}} : \overline{\Theta} \to \mathbb{T}$ follows from the proof of Proposition 3.4.

For the second claim, observe that

$$\overline{\mathfrak{X}}^{-1}(0) = \{\bar{a}_1 = \infty\}, \quad \overline{\mathfrak{X}}^{-1}\left(\pm\frac{1}{2}\right) = \{\varepsilon_1\varepsilon_2 = -,\, \bar{a}_1 = 2,\, \bar{a}_2 = \infty\},$$

and for $m \geq 3$ and $\varepsilon \in \{-1, +1\}$, we have

$$\overline{\mathfrak{X}}^{-1}\left(\frac{\varepsilon}{m}\right) = \left\{\varepsilon_1 = \varepsilon, \bar{a}_1 + \frac{1+\varepsilon\varepsilon_2}{2} = m, \bar{a}_2 = \infty\right\}.$$

All of the above are homeomorphic to $\overline{\Theta}$. For other rational numbers θ, let $k \geq 2$ be the first moment such that $\mathfrak{g}(\theta) = 0$, then $\varepsilon_j(\theta)$ and $\bar{a}_j(\theta)$ are well-defined and finite for $j \leq k-1$, $\bar{a}_{k+1} = \infty$, the sequences $\{\varepsilon_j\}_{j\geq k+1}$ and $\{\bar{a}_j\}_{j\geq k+2}$ are completely arbitrary, and the pair $(\varepsilon_k, \bar{a}_k)$ is completely determined by ε_{k+1} in the way that is similar to the $\pm\frac{1}{m}$ case. □

4.2. **The time group.**

Definition 4.5. For every $\boldsymbol{\theta} = \langle (\varepsilon_n, \bar{a}_n) \rangle_{n\geq 1} \in \overline{\Theta}$, we define an additive abelian group $\boldsymbol{T}^{\mathrm{gp}}_{\boldsymbol{\theta}}$, called the *time group of $\boldsymbol{\theta}$*, such that it is generated by the sequence $\{\mathfrak{q}_{[n]}\}_{n\geq 0}$ and subject to the relations

$$\mathfrak{q}_{[n]} = b_n \mathfrak{q}_{[n-1]} - \varepsilon_{n-1}\varepsilon_n \mathfrak{q}_{[n-2]}, \qquad \text{where } b_n = \bar{a}_n + \frac{1+\varepsilon_n\varepsilon_{n+1}}{2},$$

for all $n \geq 1$ with $\bar{a}_n < \infty$, and where $\mathfrak{q}_{[-1]} = 0$ is the identity element. For $n \geq 0$, the element $\mathfrak{q}_{[n]} = \mathfrak{q}_{[n]}(\boldsymbol{\theta}) \in \boldsymbol{T}^{\mathrm{gp}}_{\boldsymbol{\theta}}$ will be referred to as the n^{th} *return time* of $\boldsymbol{\theta}$.

Proposition 4.6 (Chronological order)**.** *For every* $\boldsymbol{\theta} = \langle(\varepsilon_n, \bar{a}_n)\rangle_{n\geq 1} \in \overline{\Theta}$*, the time group* $\boldsymbol{T}^{\mathrm{gp}}_{\boldsymbol{\theta}}$ *admits a unique translation-invariant total order* $<$*, called the chronological order of* $\boldsymbol{T}_{\boldsymbol{\theta}}$*, with the property that for all* $n \geq 0$*,*

(1) $0 < \mathfrak{q}_{[n]} < \mathfrak{q}_{[n+1]}$,
(2) *if* $\bar{a}_{n+1} = \infty$*, then* $k\mathfrak{q}_{[n]} < \mathfrak{q}_{[n+1]}$ *for all* $k \geq 1$.

The positive cone of $<$ is the commutative semigroup

$$\boldsymbol{T}_{\boldsymbol{\theta}} := \{P \in \boldsymbol{T}^{\mathrm{gp}}_{\boldsymbol{\theta}} \ : \ P > 0\},$$

which we refer to as the *time semigroup* of $\boldsymbol{\theta}$.

Proof. The order $<$ is uniquely characterized by the semigroup $\boldsymbol{T}_{\boldsymbol{\theta}}$ since $a < b$ if and only if $b - a \in \boldsymbol{T}_{\boldsymbol{\theta}}$. Hence, it is sufficient to construct the time semigroup $\boldsymbol{T}_{\boldsymbol{\theta}}$ and claim that $\boldsymbol{T}^{\mathrm{gp}}_{\boldsymbol{\theta}}$ is the disjoint union of $\boldsymbol{T}_{\boldsymbol{\theta}}$, $\{0\}$, and $-\boldsymbol{T}_{\boldsymbol{\theta}}$.

Let $\boldsymbol{T}_{\boldsymbol{\theta}}$ be the semigroup generated by

$$\{\mathfrak{q}_{[0]}\} \cup \{\mathfrak{q}_{[n+1]} - \mathfrak{q}_{[n]}\}_{n\geq 0} \cup \bigcup_{\bar{a}_{n+1}=\infty} \{\mathfrak{q}_{[n+1]} - k\mathfrak{q}_{[n]}\}_{k\geq 2}.$$

Let us prove by induction over $m \geq 0$ the following statement:

$(\heartsuit_m)$ For every $(c_0, c_1, \ldots, c_m) \in \mathbb{Z}^{m+1}$, the sum $\sum_{j=0}^{m} c_j \mathfrak{q}_{[j]}$ is contained in exactly one of the following sets: $\boldsymbol{T}_{\boldsymbol{\theta}}$, $\{0\}$, $-\boldsymbol{T}_{\boldsymbol{\theta}}$.

Since $\mathfrak{q}_{[0]}$ is in $\boldsymbol{T}_{\boldsymbol{\theta}}$, $(\heartsuit_0)$ holds. Suppose $(\heartsuit_k)$ holds for some $k \geq 0$. Pick any sequence of integers $c_0, c_1, \ldots, c_{k+1}$ with $c_{k+1} \neq 0$ and let $x = \sum_{j=0}^{k+1} c_j \mathfrak{q}_{[j]}$. There are two cases.

- Suppose $\bar{a}_{k+1} < \infty$. Since $\mathfrak{q}_{[k+1]}$ is an integral combination of $\mathfrak{q}_{[k]}$ and $\mathfrak{q}_{[k-1]}$, then by $(\heartsuit_k)$, we have $x \in \boldsymbol{T}_{\boldsymbol{\theta}} \cup \{0\} \cup -\boldsymbol{T}_{\boldsymbol{\theta}}$.
- Suppose $\bar{a}_{k+1} = \infty$. Since $\mathfrak{q}_{[k+1]}$ is in $\boldsymbol{T}_{\boldsymbol{\theta}}$, it suffices to show that if $c_{k+1} = 1$, then $x \in \boldsymbol{T}_{\boldsymbol{\theta}}$. Hence, we will now assume that $c_{k+1} = 1$. For $j \in \{0, \ldots, k\}$, denote $d_j = \sum_{j=0}^{k} \max\{0, -c_j\}$, which satisfies $d_j \geq 0$ and $c_j + d_j \geq 0$. Let $d = \sum_{j=0}^{k} d_j \geq 0$. Then,

$$\begin{aligned} x &= \sum_{j=0}^{k} c_j \mathfrak{q}_{[j]} + \mathfrak{q}_{[k+1]} \\ &= \sum_{j=0}^{k} (c_j + d_j)\mathfrak{q}_{[j]} + \sum_{j=0}^{k} d_j(\mathfrak{q}_{[k]} - \mathfrak{q}_{[j]}) + (\mathfrak{q}_{[k+1]} - d\mathfrak{q}_{[k]}). \end{aligned}$$

  The last term in the summation above is contained in $\boldsymbol{T}_{\boldsymbol{\theta}}$, and each of the remaining terms is contained in $\boldsymbol{T}_{\boldsymbol{\theta}} \cup \{0\}$. Therefore, x is in $\boldsymbol{T}_{\boldsymbol{\theta}}$.

Hence, by induction, $(\heartsuit_k)$ holds for all k, and we are done. □

If $\boldsymbol{\theta}$ is irrational, then $\mathfrak{q}_{[0]} \mapsto 1$ induces an isomorphism between $(\boldsymbol{T}^{\mathrm{gp}}_{\boldsymbol{\theta}}, <)$ and the ordered group of integers $(\mathbb{Z}, <)$ that sends each $\mathfrak{q}_{[n]}$ to the number $q_{[n]} \in \mathbb{N}$ coming from §2.2. Otherwise, $(\boldsymbol{T}^{\mathrm{gp}}_{\boldsymbol{\theta}}, <)$ is a non-Archimedean ordered group.

In practice, sometimes it is more convenient to consider a different set of generators $\{\mathfrak{q}_n = \mathfrak{q}_n(\boldsymbol{\theta})\}_{n\geq 0}$ for $\boldsymbol{T}^{\mathrm{gp}}_{\theta}$. It mimics the continuants of the regular continued fraction expansion and it is defined together with an increasing sequence of integers $\{t_n\}_{n\geq 0}$ and a sequence $\{a_n\}_{n\geq 1}$ of elements of $\overline{\mathbb{N}}$ as follows. First, set

$$t_0 = 0, \qquad a_1 = \bar{a}_1, \quad \text{and} \quad \mathfrak{q}_0 = \mathfrak{q}_{[0]}.$$

The rest is defined inductively as follows. For $n \geq 0$,

- if $\varepsilon_{n+1}\varepsilon_{n+2} = -1$, then
$$t_{n+1} = t_n + 1, \quad a_{t_n+2} = \bar{a}_{n+2}, \quad \text{and} \quad \mathfrak{q}_{t_{n+1}} = \mathfrak{q}_{[n+1]};$$
- if $\varepsilon_{n+1}\varepsilon_{n+2} = +1$, then
$$t_{n+1} = t_n + 2, \qquad a_{t_n+2} = 1, \qquad a_{t_n+3} = \bar{a}_{n+2} - 1,$$
$$\mathfrak{q}_{t_n+1} = \mathfrak{q}_{[n+1]} - \mathfrak{q}_{[n]}, \qquad \text{and} \qquad \mathfrak{q}_{t_{n+1}} = \mathfrak{q}_{[n+1]}.$$

We will call $\mathfrak{q}_n = \mathfrak{q}_n(\boldsymbol{\theta})$ the n^{th} *slow return time* of $\boldsymbol{\theta}$. The group relations become slightly simpler, namely
$$\mathfrak{q}_{n+1} = a_{n+1}\mathfrak{q}_n + \mathfrak{q}_{n-1} \qquad \text{for } n \geq 0 \text{ with } a_{n+1} < \infty,$$
where $\mathfrak{q}_{-1} = 0$ is set to be the identity for convenience. In this format, the set of generators of the time semigroup $\boldsymbol{T}_\theta$ is
$$\{\mathfrak{q}_n\}_{n\geq 0} \cup \bigcup_{a_{n+1}=\infty} \{\mathfrak{q}_{n+1} - k\mathfrak{q}_n\}_{k\geq 1}.$$

5. The natural extension

Consider the bi-infinite sequence space
$$\underleftarrow{\overline{\Theta}} := \overline{\Sigma}^{\mathbb{Z}} = \left\{ \langle(\varepsilon_n, \bar{a}_n)\rangle_{n\in\mathbb{Z}} \, : \, (\varepsilon_n, \bar{a}_n) \in \overline{\Sigma} \text{ for all } n \right\},$$
equipped with the infinite product topology. The projection map
$$\mathfrak{P} : \underleftarrow{\overline{\Theta}} \to \overline{\Theta}, \quad \langle \ldots, (\varepsilon_{-1}, \bar{a}_{-1}), (\varepsilon_0, \bar{a}_0); (\varepsilon_1, \bar{a}_1), \ldots \rangle \mapsto \langle (\varepsilon_1, \bar{a}_1), (\varepsilon_2, \bar{a}_2), \ldots \rangle$$
is continuous and the shift map $\mathfrak{s} : \underleftarrow{\overline{\Theta}} \to \underleftarrow{\overline{\Theta}}$ given by
$$\mathfrak{s}(\langle \ldots, (\varepsilon_{-1}, \bar{a}_{-1}), (\varepsilon_0, \bar{a}_0); (\varepsilon_1, \bar{a}_1), \ldots \rangle) = \langle \ldots, (\varepsilon_0, \bar{a}_0), (\varepsilon_1, \bar{a}_1); (\varepsilon_2, \bar{a}_2), \ldots \rangle$$
is a homeomorphism. Indeed, $\mathfrak{s} : \underleftarrow{\overline{\Theta}} \to \underleftarrow{\overline{\Theta}}$ is the natural extension of $\mathfrak{s} : \overline{\Theta} \to \overline{\Theta}$.

5.1. The time group.

Definition 5.1. For every $\underline{\boldsymbol{\theta}} = \langle(\varepsilon_n, \bar{a}_n)\rangle_{n\in\mathbb{Z}} \in \underleftarrow{\overline{\Theta}}$, we again define an additive abelian group $\mathbf{T}^{\text{gp}}_{\underline{\boldsymbol{\theta}}}$, called the *time group* of $\underline{\boldsymbol{\theta}}$, such that it is generated by the sequence $\{Q_{[n]}\}_{n\in\mathbb{Z}}$ and subject to the relations
$$Q_{[n]} = b_n Q_{[n-1]} - \varepsilon_{n-1}\varepsilon_n Q_{[n-2]} \quad \text{where} \quad b_n = \bar{a}_n + \frac{1 + \varepsilon_n\varepsilon_{n+1}}{2}$$
for all $n \in \mathbb{Z}$ with $\bar{a}_n < \infty$.

For every $n \in \mathbb{Z}$, $Q_{[n]} = Q_{[n]}(\underline{\boldsymbol{\theta}})$ is called the n^{th} return time of $\underline{\boldsymbol{\theta}}$. Below, we will again define another generating set $\{Q_n = Q_n(\underline{\boldsymbol{\theta}})\}_{n\in\mathbb{Z}}$ for $\mathbf{T}^{\text{gp}}_{\underline{\boldsymbol{\theta}}}$. To do so, we will need a strictly increasing sequence of integers $\{t_n\}_{n\in\mathbb{Z}}$ and a sequence of positive integers $\{a_n\}_{n\in\mathbb{Z}}$. First, we set $t_0 = 0$. For all $n \geq 1$, t_n and t_{-n} are defined recursively by
$$t_n = \begin{cases} t_{n-1} + 1 & \text{if } \varepsilon_n\varepsilon_{n+1} = -, \\ t_{n-1} + 2 & \text{if } \varepsilon_n\varepsilon_{n+1} = +, \end{cases}$$
and
$$t_{-n} = \begin{cases} t_{-n+1} - 1 & \text{if } \varepsilon_{-n+1}\varepsilon_{-n+2} = -, \\ t_{-n+1} - 2 & \text{if } \varepsilon_{-n+1}\varepsilon_{-n+2} = +. \end{cases}$$

For all $n \in \mathbb{Z}$,

- if $\varepsilon_n \varepsilon_{n+1} = -$, set $a_{t_n+1} = \bar{a}_{n+1}$;
- if $\varepsilon_n \varepsilon_{n+1} = +$, set $a_{t_n} = 1$ and $a_{t_n+1} = \bar{a}_{n+1} - 1$.

For all $n \in \mathbb{Z}$, we set

$$Q_{t_n} := Q_{[n]},$$

and whenever $\varepsilon_n \varepsilon_{n+1} = +$, we set

$$Q_{t_n-1} := Q_{[n]} - Q_{[n-1]}.$$

The corresponding set of relations for this new generating set is simpler: for every $m \in \mathbb{Z}$ with $a_m < \infty$, we have

$$Q_m = a_m Q_{m-1} + Q_{m-2}.$$

Proposition 5.2 (Chronological order)**.** *For every $\underline{\boldsymbol{\theta}} = \langle(\varepsilon_n, \bar{a}_n)\rangle_{n\in\mathbb{Z}} \in \overline{\Theta}$, the time group $\mathbf{T}^{\mathrm{gp}}_{\underline{\boldsymbol{\theta}}}$ admits a unique translation-invariant total order $<$, called the chronological order of $\mathbf{T}_{\underline{\boldsymbol{\theta}}}$, with the property that for all $n \in \mathbb{Z}$,*

1. *$0 < Q_{[n]} < Q_{[n+1]}$,*
2. *if $\bar{a}_{n+1} = \infty$, then $kQ_{[n]} < Q_{[n+1]}$ for all $k \geq 1$.*

The positive cone of $<$ is the commutative semigroup

$$\mathbf{T}_{\underline{\boldsymbol{\theta}}} := \{P \in \mathbf{T}^{\mathrm{gp}}_{\underline{\boldsymbol{\theta}}} \ : \ P > 0\},$$

which we refer to as the *time semigroup* of $\underline{\boldsymbol{\theta}}$.

Proof. The idea is similar to the proof of Proposition 4.6. Consider the slow generating set $\{Q_n\}_{n\in\mathbb{Z}}$. We define $\mathbf{T}_{\underline{\boldsymbol{\theta}}}$ to be the semigroup generated by

$$\{Q_n\}_{n\in\mathbb{Z}} \cup \bigcup_{a_{n+1}=\infty} \{Q_{n+1} - kQ_n\}_{n\in\mathbb{Z}}.$$

We need to show that $\mathbf{T}^{\mathrm{gp}}_{\underline{\boldsymbol{\theta}}}$ is the disjoint union of $\mathbf{T}_{\underline{\boldsymbol{\theta}}}$, $\{0\}$, and $-\mathbf{T}_{\underline{\boldsymbol{\theta}}}$.

Let us prove by induction over $m \geq 0$ the following statement:

($\diamondsuit_m$) For every $(n, c_0, c_1, \dots, c_m) \in \mathbb{Z}^{m+2}$, the sum $\sum_{j=0}^{k} c_j Q_{n+m}$ is contained in exactly one of the following sets: $\boldsymbol{T}_{\boldsymbol{\theta}}$, $\{0\}$, $-\boldsymbol{T}_{\boldsymbol{\theta}}$.

Similar to the proof of Proposition 4.6, ($\diamondsuit_0$) is trivial and for $m \geq 1$, ($\diamondsuit_m$) implies ($\diamondsuit_{m+1}$). It remains to prove ($\diamondsuit_1$).

Let us pick $n \in \mathbb{Z}$ and suppose x is an integral combination of Q_n and Q_{n-1}. If the coefficients of Q_n and Q_{n-1} are either zero or have the same sign, then clearly x will belong in either $\boldsymbol{T}_{\boldsymbol{\theta}}$, or $\{0\}$, or $-\boldsymbol{T}_{\boldsymbol{\theta}}$. We can now assume that

$$x = c_0 Q_n - d_0 Q_{n-1}$$

for some co-prime positive integers c_0 and d_0. We will assume that $0 < c_0 < d_0$ because otherwise x can be expressed as the sum of $(c_0 - d_0)Q_n \in \mathbf{T}_{\underline{\boldsymbol{\theta}}} \cup \{0\}$ and $d_0(Q_n - Q_{n-1}) \in \mathbf{T}_{\underline{\boldsymbol{\theta}}}$ and we are done. To proceed, there are two cases.

(i) Suppose $a_n = \infty$. We can present x as

$$x = (c_0 - 1)Q_n + (Q_n - d_0 Q_{n-1}).$$

In the expression above, the first term is in $\mathbf{T}_{\underline{\boldsymbol{\theta}}} \cup \{0\}$ and the second term is in $\mathbf{T}_{\underline{\boldsymbol{\theta}}}$. Hence, x is in $\mathbf{T}_{\underline{\boldsymbol{\theta}}}$.

(ii) Suppose $a_n < \infty$, so then we can write

$$x = -(c_{-1}Q_{n-1} - d_{-1}Q_{n-2}) \qquad \text{where } c_{-1} = d_0 - c_0 a_n \text{ and } d_{-1} = c_0.$$

There are three sub-cases.

(a) Suppose $c_{-1} \leq 0$. Since x is the sum of $(-c_{-1})Q_{n-1} \in \mathbf{T}_{\underline{\boldsymbol{\theta}}} \cup \{0\}$ and $d_{-1}Q_{n-2} \in \mathbf{T}_{\underline{\boldsymbol{\theta}}}$, then x is in $\mathbf{T}_{\underline{\boldsymbol{\theta}}}$.

(b) Suppose $c_{-1} \geq d_{-1}$. Since x is the sum of $-(c_{-1} - d_{-1})Q_{n-1} \in -\mathbf{T}_{\underline{\boldsymbol{\theta}}} \cup \{0\}$ and $-d_{-1}(Q_{n-1} - Q_{n-2}) \in -\mathbf{T}_{\underline{\boldsymbol{\theta}}}$, then x is in $-\mathbf{T}_{\underline{\boldsymbol{\theta}}}$.

(c) Suppose $0 < c_{-1} < d_{-1}$. This assumption is equivalent to

$$\frac{1}{a_n + 1} < \frac{c_0}{d_0} < \frac{1}{a_n}.$$

Analogous to (i), we have $x \in -\mathbf{T}_{\underline{\boldsymbol{\theta}}}$ if $a_{n-1} = \infty$. Suppose $a_{n-1} < \infty$. Then, we can write

$$x = c_{-2}Q_{n-2} - d_{-2}Q_{n-3}$$

where

$$c_{-2} = d_{-1} - c_{-1}a_{n-1} \qquad \text{and} \qquad d_{-2} = c_{-1}.$$

Again, analogous to (2) (a)–(b), we are done if $c_{-2} \notin (0, d_{-2})$ and this leaves us with the case where $0 < c_{-2} < d_{-2}$. This amounts to

$$\frac{1}{a_n + \frac{1}{a_{n-1}}} < \frac{c_0}{d_0} < \frac{1}{a_n + \frac{1}{a_{n-1}+1}}.$$

If we end up having to repeat the argument above $k \geq 1$ times, we are left with the case where $a_n, a_{n-1}, \ldots, a_{n-k+1} < \infty$ and x can be written as

$$x = (-1)^k (c_{-k}Q_{n-k} - d_{-k}Q_{n-k-1})$$

where

$$\begin{bmatrix} c_{-k} \\ d_{-k} \end{bmatrix} = \begin{bmatrix} -a_{n-k+1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} -a_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ d_0 \end{bmatrix}.$$

In this case, $\frac{c_0}{d_0}$ is between $[a_n, a_{n-1}, \ldots, a_{n-k+1}]$ and $[a_n, a_{n-1}, \ldots, a_{n-k+1} + 1]$. Eventually, this process has to stop at some value k because otherwise the rational number $\frac{c_0}{d_0}$ would be arbitrarily close to the irrational $[a_n, a_{n-1}, a_{n-2}, \ldots]$. We are done with the proof of $(\diamondsuit_1)$. □

Next, we will define another total order $\rhd$ on $\mathbf{T}^{\mathrm{gp}}_{\underline{\boldsymbol{\theta}}}$.

Proposition 5.3 (Left-right order)**.** *For $\underline{\boldsymbol{\theta}} \in \overleftarrow{\overline{\Theta}}$, the time group $\mathbf{T}^{\mathrm{gp}}_{\underline{\boldsymbol{\theta}}}$ admits a unique translation invariant total order $\lhd$, called the left-right order, that satisfies the following properties.*

1. *If $\varepsilon_1 = +1$, then $Q_0 \lhd 0$; otherwise, $\varepsilon_1 = -1$ and $0 \lhd Q_0$.*
2. *For $n \in \mathbb{Z}$, $-Q_n$ is always $\lhd$-between 0 and Q_{n-1}.*
3. *For $n \in \mathbb{Z}$ with $a_{n+1} = \infty$, then $-kQ_n$ is always $\lhd$-between 0 and Q_{n-1} for all $k \geq 1$.*

We will call $\rhd$ the *left-right order* on $\overleftarrow{\overline{\Theta}}$. It naturally descends to a translation-invariant total order $\rhd$ on the time semigroup $\mathbf{T}_{\underline{\boldsymbol{\theta}}}$. The geometric meaning behind $\lhd$ will become apparent in the next section.

Proof. Assume $\varepsilon_1 = -$ and we will have $0 \lhd Q_0$. Let $Q'_n = (-1)^n Q_n$. It satisfies the relation $Q'_{n-1} = a_{n+1}Q'_n + Q'_{n+1}$ for all $n \in \mathbb{Z}$ with $a_{n+1} < \infty$. Let Υ be the semigroup generated by

$$\{Q'_n\}_{n\in\mathbb{Z}} \cup \bigcup_{a_{n+1}=\infty} \{Q'_{n-1} - kQ'_n\}_{k\geq 1}.$$

With almost the same proof as in Proposition 5.2, the time group $\mathbf{T}^{\mathrm{gp}}_{\underline{\boldsymbol{\theta}}}$ is indeed equal to the disjoint union of Υ, $\{0\}$, and $-\Upsilon$. This induces a total order $\lhd$ on $\mathbf{T}^{\mathrm{gp}}_{\underline{\boldsymbol{\theta}}}$ such that Υ is the corresponding positive cone, that is, $P \lhd Q$ if and only if $Q - P \in \Upsilon$.

It remains the show that $\lhd$ satisfies the desired properties. Property (1) is clear. By the construction, we have that for every odd integer n,

$$-Q_n \lhd 0 \lhd Q_{n+1} \lhd Q_{n-1},$$

which implies property (2). For all n with $a_{n+1}\infty$ and all $k \geq 1$, since both Q'_n and $Q'_{n-1} - kQ'_n$ are in Υ, then $0 \lhd kQ'_n \lhd Q'_{n-1}$ and this implies property (3).

Lastly, if $\varepsilon_1 = +1$, then we will take Υ as the negative cone of $\lhd$. The rest of the proof above is exactly the same. ☐

The geometric meaning behind $\lhd$ will be clarified in Proposition 5.7.

5.2. **Cascade of translations.** Consider the subspace

$$\underleftarrow{\Theta} := \Sigma^{\mathbb{Z}} = \left\{ \langle(\varepsilon_n, \bar{a}_n)\rangle_{n\in\mathbb{Z}} \in \overline{\underleftarrow{\Theta}} \ : \ \bar{a}_n < \infty \text{ for all } n \right\}$$

of $\overline{\underleftarrow{\Theta}}$. Every element $\underline{\boldsymbol{\theta}}$ of $\underleftarrow{\Theta}$ induces a sequence of irrationals

$$\theta_n = \mathfrak{X} \circ \mathfrak{P} \circ \mathfrak{s}^n(\underline{\boldsymbol{\theta}}) \in \Theta, \quad n \in \mathbb{Z},$$

where $\mathfrak{g}(\theta_n) = \theta_{n+1}$ for all n. The bi-infinite tower of sector renormalizations $\{\mathfrak{r}_{\theta_n} : \bar{\mathbb{D}} \to \bar{\mathbb{D}}\}_{n\in\mathbb{Z}}$ can be put into a single dynamical plane as follows.

Definition 5.4. For $\underline{\boldsymbol{\theta}} = \langle(\varepsilon_n, \bar{a}_n)\rangle_{n\in\mathbb{Z}} \in \underleftarrow{\Theta}$, we define the *(full) cascade* associated to $\underline{\boldsymbol{\theta}}$ to be the abelian group $F = F_{\underline{\boldsymbol{\theta}}}$ of translations generated by $\{F^{[n]} = F^{[n]}_{\underline{\boldsymbol{\theta}}}\}_{n\in\mathbb{Z}}$ where for $n \in \mathbb{Z}$,

$$F^{[n]}(z) := z - \varepsilon_{n+1} l_{[n]} \qquad \text{where } l_{[n]} := l_{[n]}(\underline{\boldsymbol{\theta}}) := \begin{cases} |\theta_0 \ldots \theta_n| & \text{if } n \geq 0, \\ 1 & \text{if } n = -1, \\ |\theta_{-1} \ldots \theta_{-n+1}|^{-1} & \text{if } n \leq -2, \end{cases}$$

Let us fix $\underline{\boldsymbol{\theta}} = \langle(\varepsilon_n, \bar{a}_n)\rangle_{n\in\mathbb{Z}} \in \underleftarrow{\Theta}$. For $n \in \mathbb{Z}$, denote $b_n = \bar{a}_n + \dfrac{1 + \varepsilon_n\varepsilon_{n+1}}{2}$.

Lemma 5.5. *For all $n \in \mathbb{Z}$, we have*

$$F^{[n]} = (F^{[n-1]})^{b_n} \circ (F^{[n-2]})^{-\varepsilon_{n-1}\varepsilon_n}.$$

Proof. From Lemma 2.1, we have $\theta_n = \varepsilon_n b_n - \frac{1}{\theta_{n-1}}$. Therefore,

$$\varepsilon_{n+1}|\theta_{n-1}\theta_n| = |\theta_{n-1}| \left(\varepsilon_n b_n - \frac{1}{\theta_{n-1}} \right) = b_n \cdot \varepsilon_n |\theta_{n-1}| - \varepsilon_{n-1}\varepsilon_n \cdot \varepsilon_{n-1}.$$

By multiplying both sides by $l_{[n-2]}$, we obtain

$$\varepsilon_{n+1} l_{[n]} = b_n l_{[n-1]} - \varepsilon_{n-1}\varepsilon_n l_{[n-2]}.$$

This implies the desired equation. ☐

Consider the time group $\mathbf{T}^{\mathrm{gp}} = \mathbf{T}^{\mathrm{gp}}_{\underline{\boldsymbol{\theta}}}$ of $\underline{\boldsymbol{\theta}}$ defined in the previous section.

Proposition 5.6. *The time group $\mathbf{T}^{\mathrm{gp}}$ of $\underline{\boldsymbol{\theta}}$ is isomorphic to the full cascade F associated to $\underline{\boldsymbol{\theta}}$ via the identification $Q_{[n]} \mapsto F^{[n]}$ for all $n \in \mathbb{Z}$.*

Proof. According to the previous lemma, there is a well-defined surjective homomorphism

$$\phi : \mathbf{T}^{\mathrm{gp}} \to F, \quad P \mapsto F^P,$$

where $F^{Q_{[n]}} = F^{[n]}$ for every $n \in \mathbb{Z}$. It remains to check that ϕ is injective. Pick $P \in \mathbf{T}^{\mathrm{gp}}$, so P can be written as an integral combination of $Q_{[n]}, Q_{[n+1]}, \ldots, Q_{[n+k]}$ for some integers $n \in \mathbb{Z}$ and $k \geq 0$. By the recurrence relation, we can rewrite P as $c_0 Q_{[n]} + c_1 Q_{[n+1]}$ for some integer coefficients c_0, c_1. Hence,

$$F^P(z) = z - \varepsilon_{n+1} c_0 l_{[n]} - \varepsilon_{n+2} c_1 l_{[n+1]} = z - (\varepsilon_{n+1} c_0 + \varepsilon_{n+1} c_1 |\theta_{n+1}|)\, l_{[n]}.$$

Since θ_{n+1} is irrational, the equation $\varepsilon_{n+1} c_0 + \varepsilon_{n+1} c_1 |\theta_{n+1}| = 0$ only has trivial solution $c_0 = c_1 = 0$. This implies that F^P is the identity map if and only if $P = 0$. Therefore, ϕ is injective. □

As a consequence of this proposition, we can parametrize F in terms of elements of $\mathbf{T}^{\mathrm{gp}}$ and write $F = \{F^P\}_{P \in \mathbf{T}^{\mathrm{gp}}}$ where $F^{Q_{[n]}} = F^{[n]}$. From now on, we will denote

$$V_P := F^P(0) \qquad \text{for } P \in \mathbf{T}^{\mathrm{gp}}.$$

By Propositions 4.6 and 5.2, F admits two natural total orders, namely the chronological order $<$ and the left-right order $\lhd$. The latter is called the left-right order for the following reason.

Proposition 5.7. *For any two elements P_1 and P_2 in $\mathbf{T}^{\mathrm{gp}}$,*

$$V_{P_1} < V_{P_2} \quad \text{if and only if} \quad P_1 \lhd P_2.$$

Proof. By Proposition 5.6, there does exist a well-defined strict total order $\lhd$ such that $P_1 \lhd P_2$ if and only if $V_{P_1} < V_{P_2}$. It is clear that $\lhd$ is translation-invariant since F simply acts as translations on the real line. To prove the proposition, we just need to check that $\lhd$ satisfies properties (1) and (2) in Proposition 5.3. Property (3) is inapplicable because $a_n < \infty$ for all n in our current setting.

Property (1) is clear: since $V_{Q_0} = -\varepsilon_1 l_{[0]}$, then $V_{Q_0} < 0$ if $\varepsilon_1 = +1$ and $V_{Q_0} > 0$ if $\varepsilon_1 = -1$. Let us check property (2) for $n = 0$; the proof for every other n's will be analogous. There are two cases.

- Suppose $\varepsilon_0 \varepsilon_1 = -1$. Since $V_{Q_0} = -\varepsilon_1 l_{[0]}$ and $V_{Q_{-1}} = -\varepsilon_0 = \varepsilon_1$, then $0 < \varepsilon_1 V_{-Q_{0]}} < \varepsilon_1 V_{Q_{-1}}$. This implies that $-Q_0$ is $\lhd$-between 0 and Q_{-1}.
- Suppose $\varepsilon_0 \varepsilon_1 = +1$. Then,

$$V_{Q_{-2}} = V_{Q_{[-1]}} = -\varepsilon_1 \quad \text{and} \quad V_{Q_{-1}} = V_{Q_{[0]} - Q_{[-1]}} = \varepsilon_1 (1 - l_{[0]}).$$

Since $0 < l_{[0]} < \frac{1}{2}$, then $0 < \varepsilon_1 V_{-Q_{-2}} < \varepsilon_1 V_{Q_{-1}} < \varepsilon_1 V_{-Q_0}$. This implies that $-Q_{-1}$ is $\lhd$-between 0 and Q_0, and $-Q_{-1}$ is $\lhd$-between 0 and Q_{-2}. □

5.3. **Renormalization tiling.** For any two points x and y on the real line, we will denote by $[x, y]$ the closed real interval with endpoints x and y. For $n \in \mathbb{Z}$, denote

$$J_n = [V_{-Q_{[n]}}, V_{-Q_{[n]} + \varepsilon_n \varepsilon_{n+1} Q_{[n-1]}}].$$

Lemma 5.8. *For all $n \in \mathbb{Z}$,*

(1) $[0, V_{-2Q_{[n]} + \varepsilon_n \varepsilon_{n+1} Q_{[n-1]}}]$ *is contained in the interior of J_n;*

(2) *J_n is contained in the interior of J_{n-1}.*

Proof. We have $V_{-Q_{[n]}} = \varepsilon_{n+1} l_{[n]}$ and $V_{-Q_{[n]}+\varepsilon_n\varepsilon_{n+1}Q_{[n-1]}} = \varepsilon_{n+1}(l_{[n]} - l_{[n-1]})$ and $V_{-2Q_{[n]}+\varepsilon_n\varepsilon_{n+1}Q_{[n-1]}} = \varepsilon_{n+1}(2l_{[n]} - l_{[n-1]})$. Then, item (1) follows from the inequality $l_{[n-1]} > 2l_{[n]}$. Item (2) follows from:

$$\max_{x\in\partial J_n} |x| = l_{[n-1]} - l_{[n]} < l_{[n-1]} = \min_{x\in\partial J_{n-1}} |x|. \qquad \square$$

The first part of the lemma above tells us that J_n can be partitioned in two ways. Firstly, we have $J_n = J_{n,0} \cup J_{n,1}$ where

$$J_{n,0} = [V_{-Q_{[n]}+\varepsilon_n\varepsilon_{n+1}Q_{[n-1]}}, 0] \quad \text{and} \quad J_{n,1} = [0, V_{-Q_{[n]}}],$$

Secondly, we have $J_n = J'_{n,0} \cup J'_{n,1}$ where

$$J'_{n,j} := F^{-Q_{[n]}+j\varepsilon_n\varepsilon_{n+1}Q_{[n-1]}}(J_{n,j}), \qquad j \in \{0,1\}.$$

The two partitions are related by the commuting pair

$$\mathcal{F}_n = \left(F^{Q_{[n]}} : J'_{n,0} \to J_{n,0},\ \ F^{Q_{[n]}-\varepsilon_n\varepsilon_{n+1}Q_{[n-1]}} : J'_{n,1} \to J_{n,1}\right).$$

Let us outline how the cascade F is related to $\{\mathbb{r}_{\theta_n} : \bar{\mathbb{D}} \to \bar{\mathbb{D}}\}_{n\in\mathbb{Z}}$. Denote by $-\overline{\mathbb{H}}$ the closed lower half plane. For any closed interval $J \subset \mathbb{R}$, denote the half strip

$$W(J) := \{z \in -\overline{\mathbb{H}} \ : \ \mathrm{Re} z \in J\}.$$

For $n \in \mathbb{Z}$ and $j \in \{0,1\}$, denote

$$W_n = W(J_n), \qquad W_{n,j} = W(J_{n,j}), \qquad W'_{n,j} = W(J'_{n,j}).$$

Each W_n is a strip of horizontal width $l_{[n-1]}$. The second part of the lemma above implies that we have a bi-infinite nest of half-strips

$$\ldots \subset W_2 \subset W_1 \subset W_0 \subset W_{-1} \subset \ldots$$

whose union is $-\overline{\mathbb{H}}$ and whose nester intersection is the negative imaginary axis. The commuting pair $\mathcal{F}_n$ mentioned before extends to a commuting pair on W_n:

$$\mathcal{F}_n = \left(F^{Q_{[n]}} : W'_{n,0} \to W_{n,0},\ \ F^{Q_{[n]}-\varepsilon_n\varepsilon_{n+1}Q_{[n-1]}} : W'_{n,1} \to W_{n,1}\right).$$

The next proposition draws the bridge between cascades and sector renormalization. It just follows from elementary calculation.

Proposition 5.9. *For $n \in \mathbb{Z}$, the map*

$$\phi_n : W_n \to \overline{\mathbb{D}}, \quad \phi_n(z) = e^{-2\pi i z/l_{[n-1]}}$$

satisfies the following properties.

1. *ϕ_n conformally sends the interior of W_n to the open unit disk $\mathbb{D}$ minus the radial slit $\{\arg z = -2\pi\theta_n\}$, and sends each of the horizontal sides of W_n to this slit.*
2. *ϕ_n projects the commuting pair $\mathcal{F}_n$ to the rotation $\mathbb{r}_{\theta_n} : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$, and*
3. *For $k \geq n+1$, ϕ_n maps W_k conformally onto the sector $S^{k-n}_{\theta_n}$ and the commuting pair $\mathcal{F}_k$ projects to the first return map of $\mathbb{r}_{\theta_n}$ back to the sector $S^{k-n}_{\theta_n}$.*

In particular, the proposition above says that $\mathcal{F}_n$ is precisely the first return map of F back to W_n. This gives us a triangulation of the lower half plane.

Proposition 5.10 (Renormalization triangulation)**.** *For $n \in \mathbb{Z}$,*

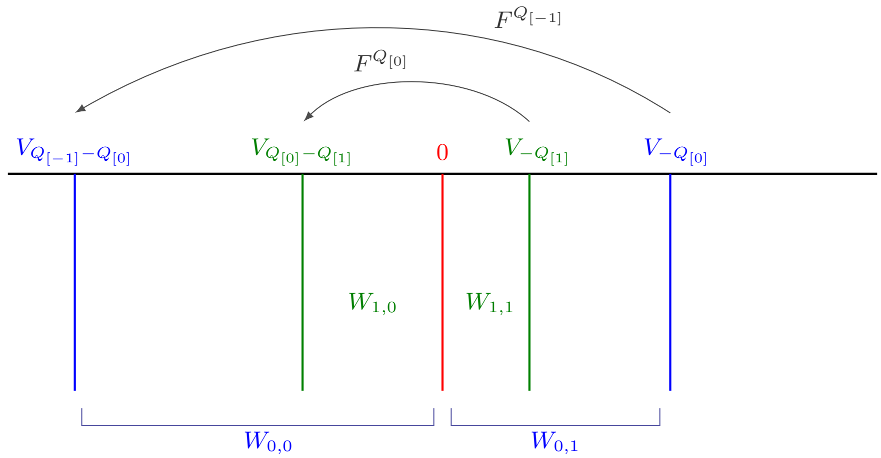

FIGURE 3. Cascade $F = F_{\underline{\boldsymbol{\theta}}}$ with combinatorics $\underline{\boldsymbol{\theta}} = \langle \dots, (+,2); (+,2), (+,2), \dots \rangle$

(1) *the collection of closed half-strips*

$$\mathcal{W}_n = \{F^{-P}(W_{n,0})\}_{0 \leq P < Q_{[n]}} \cup \{F^{-P}(W_{n,1})\}_{0 \leq P < Q_{[n]} - \varepsilon_n \varepsilon_{n+1} Q_{[n-1]}}$$

partitions the closed lower half plane, i.e. the elements of $\mathcal{W}_n$ have pairwise disjoint interior and their union is $\mathbb{R}$;

(2) *$\mathcal{W}_n$ is a refinement of $\mathcal{W}_{n-1}$: every element in $\mathcal{W}_{n-1}$ is a union of at least two consecutive half-strips in $\mathcal{W}_n$.*

As a consequence, we also have:

Corollary 5.11 (Proper discontinuity)**.** *For $n \in \mathbb{Z}$,*

$$\text{if} \quad 0 < P < Q_{[n]} \ \text{or} \ -Q_{[n]} < P < 0, \qquad \text{then} \quad |V_P| > l_{[n]}.$$

In particular, if P is small, then $|V_P|$ is large.

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