Quadratic-Like Renormalisation in Holomorphic Dynamics

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An M4R project in partial fulfilment of
the requirements for the degree of MSci Mathematics.
June 2019

Abstract

Holomorphic dynamics is the study of the behaviour of iterations of holomorphic endomorphisms of a Riemann surface. Quadratic maps are an epitome of how even the simplest non-linear system can admit a highly complicated dynamical behaviour.

In the study of the dynamics of quadratic maps, quadratic-like renormalisation is the process of restricting a quadratic discrete dynamical system to a smaller scale to obtain a new dynamical system behaving in a topologically similar way to quadratic maps. In this project, we aim to study in depth the concepts of renormalisation and explore its significance in tackling two distinct problems in holomorphic dynamics, namely the problem of local connectivity of Julia sets and the Mandelbrot set as well as the problem of existence of a fixed point of renormalisation.

Contents

Chapter 1

Introduction

1.1 Brief History

Complex dynamics is the study of the behaviour of iterations of a holomorphic map $f : X \to X$ defined on a Riemann surface X. The domain X is split into the stable subset, characterised by equicontinuity of iterations of f, and the chaotic subset, where the iterates exhibit sensitive dependence on initial conditions.

The modern framework of holomorphic dynamics was formulated in the 1920s by Pierre Fatou and Gaston Julia as a highly successful application of Montel's theory of normal families of holomorphic functions. However, a surge of theoretical developments only appeared in the 1970s due to its connections to other fields, such as the study of hyperbolic 3-manifolds and Kleinian groups.

Inevitably, the dynamical object that receives the most attention is the Mandelbrot set $\mathbb{M}$. The bifurcation locus of the family $\{f_c(z) = z^2 + c \,|\, c \in \mathbb{C}\}$ forms the boundary of $\mathbb{M}$. Theoretically, its attractiveness lies in its universality — a copy of $\mathbb{M}$ can be found in other families of holomorphic maps which are seemingly unrelated to the quadratics. Much of the current research is also motivated by the following conjecture by Douady and Hubbard ([DH84]).

Conjecture (MLC)**.** *The Mandelbrot set* $\mathbb{M}$ *is locally connected.*

If true, the MLC gives us a complete topological description of all quadratic maps and implies another central conjecture, namely the density of hyperbolicity for the quadratic family. While we are now aware of many topological properties of $\mathbb{M}$ (compactness, connectedness, etc), proving local connectivity has been extremely difficult. So far, many cases have been settled through the theory of *renormalisation*.

Renormalisation can be thought as the process of restricting a dynamical system to a smaller scale and obtaining a new dynamical system of the same type. In the 1980s, Douady and Hubbard ([DH85]) developed the *quadratic-like renormalisation* which concerns dynamical systems behaving in a topologically similar way to quadratic maps. Yoccoz ([Hub93]) then pioneered an innovative approach to prove the MLC at parameter values c which are at most *finitely renormalisable.* It took about 15 years for Yoccoz's result to be generalised to unicritical polynomials of arbitrary degree $d \geq 2$ (see [AKLS09], [KL09a]).

The theory of renormalisation of quadratic maps, however, was initiated by physicists Coullet, Tresser, and Feigenbaum in the 1970s ([CT78] and [Fei78]). Their work numerically explained the universality in period-doubling cascades inspired by Robert May's quadratic model for population dynamics ([May76]). Specifically, consider the real quadratic family $\{f_c\}_{c\in[-2,1/4]}$ and the parameters $\{c_n\}_{n\in\mathbb{N}}$ at which period-doubling bifurcation occurs, labelled in decreasing order. They discovered that c_n converges to $c_F \approx -1.4011552$, known as the Feigenbaum parameter, and that the ratio $(c_n - c_{n-1})/(c_{n+1} - c_n)$ converges to $\delta \approx 4.669201$, known as the Feigenbaum constant. Astoundingly, δ appears in other generic families of unimodal maps.

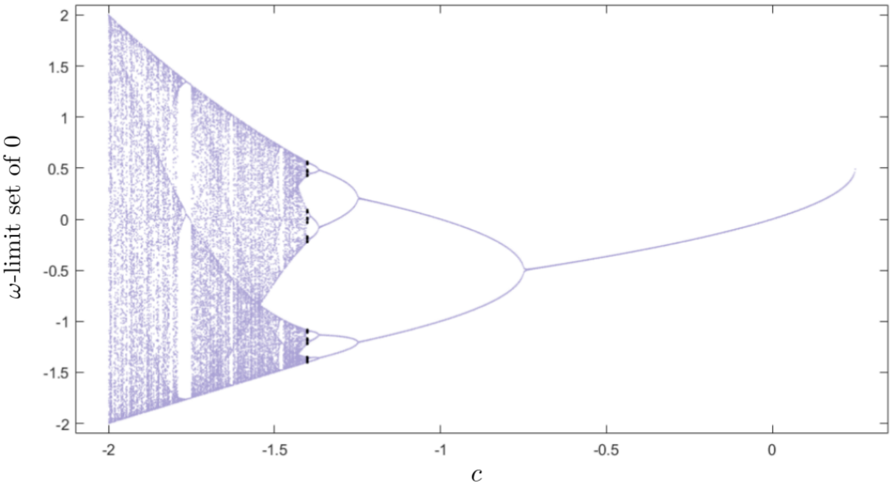

Figure 1.1: Bifurcation diagram for $\{f_c\}_{c\in\mathbb{R}}$ showing the transition from stable dynamics on the right to chaotic dynamics on the left through period-doubling cascade. The limit of the cascade shown in black is a Cantor set.

The three physicists left behind some conjectures, one of which says that the renormalisation operator $\mathcal{R} : f \mapsto af^2(a^{-1}z)$ for some normalising constant a has a unique fixed point. The conjectures motivated much of the remarkable progress made by Sullivan, McMullen and Lyubich on the renormalisation theory for *infinitely renormalisable* quadratic maps satisfying a certain precompactness property called *a priori bounds* ([Sul88], [McM96], and [Lyu97]).

Since then, significant progress has been made in [Lev11] and [CS15] in the case of infinitely renormalisable quadratic maps without a priori bounds. Cheraghi and Shishikura, in particular, applied another type of renormalisation called *near-parabolic renormalisation* invented by Inou and Shishikura in [IS06].

1.2 Summary of Contents

The main goal of the project is to study the theory of renormalisation within the framework of the dynamics of quadratic-like maps, i.e. those which behave topologically like quadratic maps. Other types of renormalisation, such as parabolic and near-parabolic renormalisations in holomorphic dynamics, renormalisation in billiard maps, as well as renormalisation groups in quantum field theory, bear similarity in ideas but these will not be covered here. The author recognises that the topic is very broad, and only intends to capture fundamental parts of quadratic-like renormalisation and how they contribute to the progress on local connectivity and the renormalisation conjectures. All illustrations presented here were originally produced using MATLAB.

Chapter 2 begins by reviewing many essential tools from complex analysis. This includes the theory of quasiconformal maps - undoubtedly one of the most important modern tools in complex dynamics. The second half will emphasise on applying quasiconformal techniques to the geometry of annular domains.

In chapter 3, we will review preliminary concepts in holomorphic dynamics. We emphasise on the dynamics of quadratic maps as well as the properties of the Mandelbrot set $\mathbb{M}$. We conclude with external rays on the dynamical space.

In chapter 4, we begin studying from [DH85] and [McM94a] the objects of high interest: polynomial-like maps and renormalisations of quadratic-like maps. We first prove Douady and Hubbard's straightening theorem, a result which is central to almost every renormalisation argument. We then discuss the dynamical properties of renormalisable maps and the existence of copies of the Mandelbrot set in itself.

We introduce in chapter 5 a way to construct renormalisations of quadratic maps. This is done through *puzzles*, a powerful combinatorial tool introduced by Yoccoz to prove local connectivity of Julia sets of at most finitely renormalisable quadratic maps having no irrationally indifferent periodic cycles, and the MLC at at most finitely renormalisable parameters.

Chapter 6 focuses on the classical problem of the existence of a *renormalisation fixed point*. To study infinitely renormalisable maps, we define *a priori bounds* and discuss its importance. We then present two known results with our own proofs in Theorems 6.7 and 6.11. In short, we prove that if an infinitely renormalisable quadratic map f has a priori bounds, we have the following:

1. The postcritical set $P(f)$, i.e. the closure of the forward orbit of the critical value, is a Cantor set.
2. The map f has an infinite sequence of distinct repelling periodic cycles with multiplier uniformly bounded by a constant.

Lastly, we will discuss the existence of a renormalisation fixed point, i.e. a solution of the Cvitanovic-Feigenbaum equation

$$f^p(z) = af(a^{-1}z)$$

for some normalising constant $a \in \mathbb{C}^*$ and integer $p \geq 2$.

1.3 Notation and Terminology

The Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the usual one point compactification of the complex plane $\mathbb{C}$. Below is a list of common subsets of $\hat{\mathbb{C}}$:

$$\begin{aligned} \mathbb{H} &= \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}, & \mathbb{D}_r(w) &= \{z \in \mathbb{C} \mid |z-w| < r\}, \\ \mathbb{D} &= \{z \in \mathbb{C} \mid |z| < 1\}, & \mathbb{D}_r &= \{z \in \mathbb{C} \mid |z| < r\}, \\ \mathbb{T} &= \{z \in \mathbb{C} \mid |z| = 1\}, & \mathbb{T}_r &= \{z \in \mathbb{C} \mid |z| = r\}, \\ \mathbb{A}_{a,b} &= \{z \in \mathbb{C} \mid a < |z| < b\}. \end{aligned}$$

We denote by $\overline{A}$ the closure of A, $\operatorname{int}(A)$ the interior of A, and ∂A the boundary of A respectively. A non-empty subset A of the complex plane $\mathbb{C}$ is:

- *compactly contained* in B, i.e. $A \Subset B$, if $\overline{A} \subset \operatorname{int}(B)$,
- a *topological disk* if A is open, simply connected, and $A \neq \mathbb{C}$,
- a *topological annulus* if A is open and doubly connected,
- a *Jordan curve* if A is a simple closed curve,
- a *Jordan domain* if A is a topological disk and ∂A is a Jordan curve,
- *full* if A is compact in $\mathbb{C}$ and $\mathbb{C} \backslash A$ is connected,
- a *hull* if A is full non-degenerate connected set,
- a *Cantor set* if A is metrisable and as a metric space, A is a compact, perfect, and totally disconnected.

Let U and V be open sets in $\mathbb{C}$ and $f : U \to V$ be a smooth function. The complex partial derivatives of f at a point $z = x + iy$ are

$$f' = f_z = \frac{\partial f}{\partial z} := \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big), \qquad f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big)$$

The smooth function $f : U \to V$ is:

- *holomorphic* if $f_{\bar{z}} \equiv 0$,
- *conformal* if f is holomorphic and $f'(z) \neq 0$ for all $z \in U$,
- *univalent* if f is holomorphic and injective,
- a *biholomorphism* if f is holomorphic, bijective, and has a holomorphic inverse.

A point $z \in U$ is a *critical point* of a holomorphic function $f : U \to V$ if $f'(z) = 0$. If so, $f(z)$ is called a *critical value* of f. The holomorphic function f is:

- *proper* if any compact subset $K \subset V$ has a compact preimage $f^{-1}(K) \subset U$,
- a *covering map* of degree $d > 1$ if f is a surjective open holomorphic local homeomorphism and each fibre $f^{-1}(z)$ has cardinality d,
- a *branched covering map* of degree $d > 1$ if $f : U \backslash S \to V \backslash f(S)$ is a covering map of degree d where S is the set of critical points of f.

Unless otherwise stated, we will always assume that a space of functions between two open sets $U, V \subset \hat{\mathbb{C}}$ is endowed with the *compact-open topology*, i.e. $f_n \to f$ if and only if f_n converges uniformly to f on any compact subsets of U.

Chapter 2

Complex Analysis

This chapter reviews basic results in complex analysis on conformal and quasiconformal maps. Later parts will put emphasis on the quasiconformal geometry of annuli.

2.1 Conformal Maps

Definition 2.1. A *Riemann surface* is defined as a one-dimensional complex manifold. A map $f : X \to Y$ between two Riemann surfaces is *holomorphic* if f is holomorphic in the corresponding coordinate charts. Two Riemann surfaces are *biholomorphic* when there is a biholomorphism (bijective holomorphic map with holomorphic inverse) between them.

Theorem 2.1 (Riemann Mapping Theorem)**.** *Any topological disk $X \subset \mathbb{C}$ is biholomorphic to the unit disk $\mathbb{D}$.*

The Riemann mapping theorem is a special case of the uniformisation theorem.

Theorem 2.2 (Uniformisation Theorem)**.** *Any simply connected Riemann surface X is biholomorphic to either the Riemann sphere $\hat{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the unit disk $\mathbb{D}$.*

Definition 2.2. We say that a Riemann surface X is *hyperbolic* if its universal cover is biholomorphic to $\mathbb{D}$.

The Poincaré metric $\rho_{\mathbb{D}}(z) := \frac{4}{1-|z|^2}$ induces a hyperbolic distance on $\mathbb{D}$ defined as

$$d_{\mathbb{D}}(z,w) := \inf\{L_{\rho_{\mathbb{D}}}(\gamma) \mid \gamma \text{ is a curve joining } z \text{ and } w\},$$

where $L_{\rho_{\mathbb{D}}}(\gamma) := \int_\gamma \rho_{\mathbb{D}}|dz|$ is the $\rho_{\mathbb{D}}$-length of γ. Any hyperbolic Riemann surface X can be endowed with a *hyperbolic distance* d_X induced by the distance $d_{\mathbb{D}}$ on its universal cover.

Lemma 2.3 (Schwarz-Pick)**.** *Let $f : X \to Y$ be a holomorphic map between two hyperbolic Riemann surfaces endowed with their respective hyperbolic distances d_X and d_Y. If f is a covering map, then it is a local isometry. Else, f is local uniform contraction, i.e. for any compact $K \subset X$, there is a contraction factor $r_K \in (0, 1)$ such that $d_Y(f(z), f(w)) \leq r_K d_X(z, w)$ for all $z, w \in K$.*

Definition 2.3. A family $\mathcal{F}$ of holomorphic maps from a Riemann surface X to another surface Y is *normal* if $\mathcal{F}$ is precompact in the compact-open topology. In other words, every sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{F}$ admits a subsequence which converges uniformly on compact subsets.

Theorem 2.4 (Montel)**.** *A family $\mathcal{F}$ of holomorphic maps between hyperbolic Riemann surfaces X and Y is a normal family.*

Definition 2.4. A map $f : \mathbb{D} \to \mathbb{C}$ is a *Schlicht map* if it is univalent, $f(0) = 0$, and $f'(0) = 1$.

Theorem 2.5 (Koebe Distortion)**.** *Let $f : \mathbb{D} \to \mathbb{C}$ be a Schlicht map. Then, for any $|z| \in \mathbb{D}$, if $r = |z|$,*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \qquad \frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}.$$

Corollary 2.6. *The family of Schlicht maps is precompact.*

Proof. The first inequality in the previous theorem provides uniform boundedness on compact subsets, hence, by Montel's Theorem 2.4, the family is normal. □

2.2 Quasiconformal Maps

Holomorphic maps are often too restrictive for our analysis. A generalisation of conformal maps is those which distort angles locally in a controlled manner. Such maps are called quasiconformal maps. This section aims to review the properties of quasiconformal maps as well as their relationship with quasisymmetric maps. Many results stated without proof can be found in [Ahl06] and [LV73].

Definition 2.5. A *K-quasiconformal map* $f : U \to V$ between two open subsets of CC is an orientation preserving homeomorphism such that:

1. f is absolutely continuous on lines in U,
2. the *complex dilatation* $\mu_f(z) := \frac{f_{\bar{z}}(z)}{f_z(z)}$ satisfies $\|\mu_f\|_\infty < \frac{K-1}{K+1}$.

We say that f is a *K-quasiregular* map if it is the composition of a non-constant holomorphic map and a K-quasiconformal map.

Theorem 2.7. *Suppose $f : U \to V$ is a K-quasiconformal map.*

- *(A) If $K = 1$, then f is conformal.*
- *(B) The inverse $f^{-1} : V \to U$ is also K-quasiconformal.*
- *(C) If $g : V \to W$ is a L-quasiconformal map, then the composition $g \circ f : U \to W$ is KL-quasiconformal.*

Remark. Item (A) is popularly known as Weyl's lemma. Items (A) and (C) of the proposition can be generalised to quasiregular maps.

Theorem 2.8. *For any $K \geq 1$, the space of K-quasiconformal maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ fixing* 0, 1 *and* ∞ *is compact.*

Definition 2.6. A *Beltrami coefficient* on an open subset $U \subset \hat{\mathbb{C}}$ is a measurable $\mu \in L^\infty(U)$ where $\|\mu\|_\infty < 1$.

Every quasiconformal map $f : U \to V$ has an associated Beltrami coefficient, which is its complex dilatation μ_f. The following theorem by Ahlfors and Bers gives us the converse.

Theorem 2.9 (Measurable Riemann Mapping Theorem (MRMT))**.** *For any Beltrami coefficient μ on $\hat{\mathbb{C}}$, there is a quasiconformal map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with complex dilatation $\mu_f = \mu$. Moreover, f is unique up to post-composition of biholomorphisms of $\hat{\mathbb{C}}$ (in particular if we require f to fix* 0, 1 *and* ∞*).*

Remark. MRMT also applies to Beltrami coefficients on open domains in $\hat{\mathbb{C}}$. However, the uniqueness criterion may vary depending on the domain.

Definition 2.7. Let $f : U \to V$ be a quasiconformal holomorphic map between open subsets of $\hat{\mathbb{C}}$, μ be a Beltrami coefficient on V and ϕ be the unique quasiconformal map with $\mu_\phi = \mu$ fixing 0, 1, and ∞. The *pullback* of μ via f is defined as $f^*\mu := \mu_{\phi \circ f}$, a Beltrami coefficient on U associated to $\phi \circ f$.

We now turn to a more geometric characterisation of quasiconformal maps using the concept of extremal lengths. Consider a set of paths Γ (curves or arcs) in an open and connected domain $U \subset \mathbb{C}$. We wish to construct a conformal invariant measure of the size of Γ.

Definition 2.8. A measurable function $\rho : U \to [0, \infty)$ is *allowable* for $U \subset \mathbb{C}$ if the ρ-area of U, $A_\rho(U) = \iint_U \rho^2 dxdy$, is non-zero and finite. The set of allowable functions on U is denoted by $\mathcal{A}(U)$.

Let Γ be some family of rectifiable curves in U. For $\rho \in \mathcal{A}(U)$, define the *ρ-length* of Γ to be $L_\rho(\Gamma) = \inf_{\gamma \in \Gamma} L_\rho(\gamma)$, where $L_\rho(\gamma) = \int_\gamma \rho |dz|$ denotes the ρ-length of γ. Set

$L_\rho(\Gamma) = \infty$ if Γ is empty. The *extremal length* of Γ is defined as

$$\lambda(\Gamma) = \sup_{\rho \in \mathcal{A}(U)} \frac{L_\rho(\Gamma)^2}{A_\rho(U)}.$$

Remark. The extremal length can be seen as an average minimal length for a curve family. Notice that the fractional expression is invariant under rescaling of ρ. Sometimes it is convenient to normalise ρ such that $L_\rho(\Gamma) = A_\rho(U)$.

Theorem 2.10. *An orientation preserving homeomorphism $f : U \to V$ is K-quasiconformal if and only if any family of curves Γ in U satisfies*

$$\frac{1}{K}\lambda(f(\Gamma)) \leq \lambda(\Gamma) \leq K\lambda(f(\Gamma)).$$

Corollary 2.11. *Extremal length is a conformal invariant.*

Definition 2.9. A homeomorphism $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is *S-quasisymmetric* if there is $S > 0$ such that for all $x, y, z \in X$,

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq S\frac{d_X(x,y)}{d_X(x,z)}.$$

Example 2.1. Quasisymmetric homeomorphisms are a generalisation of bi-Lipschitz maps. Indeed, any L-bi-Lipschitz map is a L^2-quasisymmetric homeomorphism.

The following theorem asserts the relation between quasisymmetry and quasiconformality. The proof can be found on [AB56].

Theorem 2.12 (Ahlfors-Beurling Extension)**.** *Any S-quasisymmetric homeomorphism $h : \mathbb{R} \to \mathbb{R}$ can be extended to a D-quasiconformal homeomorphism $H : \mathbb{C} \to \mathbb{C}$ such that $H = h$ on $\mathbb{R}$, the dilatation D depends only on S, H is smooth on $\mathbb{C}\backslash\mathbb{R}$, and $h \mapsto H$ is linear.*

Corollary 2.13. *Let h be an S-quasisymmetric homeomorphism on $\mathbb{T}$ onto itself and also on $\mathbb{T}_r$ onto itself, for some $r > 1$. Then, h extends to a D-quasiconformal homeomorphism $H : \mathbb{A}_{1,r} \to \mathbb{A}_{1,r}$ where D depends only on r and S.*

Proof. Consider the covering map $g : \mathbb{H} \to \mathbb{A}_{1,r}, z \mapsto z^{-\frac{i \ln r}{\pi}}$ with deck transformation group generated by $\phi(z) = \lambda z$ where $\lambda = e^{\frac{2\pi^2}{\ln r}}$. The map g can be extended such that $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$ cover $\mathbb{T}$ and $\mathbb{T}_r$ respectively.

The map h lifts to $\tilde{h} : \mathbb{R} \to \mathbb{R}$ where $h(0) = 0$ and $g \circ \tilde{h} = h \circ g$ on $\mathbb{R}^*$. Moreover, $\tilde{h}$ can be chosen so that $\tilde{h}(1), -\tilde{h}(-1) \in [1, \lambda)$ and $\tilde{h}$ commutes with ϕ.

As the function ln has bounded first derivative on $(-\lambda, 1] \cup [1, \lambda)$, it is bi-Lipschitz and thus quasisymmetric. Consequently, g and $\tilde{h}$ are quasisymmetric on $\mathbb{R}^*$. In fact, $\tilde{h}$ is λ^2-quasisymmetric around 0 since for any interval I containing 0, $|I| \leq \tilde{h}(|I|) \leq \lambda^2|I|$.

Thus, $\tilde{h}$ is S'-quasisymmetric on $\mathbb{R}$ where S' depends only on r and S.

By Theorem 2.12, we can extend $\tilde{h}$ to a D-quasiconformal $\tilde{H} : \mathbb{H} \to \mathbb{H}$ where D depends only on S'. By linearity of the extension operator, $\tilde{H}$ commutes with ϕ too. Lift back to get a D-quasiconformal homeomorphism $H : \mathbb{A}_{1,r} \to \mathbb{A}_{1,r}$. □

2.3 Conformal Modulus

Consider a regular annulus $\mathbb{A}_{r_1,r_2}$ and let Γ be the set of all curves on $\mathbb{A}_{r_1,r_2}$ joining the two boundaries $\mathbb{T}_{r_1}$ and $\mathbb{T}_{r_2}$ at the endpoints. Let $\gamma_\theta = \{re^{i\theta} \mid r \in (r_1, r_2)\}$, then

$$2\pi L_\rho(\Gamma) \leq \int_0^{2\pi} L_\rho(\Gamma) d\theta \leq \int_0^{2\pi} L_\rho(\gamma_\theta) d\theta \leq \int_0^{2\pi} \int_{r_1}^{r_2} \rho(re^{i\theta}) dr d\theta.$$

By triangle inequality, we have

$$(2\pi)^2 L_\rho(\Gamma)^2 \leq \iint_{\mathbb{A}_{r_1,r_2}} \frac{1}{r} d\theta dr \iint_{\mathbb{A}_{r_1,r_2}} \rho^2 r d\theta dr = 2\pi \log\left(\frac{r_2}{r_1}\right) A_\rho(\mathbb{A}_{r_1,r_2})$$

Rearranging, we then obtain $\lambda(\Gamma) \leq \log(r_2/r_1)/2\pi$. This upper bound is achieved if we take $\rho(z) = 1/|z|$, so then the extremal length is $\lambda(\Gamma) = \log(r_2/r_1)/2\pi$.

Since extremal length is a conformal invariant, we see that two regular annuli $\mathbb{A}_{r_1,r_2}$ and $\mathbb{A}_{s_1,s_2}$ are biholomorphic if and only if $r_1 s_2 = r_2 s_1$.

Definition 2.10. Define the *conformal modulus* $\mathrm{mod}(A)$ of a topological annulus A as

$$\mathrm{mod}(A) = \frac{1}{2\pi} \log r,$$

where A is biholomorphic to $\mathbb{A}_{1,r}$ for some unique $r > 1$. If $r = \infty$, then set $\mathrm{mod}(A) = \infty$.

Note that in the analysis of annuli, sometimes it is more convenient to work with the set of all closed curves separating the two boundary components of the annulus. A similar computation will tell us that the corresponding extremal length is equal to the reciprocal of the modulus.

The modulus can be thought of as a measure of thickness of an annulus. By definition, it is a conformal invariant, and under quasiconformal homeomorphisms, its change is controlled by the dilatation due to Theorem 2.10. As a conformal invariant, the modulus is a highly valued measure in holomorphic dynamics.

Proposition 2.14. *Let $f : A \to A'$ be a holomorphic covering map of degree $d < \infty$ between two topological annuli A and A' in $\mathbb{C}$, then*

$$\mathrm{mod}(A') = d \, \mathrm{mod}(A).$$

Proof. Since any topological annulus is biholomorphic to some regular annulus, we can assume without loss of generality that f is a covering map from $\mathbb{A}_{1,r}$ to $\mathbb{A}_{1,R}$ for some $r, R > 1$ and that $f(1) = 1$ on the boundary. The map f can be lifted via the universal covers $g_s : \mathbb{H} \to \mathbb{A}_{1,s}, z \mapsto z^{-\frac{i \ln s}{\pi}}$ for $s \in \{r, R\}$, to a unique holomorphic map $\tilde{f} : \mathbb{H} \to \mathbb{H}$ fixing $(2\pi)^k$ for all $k \in \mathbb{Z}$. It turns out that $\tilde{f}$ has to be the identity and consequently

$$f(z) = g_R \circ \tilde{f} \circ g_r^{-1}(z) = z^{\log_r R} \exp\big(2\pi i k \log_r R\big),$$

where each $k \in \mathbb{Z}$ indicates a choice of branch of g_r^{-1}. As the expression must be independent of the choice of $k \in \mathbb{Z}$, then $\log_r R$ must be some positive integer d and f simplifies to $f(z) = z^d$, a holomorphic covering map of degree d. The equation in the proposition follows immediately from $R = z^d$. ☐

In hyperbolic geometry, concentric circles are geodesic curves in a regular annulus.

Definition 2.11. Let A be a topological annulus in $\mathbb{C}$ and let $\phi : \mathbb{A}_{1,r} \to A$ be a biholomorphism. A closed curve $\gamma \subset \mathbb{A}$ is a *geodesic curve* of A if $\gamma = \phi(\mathbb{T}_t)$ for some $t \in (1, r)$. We say that γ is the *core curve* of A if $t = \sqrt{r}$, i.e. the unique geodesic curve splitting A into two annuli of equal moduli.

Proposition 2.15 (Grötzsch Inequality)**.** *If A and B are two topological annuli in $\mathbb{C}$ such that $B \subset A$, then* $\operatorname{mod}(B) \le \operatorname{mod}(A)$*. Moreover, if A_1 and A_2 are two disjoint topological annuli in $\mathbb{C}$, then for any annulus A such that $A_1 \cup A_2 \subset A$,*

$$\operatorname{mod}(A_1) + \operatorname{mod}(A_2) \le \operatorname{mod}(A).$$

Proof. If $\operatorname{mod}(B) = \infty$, then both A and B are punctured disks and the result is trivial. Therefore, assume that $\operatorname{mod}(B) \in (0, \infty)$. Consider curve families Γ_A and Γ_B consisting of closed curves separating the two boundary components of A and B respectively. Pick any arbitrary $\rho_A \in \mathcal{A}(A)$ and let ρ_B be its restriction on B, then $A_{\rho_A}(A) \ge A_{\rho_B}(B)$. Since $\Gamma_B \subset \Gamma_A$, it follows that $L_{\rho_B}(\Gamma_B) \ge L_{\rho_A}(\Gamma_A)$, so then $\lambda(\Gamma_B) \ge \lambda(\Gamma_A)$. The modulus is the reciprocal of the extremal length of this curve family, hence we have proven our first statement.

To prove the second, it is sufficient to consider the case the inner boundary of the annulus A_1 is the same as the outer boundary of another A_2 and let $A = \operatorname{int}(\overline{A_1 \cup A_2})$ be obtained by gluing the two. Assume as well that $\operatorname{mod}(A_1) = \operatorname{mod}(A_2) \in (0, \infty)$.

Consider three curve families Γ, Γ_1 and Γ_2 consisting of paths joining the inner and outer boundaries of A, A_1 and A_2 respectively. For any allowable $\rho \in \mathcal{A}(A)$, we denote by ρ_1 and ρ_2 its restrictions to $\mathcal{A}(A_1)$ and $\mathcal{A}(A_2)$, then $A_\rho(A) \ge A_{\rho_1}(A_1) + A_{\rho_2}(A_2)$. Pick any curve $\gamma \in \Gamma$ and subcurves γ_i of γ in Γ_i for $i = 1, 2$. Taking the infimum across all curves in Γ, we have

$$L_\rho(\gamma) \ge L_\rho(\gamma) \ge L_{\rho_1}(\gamma_1) + L_{\rho_2}(\gamma_2) \ge L_{\rho_1}(\Gamma_1) + L_{\rho_2}(\Gamma_2)$$

Assume w.l.o.g. that ρ_i is normalised, i.e. $L_{\rho_i}(\Gamma_i) = A_{\rho_i}(A_i)$ for $i = 1, 2$, then it is immediate that $\lambda(\Gamma) \ge \lambda(\Gamma_1) + \lambda(\Gamma_1)$. ☐

Remark. From the proof, it is obvious that we can generalise the proposition to arbitrary curve families. The precise statement can be found from [Ahl06].

Proposition 2.16. *Let $A \subset \mathbb{C}$ be a topological annulus with inner and outer boundaries I and O. Then,*

$$\operatorname{mod}(A) \leq \frac{\pi}{4} + \frac{\operatorname{dist}(I, O)}{\operatorname{diam} I}.$$

Proof. Rescale for convenience such that $\operatorname{diam} I = 1$ and $\operatorname{dist}(I, O) = d > 0$. Let $a \in O$ and $b \in I$ such that $|a - b| = d$. There exists some $c \in I$ such that $|b - c| \geq \frac{1}{2}$. Consider a family of simple closed curves Γ separating I from $\{a, \infty\}$. Then, the extremal length of $\lambda(\Gamma)$ satisfies $\operatorname{mod}(A) \leq \lambda(\Gamma)^{-1}$.

Let $\rho : \mathbb{C} \to \{0, 1\}$ be the characteristic function of $U := \{z \in \mathbb{C} \mid \operatorname{dist}(z, [a, b]) < \frac{1}{2}\}$, the $\frac{1}{2}$-neighbourhood of the line segment $[a, b]$. Then, any $\gamma \in \Gamma$ has to pass through U as well as $[a, b]$ and consequently $L_\rho(\Gamma) \geq 1$. Thus,

$$\lambda(\Gamma) \geq \frac{L_\rho(\Gamma)}{A(\rho)} \geq \frac{1}{\frac{\pi}{4} + d}.$$

Combining the two inequalities, $\operatorname{mod}(A) \leq \frac{\pi}{4} + d$. □

Lemma 2.17. *Suppose K is a compact, simply connected subset of the unit disk $\mathbb{D}$ containing 0 with $\operatorname{mod}(\mathbb{D} \backslash K) \geq \mu$ for some $\mu > 0$. Then, there is a radius $r_\mu \in [0, 1)$ depending only on μ such that $K \subset \mathbb{D}_r$.*

Proof. Suppose instead that we have a sequence of compact subsets K_n all satisfying the assumption for the same μ and $\sup\{|z| : z \in K_n\} \to 1$. Take a sequence of biholomorphisms $\phi_n : \mathbb{A}_{\delta_n, 1} \to \mathbb{D} \backslash K$ where $\phi_n(\partial \mathbb{D}) = \partial \mathbb{D}$ and $\delta_n \leq r_0 := e^{-2\pi\mu} < 1$. By Montel's theorem, the sequence ϕ_n restricted to $\mathbb{A}_{r_0, 1}$ is normal, so it has a subsequence ϕ_{n_i} compactly converging to some ϕ on compact subsets. Take the core curve γ of $\mathbb{A} - r_0, 1$, then $\phi(\gamma)$ separates K_{n_i} from $\partial \mathbb{D}$ for sufficiently large i. This is a contradiction. □

Lemma 2.18. *Let $U \Subset V$ be a pair of simply connected open subsets of $\mathbb{C}$ and let $\operatorname{mod}(V \backslash \overline{U}) \geq \mu > 0$. Then, any univalent $f : V \to \mathbb{C}$ has a bounded distortion on U depending only on μ, i.e. there is C_μ such that for all $z, w \in U$,*

$$|f'(z)| \leq C_\mu |f'(w)|.$$

Proof. Let $g : \mathbb{D} \to V$ be a Riemann map such that $g(0) \in U$. Let $\tilde{U} = g^{-1}(\overline{U})$, then by Lemma 2.17, $\tilde{U}$ is contained in $\mathbb{D}_{r_\mu}$. By Koebe distortion, g and $f \circ g$ have distortion bounded by some $\tilde{C}_\mu$ on $\tilde{U}$. By chain rule, f has distortion bounded by $\tilde{C}_\mu^2$ on U. □

Definition 2.12. Let $U \subset \mathbb{C}$ be an open subset. The *inner radius* $r_{U,z}$ and the *outer radius* $R_{U,z}$ of U about a point $z \in U$ are

$$r_{U,z} := \sup\{r > 0 \mid \mathbb{D}_r(z) \subset U\}, \qquad R_{U,z} := \inf\{R > 0 \mid U \subset \mathbb{D}_R(z)\}.$$

The *eccentricity* of U at z is the ratio $R_{U,z}/r_{U,z}$.

Lemma 2.19. *Let $f : U \to V$ be a D-quasiconformal homeomorphism between two open subsets of $\mathbb{C}$. Let an open ball $\mathbb{D}_t(z) \subset U$ satisfy $\mathbb{D}_R(f(z)) \subset V$ where R is the outer radius of $f(\mathbb{D}_t(z))$ about $f(z)$, then $f(\mathbb{D}_t(z))$ has eccentricity bounded by a constant depending only on D.*

Proof. Let r and R be the inner and outer radii of $f(\mathbb{D}_t(z))$. Label $w_1, w_2 \in \partial\mathbb{D}_t(z)$ such that $|f(w_1) - f(z)| = r$ and $|f(w_2) - f(z)| = R$. Let $A = \{w \in V : r < |w - f(z)| < R\}$. Let I and O be the inner and outer boundaries of $f^{-1}(A)$, then $\mathrm{diam} I \geq t$ and $\mathrm{dist}(I, O) \leq t$. By Proposition 2.16, $f^{-1}(A)$ has modulus bounded by $\pi/4 + 1$. In short,

$$\frac{1}{2\pi} \log \frac{R}{r} = \mathrm{mod}(A) \leq D \ \mathrm{mod}\big(f^{-1}(A)\big) \leq D\Big(\frac{\pi}{4} + 1\Big).$$

As such, the eccentricity is bounded by $\exp\big(2\pi D(\frac{\pi}{4} + 1)\big)$. $\square$

2.4 Quasicircles and Quasidisks

Definition 2.13. A *C-quasicircle* is a Jordan curve γ in $\mathbb{C}$ such that γ is the image of a circle S^1 under a C-quasiconformal homeomorphism $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. We will call the Jordan domain bounded by some C-quasicircle γ a *C-quasidisk.*

Definition 2.14. Let γ be a Jordan curve in $\mathbb{C}$. Define a metric $dd : \gamma \times \gamma \to [0, \infty)$ on γ by setting $dd(x, y)$ as the minimum diameter of a subarc joining x and y. We say that γ has *C'-bounded turning* if for all $x, y \in \gamma$. $dd(x, y) \leq C'|x - y|$.

Lemma 2.20. *Let γ be a Jordan curve in $\mathbb{C}$. If there is some $\epsilon > 0$ such that for all $x, y \in \gamma$, the subarc $\gamma_{x,y}$ joining x and y satisfies* $\mathrm{diam}(\gamma_{x,y}) \leq C'|x - y|$ *whenever $|x - y| \leq \epsilon$, then γ has $C'N$-bounded turning, where N is the minimum number of subarcs of diameter ϵ needed to cover γ.*

Proof. If $|x-y| \leq \epsilon$, then obviously $dd(x, y) \leq C'|x-y|$ by the assumption. If $|x-y| > \epsilon$, then we can pick a number of points $x_0, x_1, \dots x_m$ along γ where $x_0 = x$, $x_m = y$ and $|x_i - x_{i+1}| \leq \epsilon$ for some $m \leq N$. By the triangle inequality,

$$\mathrm{diam}(\gamma_{x,y}) \leq \sum_{i=1}^{m} \mathrm{diam}(\gamma_{x_{i-1},x_i}) \leq \sum_{i=1}^{m} C'|x_{i-1} - x_i| \leq \epsilon m C'$$

Applying $\epsilon < |x - y|$ and $m \leq N$, we then have our desired inequality. $\square$

Proposition 2.21. *Any* 0*-symmetric Jordan domain* D *bounded by a* C'*-bounded turning curve has bounded eccentricity with constant* $2C' + 1$*.*

Proof. Let r and R be the inner and outer radii of D about 0 and let $x, y \in \partial D$ be such that $|x| = r$ and $|y| = R$. By symmetry and bounded turning condition, $dd(x, y) \leq dd(y, -y)$ and $dd(y, -y) \leq 2C|y|$. Consequently,

$$\frac{R}{r} = \frac{|x|}{|y|} \leq \frac{|x-y|}{|y|} + 1 \leq \frac{dd(x,y)}{\frac{1}{2C} dd(y,-y)} + 1 \leq 2C + 1.$$

The inequality above gives us the desired result. □

It turns out that the bounded turning condition is just a more geometric way of characterising quasicircles.

Theorem 2.22. *A Jordan curve* γ *is a* C*-quasicircle if and only if it has* C'*-bounded turning. Moreover,* C *and* C' *depend only on each other.*

Refer to [LV73] or [Ahl06] for the details of the proof of the theorem. One particular example of quasicircles that will render useful is the following.

Proposition 2.23. *The core curve of an annulus* A *where* $\operatorname{mod}(A) \geq \mu > 0$ *is a* $C(\mu)$*-quasicircle.*

Proof. Consider a biholomorphism $\phi : \mathbb{A}_{1,r^2} \to A$ where $r = \exp\big(\pi \operatorname{mod}(A)\big)$ and pick an arbitrary point x along the core curve $\mathbb{T}_r$. The disk $\mathbb{D}_{r-1}(x)$ contains a subarc of $\mathbb{T}_r$ of some angle $\theta(r)$. By Koebe 1/4, the open disk $\mathbb{D}_{\frac{1}{4}(r-1)|\phi'(x)|}(\phi(x))$ is a subset of the image $\phi(\mathbb{D}_{r-1}(x))$.

Take $\epsilon(r) = \frac{1}{8}(r-1)\min_{|z|=r}|\phi'(z)|$, and pick any $y \in \mathbb{T}_r$ such that $|\phi(x) - \phi(y)| \leq \epsilon$. By Koebe distortion, there is some $R = R(r) > 0$ and $c = c(r) > 0$ such that for all $z \in \mathbb{D}_R(x)$,

$$\phi(z) \in \mathbb{D}_\epsilon(\phi(z)), \quad |\phi'(z)| \leq c|\phi'(x)|.$$

The second inequality gives us a bound on the length of subarc $\gamma_{\phi(x),\phi(z)}$. Specifically, if $L_{x,z}$ denotes the arc length of the subarc of $T_r \cap \mathbb{D}_R(x)$ joining x and r, then $L(\gamma_{\phi(x),\phi(z)}) \leq c|\phi'(x)|L_{x,r}$. The ratio of arc to chord of a circle is always bounded by $\pi/2$, so then we can improve our inequality to

$$L(\gamma_{\phi(x),\phi(z)}) \leq \frac{\pi}{2} c|\phi'(x)||x - z|. \tag{2.1}$$

Again, Koebe 1/4 on $\mathbb{D}_{|x-z|}(x)$ yields

$$|\phi'(x)||x - z| \leq 4|\phi(x) - \phi(z)|. \tag{2.2}$$

Combining (2.1) and (2.2) gives us $\operatorname{diam}(\gamma_{\phi(x),\phi(z)}) \leq C'|\phi(x) - \phi(z)|$ whenever $|\phi(x) - \phi(z)| \leq \epsilon$, where $C' = C'(r)$. As x and z are arbitrary, Lemma 2.20 gives our desired result. □

Chapter 3

Holomorphic Dynamics

We will review the basic theory of rational dynamics. Most of the results in this chapter can be found in classical textbooks, e.g. [Mil11] and [CG05], as well as Douady and Hubbard's Orsay notes [DH].

3.1 Dynamics of Rational Maps

Rational maps make up $\text{Hol}(\hat{\mathbb{C}})$, the space of all holomorphic maps from $\hat{\mathbb{C}}$ to itself. Möbius maps, i.e. rational maps of degree 1, make up the space of all biholomorphisms from $\hat{\mathbb{C}}$ to itself. The dynamics of Mobius maps are very well understood and rather uninteresting. From now on all rational maps are taken to be of degree ≥ 2.

Definition 3.1. Let $f \in \text{Hol}(\hat{\mathbb{C}})$. The *forward orbit* of a point $z \in \hat{\mathbb{C}}$ is the sequence

$$O_f^+(z) := \{f^n(z) \mid n \geq 0\},$$

and the *backward orbit* of z is the set

$$O_f^-(z) := \{w \mid f^n(w) = z \text{ for some } n \geq 0\}.$$

Definition 3.2. Let $f \in \text{Hol}(\hat{\mathbb{C}})$. A point z_0 is a *periodic point* of f of period p if $f^p(z_0) = z_0$ for some positive integer p. We say that z is *preperiodic* if $f^{p+m}(z_0) = f^m(z_0)$ for some positive integers p and m.

Remark. We see that the forward orbit $O_f^+(z)$ is finite if and only if z is a periodic or preperiodic point of f.

Definition 3.3. The *multiplier* of a periodic point z_0 of period p is the value $\lambda := (f^p)'(z_0)$. If $z_0 = \infty$, we can define the derivative on the local chart $z \to \frac{1}{z}$ by letting $\lambda := (g^p)'(0)$ where $g(z) = f(z^{-1})^{-1}$. We can classify periodic points according to its multiplier:

$\lvert\lambda\rvert$	z_0
0	superattracting
< 1	attracting
1	indifferent
> 1	repelling

Additionally, we say that z_0 is *parabolic* if λ is a root of unity and *irrationally indifferent* if otherwise.

The classification above is in sync with the local topological dynamics near the periodic point.

Proposition 3.1. *Let z_0 be a periodic point of f of period p. Then, z_0 is attracting (possibly superattracting) if and only if for any open neighbourhood U of z_0 there is an open neighbourhood U of z_0 such that*

$$f^p(U) \subset U \text{ and for all } z \in U, \lim_{n\to\infty} f^{np}(z) = z_0.$$

Moreover, z_0 is repelling if and only if there is an open neighbourhood V of z_0 such that

$$\text{for all } z \in V\backslash\{z_0\}, \text{ there is some } n_z \in \mathbb{N} \text{ such that } f^{n_z}(z) \notin V.$$

Example 3.1. Any polynomial f of degree $d \geq 2$ has a superattracting fixed point at ∞. Indeed, its multiplier is

$$\lambda := \frac{d}{dz} f'(z^{-1})^{-1}\bigg|_{z=0} = \frac{f'(z^{-1})}{z^2 f(z^{-1})^2}\bigg|_{z=0} = 0.$$

It turns out that in the case where a periodic point z_0 of a map f is not indifferent, f is locally holomorphically conjugate to either a linear map $z \mapsto \lambda z$ or a power map $z \mapsto z^d$.

Theorem 3.2 (Koenigs). *Let $f : U \to V$ be a holomorphic map with fixed point z_0 with multiplier λ. If z_0 is attracting or repelling, i.e. $|\lambda| \notin \{0, 1\}$, then there is an open neighbourhood W of z_0 and a univalent map $\phi : f(W) \to \mathbb{C}$ such that $\phi(z_0) = 0$, $W \Subset f(W)$, and $\phi \circ f \circ \phi^{-1}(z) = \lambda z$ for all $z \in \phi(W)$. The map ϕ is called the lineariser of f about z_0 and it is unique up to multiplication by a nonzero constant.*

Theorem 3.3 (Böttcher). *Let $f : U \to V$ be a holomorphic map with a superattracting fixed point z_0 of order d, i.e. $f - z_0$ has a zero at z_0 of order d. Then, there is an open neighbourhood W of p and a univalent map $\phi : f(W) \to \mathbb{C}$ such that $\phi(z_0) = 0$, $W \Subset f(W)$, and $\phi \circ f \circ \phi^{-1}(z) = z^d$ for all $z \in \phi(W)$. The map ϕ is called the Böttcher coordinate of f at z_0 and it is unique up to multiplication by a $(d-1)$-th root of unity.*

Definition 3.4. The *Fatou set* $F(f)$ of f is the set of points $z \in \hat{\mathbb{C}}$ such that $\{f^n\}_{n\in\mathbb{N}}$ is normal on some neighbourhood of z. The *Julia set* of f is the complement $J(f) := \hat{\mathbb{C}} \backslash F(f)$.

We will summarise some important and nontrivial properties of the Julia set.

Theorem 3.4. *Let f be a rational map of degree $d \geq 2$.*

(A) The Julia set $J(f)$ is completely invariant, i.e. $f^{-1}(J(f)) = J(f) = f(J(f))$.

(B) The set of repelling periodic points of f is dense in $J(f)$.

(C) The backward orbit $O^-_f(z)$ of any point $z \in J(f)$ is dense in $J(f)$.

(D) For any $z \in J(f)$ and any neighbourhood U of z, $J(f) \subset f^N(U)$ for sufficiently large $N \in \mathbb{N}$.

Definition 3.5. Let z_0 be an attracting or superattracting periodic point of f of period p. We define the *basin of attraction* of the attracting cycle $\{f^k(z_0)\}_{k=0,1,\dots p-1}$ by the set

$$\mathcal{A}_f(z_0) := \{z \mid f^{np}(z) \to f^k(z_0) \text{ for some } k\}.$$

The *immediate basin of attraction* of the cycle is the union of p connected components of $B_f(z_0)$ containing $f^k(z_0)$ for some k. The basin of attraction is completely invariant open subset of the Fatou set of f.

Theorem 3.5 (Sullivan)**.** *Let U be a connected component of the Fatou set $F(f)$ of a rational map f of degree $d \geq 2$. Then, U is either periodic or preperiodic, i.e. $f^{p+m}(U) = f^m(U)$ for some integers $p > 0$ and $m \geq 0$. Moreover, if $m = 0$, exactly one of the following holds:*

(A) U is attracting: there is a unique attracting periodic point of period p in U;

(B) U is parabolic: there is a parabolic periodic point z_0 of period p on ∂U such that $f^{np}(z) \to z_0$;

(C) U is a Siegel disk: $f^p|_U$ is conformally conjugate to some irrational rotation of the unit disk $\mathbb{D}$ about 0;

(D) U is a Herman ring: U is a topological annulus and $f^p|_U$ is conformally conjugate to some irrational rotation on a regular annulus.

The first part of the theorem is known as Sullivan's theorem of no wandering domains ([Sul85]). In proving this theorem, Sullivan pioneered the use of quasiconformal maps in holomorphic dynamics which has now led to many other important results.

To complete the picture, we would like to point out that a rational map f may not be linearisable around an indifferent periodic point z_0. In the parabolic case, Leau-Fatou flower theorem gives us a conjugacy with translation $z \mapsto z + 1$ on the parabolic basin. If z_0 is irrationally indifferent, then if z_0 lies in $F(f)$, it must lie in the Siegel disk and we have some conjugacy described in case (C) of the theorem above. In fact, this is the only case where an irrationally indifferent point z_0 is linearisable.

Definition 3.6. An irrationally indifferent periodic point z_0 of period p of a rational map f is called a *Cremer point* if and only if $z_0 \in J(f)$.

Proposition 3.6. *Every immediate basin of an attracting or parabolic periodic point of a rational map f contains at least one critical point.*

Since a rational map of degree d has at most $2d-2$ critical points, we have an upper bound on the number of periodic cycles of f. The following theorem gives a sharp upper bound on the number of non-repelling cycles.

Theorem 3.7 (Fatou-Shishikura Inequality). *Any rational map $f \in \mathrm{Hol}(\hat{\mathbb{C}})$ of degree d has at most $2d-2$ non-repelling periodic cycles.*

The proof of the theorem relies on quasiconformal surgery, a technique also used by Sullivan to prove the nonexistence of wandering domains for rational maps. We will use similar surgery techniques later on in Theorem 4.3.

Definition 3.7. The *postcritical set* $P(f)$ is the closure of the forward orbit of critical values of f. A map f is *postcritically finite* if $|P(f)| < \infty$.

Proposition 3.8. *Let f be a rational map with some irrationally indifferent periodic point z_0.*

(A) If z_0 lies in a Siegel disk U, then $\partial U \subset P(f)$.
(B) If z_0 is a Cremer point, then $z_0 \in P(f)$.

3.2 Dynamics of Quadratic Maps

Definition 3.8. The *filled Julia set* $K(f)$ of a non-constant polynomial f is the complement of the attracting basin of infinity $\mathcal{A}_f(\infty)$ in $\hat{\mathbb{C}}$.

As $\mathcal{A}_f(\infty)$ is an open neighbourhood of ∞, $K(f)$ is compact in $\mathbb{C}$ and completely invariant under f.

The following is a list of important results on polynomial dynamics. The proof relies on the maximum modulus principle and Montel's theorem.

Proposition 3.9. *If f is a polynomial of degree $d \geq 2$ with filled Julia set $K(f)$,*

(A) f has no Herman rings;
(B) $K(f)$ is full;
(C) $\partial K(f) = J(f)$.

The following is an improvement of Böttcher's theorem in the context of polynomials.

The details are explained thoroughly in [Mil11, §9].

Theorem 3.10. *Let f be a polynomial of degree d and let the filled Julia set $K(f)$ contain all finite critical points of f. Then, if ϕ is a Böttcher coordinate of f at ∞, then ϕ extends to a biholomorphism $\phi : \mathcal{A}_f(\infty) \to \mathbb{D}$.*

We will focus our attention to the dynamics of quadratic maps. Quadratic maps enjoy certain properties which polynomials of higher degrees do not. For instance, the uniqueness criterion in Böttcher's theorem implies that the Böttcher coordinate of a quadratic map is unique.

Definition 3.9. In the context of Theorem 3.10, we define the *Böttcher map* of a quadratic map f as the unique biholomorphism $B_f : \mathbb{C}\backslash K(f) \to \mathbb{C}\backslash\mathbb{D}, z \mapsto \phi(z)^{-1}$ satisfying the conjugacy $B_f \circ f(z) = B_f(z)^2$.

Having only one finite critical point, quadratic maps also satisfy a stronger version of Fatou-Shishikura inequality.

Corollary 3.11. *Any quadratic map f has at most one finite non-repelling periodic cycle in $\mathbb{C}$.*

Therefore, it makes sense for us to say that a quadratic map is *attracting* or *indifferent* when it has a finite attracting or indifferent cycle, or *repelling* when otherwise.

Proposition 3.12. *Any quadratic map $f(z) = az^2 + bz + d$ is conformally conjugate to a unique quadratic f_c of the form $f_c(z) = z^2 + c$.*

Proof. Set $c := ad + \frac{b}{2}\left(1 - \frac{b}{2}\right)$. The affine map $g(z) = az + \frac{b}{2}$ satisfies $g \circ f = f_c \circ g$. $\square$

To determine the dynamics of all quadratic maps, it is equivalent to studying those of the form $f_c(z) = z^2 + c$. All quadratic maps will now be assumed to be of the form f_c, unless otherwise stated.

The map f_c has a unique finite critical point 0 and critical value c. If 0 lies in $K(f_c)$, then the Böttcher map is well-defined and we can say more about the topology of $K(f_c)$.

Theorem 3.13 (Dichotomy Theorem)**.** *Let f_c be a qudaratic map with filled Julia set $K(f_c)$. If $0 \in K(f_c)$, $K(f_c)$ is connected. Else, $K(f_c)$ is a Cantor set.*

Definition 3.10. The *Mandelbrot set* $\mathbb{M}$ is the set of all parameters $c \in \mathbb{C}$ such that the filled Julia set $K(f_c)$ is connected.

Remark. Equivalently, we can say that $c \in \mathbb{M}$ if and only if $K(f_c)$ is a hull if and only if $f_c^n(0) \not\to \infty$ as $n \to \infty$.

Proposition 3.14. *$\mathbb{M} := \{c \in \mathbb{C} \,|\, |f_c^n(0)| \leq 2$ for all $n \in \mathbb{N}\}$ and in particular, $\mathbb{M} \subset \overline{\mathbb{D}_2}$. If $c \in \overline{\mathbb{D}_2}$, then the Julia set also satisfies $J(f_c) \subset \overline{\mathbb{D}_2}$.*

Proof. Suppose $|c| = 2+\epsilon$ for some $\epsilon > 0$. Claim that $|f_c^n(0)| \geq 2+2^n\epsilon$ for all n. Indeed, we can prove this inductively. If it's true for $k-1$, then

$$|f_c^k(0)| = |f_c^{k-1}(0)^2 + c| \geq (2+2^{k-1}\epsilon)^2 - |c| \geq 4 + 2^{k+1}\epsilon - 2 - \epsilon \geq 2 + 2^k\epsilon.$$

Thus, $f_c^n(0) \to \infty$ as $n \to \infty$, and $\{|z| > 2\} \subset \hat{\mathbb{C}}\backslash\mathbb{M}$.

Suppose now that $|c| \leq 2$ and $|z| = 2+\epsilon$ for some $\epsilon > 0$. Similar to above, we can use triangle inequality to inductively prove that $|f_c^n(z)| \geq 2+2^n\epsilon$ and consequently conclude that $f_c^n(z) \to \infty$ as $n \to \infty$. ☐

The proposition gives rise to the *escape time* algorithm, one of the simplest procedures to illustrate the Mandelbrot set, as well as all connected Julia sets of quadratic maps up to biholomorphism simply by plotting points lying in the closed disk $\overline{\mathbb{D}_2}$ which does not escape outside $\overline{\mathbb{D}_2}$ for a high number of iterates.

The following is a theorem by Douady and Hubbard.

Theorem 3.15. *The map $\Phi : \hat{\mathbb{C}}\backslash\mathbb{M} \to \hat{\mathbb{C}}\backslash\overline{\mathbb{D}}, c \mapsto B_{f_c}(c)$, where B_{f_c} denotes the Böttcher map of f_c, is a biholomorphism.*

Corollary 3.16. *The Mandelbrot set $\mathbb{M}$ is a hull in $\mathbb{C}$.*

Proof. We know that $\hat{\mathbb{C}}\backslash\mathbb{M}$ is a simply connected open set containing ∞, so $\mathbb{M}$ must be connected, compact, and in particular full. Non-degeneracy is obvious as $\hat{\mathbb{C}}\mathbb{M}$ cannot be biholomorphic to the complex plane. ☐

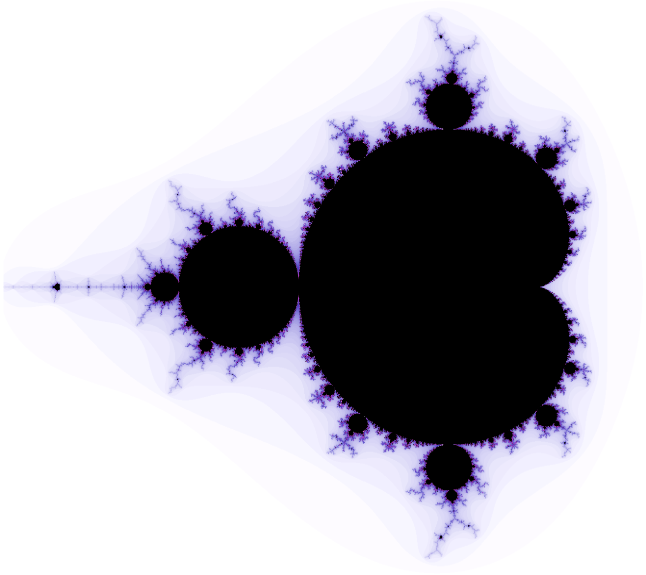

Figure 3.1: The Mandelbrot set $\mathbb{M}$ is rendered using the escape time algorithm.

Definition 3.11. A rational map $f \in \mathrm{Hol}(\hat{\mathbb{C}})$ of degree $d \geq 2$ is *hyperbolic* if every critical point of f lies in some attracting basin.

Proposition 3.17. *A quadratic map f_c where is hyperbolic if and only if either $c \notin \mathbb{M}$ or f_c has a finite attracting cycle.*

Proof. If $c \notin \mathbb{M}$, the critical point 0 lies in the basin of infinity and hyperbolicity is obvious. Suppose $c \in \mathbb{M}$. If f_c is hyperbolic, then 0 must tend to some finite attracting cycle which is not ∞. If f_c has a finite attracting cycle, then by Proposition 3.6, it must contain some critical point. Since 0 is the only finite critical point, this has to be 0. ☐

By fullness, the connected components of $\mathrm{int}(\mathbb{M})$ are topological disks. When $c \in \mathbb{M}$ is a hyperbolic parameter, the postcritical set $P(f)$ lies entirely in the attracting basin of a finite attracting cycle. It is not difficult to show that hyperbolicity is preserved under small pertubations of c, and thus hyperbolic parameters are contained in $\mathrm{int}(\mathbb{M})$.

Definition 3.12. A connected component H of $int(\mathbb{M})$ is *hyperbolic* if all parameters in H are hyperbolic. Otherwise, H is called *queer.*

Proposition 3.18. *Let H be a connected component of* $\mathrm{int}(\mathbb{M})$*.*

(A) If H is queer, H contains no hyperbolic parameters.

(B) If H is hyperbolic, H contains a unique parameter c such that 0 *is a superattracting periodic point of f_c of some period p.*

Definition 3.13. In the case (B) above, we call such a parameter c a *superstable* parameter or *centre* of H, and p the *period* of H.

3.3 External Rays

Let K be a hull in $\mathbb{C}$. By Riemann mapping theorem, we have a biholomorphism $\phi : \mathbb{C}\backslash K \to \mathbb{C}\backslash\overline{\mathbb{D}}$. Pull back via ϕ the foliations of geodesic rays and potentials in $\mathbb{C}\backslash\overline{\mathbb{D}}$ to $\mathbb{C}\backslash K$ external rays and equipotentials.

Definition 3.14. An *external ray* for K of external angle θ is of the form

$$R_\theta = \{\phi^{-1}(re^{2\pi i\theta}) \mid r \in (1, \infty)\},$$

and an *equipotential* for K of radius $r > 1$ is of the form

$$E_r = \{\phi^{-1}(re^{2\pi i\theta}) \mid \theta \in [0, 1)\}.$$

A point $x \in \partial K$ is a *landing point* of K if there exists an external ray R_θ such that $\phi^{-1}(re^{2\pi i\theta}) \to x$ as $r \to 1$.

The following is a result by Douady.

Theorem 3.19 (Landing Theorem). *Suppose $K \subset \mathbb{C}$ is a hull and $x \in \partial K$ is a landing point of n external rays. Then, $K \backslash \{x\}$ has n components.*

Now, consider a polynomial $f(z)$ of degree d with connected filled Julia set $K(f)$ (in particular, this set is a hull). The Böttcher map $B_f : \mathbb{C}\backslash K(f) \to \mathbb{C}\backslash\overline{\mathbb{D}}$ is a biholomorphism and it induces foliations of external rays and equipotentials for $K(f)$. f will act on these foliations by $f(R_\theta) = R_{2\theta}$ and $f(E_r) = E_{r^2}$.

An external ray R_θ is *periodic* when there is some positive $m \in \mathbb{N}$ where $f^m(R_\theta) = R_{d^m\theta} = R_\theta$. If so, the angle θ must be rational of the form $\frac{d^k}{d^m-1}$ where $k = 0, 1, \dots m-1$.

Theorem 3.20. *Let f be a polynomial with connected filled Julia set $K(f)$. Any periodic external ray lands on $J(f)$ at a repelling or parabolic periodic point of f. Moreover, any repelling or parabolic periodic point x is a landing point of m external rays, where m is the number of components of $K(f)\backslash\{x\}$.*

Now consider a quadratic map $f(z) = z^2 + c$ with connected Julia set. The external ray R_0 of $K(f)$ of angle 0 is a fixed ray under f and it must land at a repelling or parabolic fixed point. We call it the β fixed point, or the zero angle fixed point.

If $c = \frac{1}{4}$, then β is the only fixed point of f. Otherwise, we have another fixed point, and we shall call it α. β is not a dividing fixed point, and if α is parabolic or repelling, it is dividing and it disconnects $K(f)$ into a number of components equal to the number of external rays landing at α.

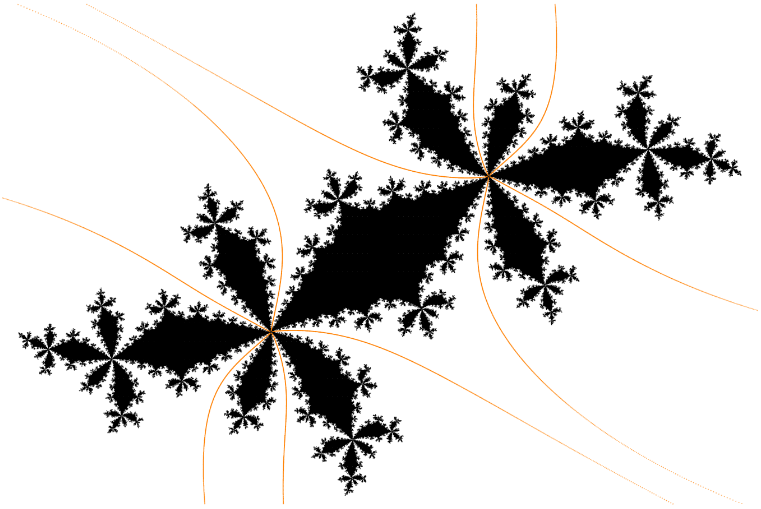

Figure 3.2: External rays landing at the α fixed point of $f(z) = z^2 - 0.52 - 0.58i$ and its preimage.

Lemma 3.21. *Let f be a quadratic map. The critical point 0 and the β fixed point lie in different components of $K(f)\backslash\{-\alpha\}$.*

Proof. Let the external rays landing at α divide the plane into q open sectors S_i for $i = 1, \dots q$. Let S_q be the unique one containing 0, then we can label other sectors such that f univalently maps each S_i onto S_{i+1} where $i < q$. Clearly, the β fixed point must lie in S_q, since $S_q \subset f(S_q)$. The external rays landing at $-\alpha$ will divide the S_q into $q-1$ components, namely S_i' for $i = 1, \dots q-1$ and an infinite strip S_q' containing 0. As $f(S_q') = S_1$, β must lie in one of the sectors S_i'. ☐

Theorem 3.22 (Carathéodory Theorem)**.** *Let f be a polynomial of degree $d \geq 2$ with connected filled Julia set $K(f)$. The following are equivalent:*

- *(A) $K(f)$ is locally connected;*
- *(B) the inverse Böttcher map B_f extends continuously to $B_f : \mathbb{C}\backslash\mathbb{D} \to \mathbb{C}\backslash\text{int}K(f)$;*
- *(C) every external ray R_θ lands at some point on $z(\theta) \in \partial K(f)$.*

Remark. Items (A) and (B) can be generalised to arbitrary hulls in $\mathbb{C}$. See [DH, Chapter 2 §3].

The theorem hints at the extreme importance of local connectivity of these dynamical objects. Specifically, it enables us to extend our knowledge on the dynamics in $\mathbb{C}\backslash K(f)$ to the Julia set itself and retrieve a complete dynamical information on the whole dynamical plane. Local connectivity of $\mathbb{M}$ also enables us to deduce more dynamical information on the maps f_c where $c \in \partial\mathbb{M}$. Thus, the MLC is a natural and vital problem for dynamicists to solve.

Conjecture (MLC)**.** *The Mandelbrot set $\mathbb{M}$ is locally connected.*

The MLC conjecture is one of the central problems in the study of holomorphic dynamics. In particular, the MLC implies the conjecture of density of hyperbolic parameters in the Mandelbrot set.

Chapter 4

Quadratic-Like Maps and Renormalisation

4.1 Polynomial-Like Maps

Recall that a non-constant polynomial of degree d is a branched covering map of degree d having $d-1$ finite critical points counting multiplicity. Moreover, by basic complex analysis, any entire function $f : \mathbb{C} \to \mathbb{C}$ is proper if and only if f is a non-constant polynomial. In this section, we wish to introduce a topological generalisation of polynomials following [DH85] and [McM94a, Chapter 5].

Definition 4.1. A *polynomial-like* map $f : U \to V$ of degree d is a proper holomorphic branched covering map of degree d such that both U and V are topological disks where $U \Subset V \subset \mathbb{C}$. The map $f : U \to V$ is a *quadratic-like* map if it is a polynomial-like map of degree 2.

The Riemann-Hurwitz formula can be used to show that any polynomial-like map of arbitrary degree d must have $d-1$ critical points counting multiplicity. In particular, any quadratic-like map $f : U \to V$ has a unique critical point.

From now on, unless otherwise stated, we shall normalise any quadratic-like map f such that its critical point is 0, the domains U and V are 0-symmetric and f is an even function. This allows us to express f as a composition $h \circ f_0$, where $f_0(z) = z^2$ is the doubling map on U and $h : f_0(U) \to V$ is a biholomorphism.

Example 4.1. Let $f : U \to V$ be a quadratic-like map such that $f^n(0) \in V$ for some integer $n > 1$. Then, $f^n : f^{-n+1}(U) \to V$ is a polynomial-like map of degree 2^n.

Example 4.2. Let f be a non-constant polynomial of arbitrary degree d with filled Julia set $K(f)$ defined in §3.2. Recall that f has a superattracting fixed point at ∞, so then we can always find a large enough bounded open domain U containing $K(f)$ and all finite critical points of f such that $f : U \to f(U)$ is polynomial-like of degree d.

In the example above, notice that regardless of the choice of the domain U, the filled Julia set $K(f)$ of the polynomial f coincides with the set $\bigcap_{n\in\mathbb{N}} f^{-n}(U)$. We can therefore define invariant sets corresponding to a polynomial-like map in a similar way.

Definition 4.2. The *filled Julia set* of a polynomial-like map $f : U \to V$ is the invariant set $K(f) := \bigcap_{n\in\mathbb{N}} f^{-n}(U)$. The *Julia set* of f is $J(f) := \partial K(f)$.

The filled Julia set $K(f)$ of a polynomial-like map $f : U \to V$ is non-empty since $K(f)$ can be expressed as the limit $\bigcap_{n\in\mathbb{N}} f^{-n}(\overline{U})$ and each $f^{-n}(\overline{U})$ is compact due to properness of f. By maximum modulus principle, we can also easily deduce that $K(f)$ is full.

The domains U and V are only required to be open, simply connected, and $U \Subset V$. It turns out that we can in fact assume that both have smooth boundaries.

Lemma 4.1. *Let $f : U \to V$ be a polynomial-like map of degree d. For any $\epsilon \in (0, 1)$, there are open domains U' and V' with smooth boundaries such that $f : U' \to V'$ is a polynomial-like map of degree d with the same filled Julia set $K(f)$ and*

$$\epsilon \operatorname{mod}(V\backslash\overline{U}) \leq \operatorname{mod}(V'\backslash\overline{U'}) \leq \operatorname{mod}(V\backslash\overline{U}).$$

Proof. Let $g : \mathbb{A}_{1,r} \to V\backslash\overline{U}$ be a biholomorphism where $r = 2\pi \exp\big(\operatorname{mod}(V\backslash\overline{U})\big) > 1$. For any $t \in (r^{\epsilon}, r)$, the geodesic curve $\mathbb{T}_t$ of $\mathbb{A}_{1,r}$ is sent to a smooth curve $g(\mathbb{T}_t)$. The number of critical values of f is finite, so we can pick t such that the open domain V' bounded by the smooth curve $g(\mathbb{T}_t)$ contains all the critical values. Letting $U' := f^{-1}(V')$, $f : U' \to V'$ remains polynomial-like of the same degree. Note that this restriction still has the same filled Julia set $K(f)$ since $f^n(z) \in U$ for all n if and only if $f^n(z) \in V'$ for all n. The bounds on $\operatorname{mod}(V'\backslash U')$ follow from Grötzsch inequality. $\square$

Definition 4.3. We say that two polynomial-like maps $f : U \to V$ and $g : U' \to V'$ are *hybrid conjugate* if there is a quasiconformal homeomorphism ϕ from a neighbourhood of the filled Julia set $K(f)$ to a neighbourhood of $K(g)$ such that $\phi \circ f = g \circ \phi$ and ϕ is conformal on on $K(f)$.

Hybrid conjugacy is indeed an equivalence relation. We call the corresponding equivalence class the *hybrid class* of f. This conjugacy turns out to be the right type of conjugacy to consider when comparing quadratic-like maps since most dynamical information are contained near the filled Julia set. Note that the concept of hybrid conjugacy can be naturally extended to non-constant polynomials.

Theorem 4.2. *Let f and g be two polynomials of the same degree d with connected filled Julia set $K(f)$ and $K(g)$ respectively. If f and g are hybrid conjugate, then f and g are affinely conjugate.*

Proof. Assume without loss of generality that f and g are monic of degree d. Let ϕ be the corresponding hybrid conjugation and let B_f and B_g be Böttcher maps of f and g respectively. Pick any $r > 1$ and define domains W_f and W_g by

$$W_f = K(f) \cup B_f^{-1}(\mathbb{A}_{1,r}) \text{ and } W_g = K(g) \cup B_g^{-1}(\mathbb{A}_{1,r}).$$

Let N be an open neighbourhood of $K(f)$ compactly contained in W_f. Define a quasiconformal homeomorphism $\phi_0 : \mathbb{C} \to \mathbb{C}$ as

$$\phi_0(z) = \begin{cases} \phi(z), & \text{if } z \in N, \\ B_g^{-1} \circ B_f(z), & \text{if } z \in \mathbb{C} \backslash \overline{W_f}, \end{cases}$$

and let ϕ_0 to be quasiconformal on $W_f \backslash N$ such that it is continuous on the boundaries ∂W_f and ∂N, which are smooth.

By the Böttcher conjugacy, we have that $f_0 = B_f \circ f \circ B_f^{-1} = B_g \circ g \circ B_g^{-1}$, thus ϕ_0 is a holomorphic conjugation between f and g outside $\overline{W_f}$. Moreover, both ϕ and ϕ_0 are conjugacies along the critical orbits of f and g.

Let $\{\phi_n\}_{n \in \mathbb{N}}$ be a sequence of quasiconformal homeomorphisms such that $g \circ \phi_{n+1} = \phi_n \circ f$ and each ϕ_n is a conjugation along the critical orbits of f and g. By hybrid and Böttcher conjugacies, for each n, $\phi_n = \phi$ on $K(f)$ and $\phi_n = B_g^{-1} \circ B_f$ outside $\overline{W_f}$.

By construction, all ϕ_n, $n \geq 0$ have the same dilatation. By Theorem 2.8, there is a subsequence converging to some limit ϕ_∞. Any $z \in \mathbb{C} \backslash K(f)$ will eventually escape W_f via iterations of f, so $\phi_n(z)$ eventually coincides with $B_g^{-1} \circ B_f(z)$ due to the Böttcher conjugacy. Thus, $\phi_\infty = \phi$ on $K(f)$ and $\phi_\infty = B_g^{-1} \circ B_f$ in $\mathbb{C} \backslash K(f)$. As ϕ_∞ is conformal almost everywhere, it has to be a conformal automorphism of $\mathbb{C}$, hence an affine map conjugating f and g. ☐

Theorem 4.3 (Straightening Theorem). *Let $f : U \to V$ be a polynomial-like map of degree d. Then, f is hybrid conjugate to some polynomial g of degree d. Moreover, if $K(f)$ is connected, g is unique up to affine conjugacy. If $d = 2$, there is a unique parameter $c \in \mathbb{M}$ such that f is hybrid conjugate to the quadratic map f_c.*

Proof. From Lemma 4.1, we can assume U and V have a smooth boundaries. Pick any $r > 1$ and a Riemann map $\phi : U \to \mathbb{D}_r$. By smoothness, the map ϕ can be extended continuously along the boundary ∂U. Define ϕ on ∂V such that ϕ is equivariant on the boundary, i.e. $\phi \circ f(z) = \phi(z)^2$ on $z \in \partial U$. By Corollary 2.13, we can extend ϕ to be quasiconformal on $\overline{V} \backslash U \to \overline{\mathbb{D}_{r^2}} \backslash \mathbb{D}_r$.

We now wish to construct a function $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which has the dynamics of ϕ near 0 and of $f_0(z) = z^2$ near ∞. Define F as follows:

$$F(z) = \begin{cases} \phi \circ f \circ \phi^{-1}(z), & \text{if } z \in \mathbb{D}_r, \\ f_0(z), & \text{if } z \in \hat{\mathbb{C}} \backslash \mathbb{D}_r. \end{cases}$$

By equivariance, F is continuous on the boundaries and is in fact quasiregular on $\hat{\mathbb{C}}$. Moreover, F is holomorphic on $F^{-1}(\mathbb{D}_r) = \phi(f^{-1}(U))$ and $\hat{\mathbb{C}} \backslash \overline{\mathbb{D}_r}$. Using complex chain rule, the complex dilatation of F on $\mathbb{D}_r \backslash F^{-1}(\mathbb{D}_r)$ is

$$\mu_F(z) = \mu_\phi\big(f \circ \phi^{-1}(z)\big)\frac{\overline{(\phi^{-1})_z(z)}}{(\phi^{-1})_z(z)}. \tag{4.1}$$

We will now seek an F-invariant Beltrami coefficient μ. Define $\mu = \sigma$ on $\phi(K(f))$ and outside of $\mathbb{D}_r$, where $\sigma \equiv 0$. On $\mathbb{D}_r \backslash \phi(K(f))$, we define μ by its pullback, i.e. if $z \in \mathbb{D}_r$, $F^n(z) \in \hat{\mathbb{C}} \backslash \mathbb{D}_r$, where n is the first escape time of z out of the disk $\mathbb{D}_r$, then $\mu(z) = (F^n)^*\mu(z)$.

For $z \in \mathbb{D}_r \backslash F^{-1}(\mathbb{D}_r)$, $\mu(z)$ coincides with the complex dilatation $\mu_F(z)$ of F. Moreover, for $z \in F^{-1}(\mathbb{D}_r) \backslash \phi(K(f))$, as F is holomorphic, chain rule leads us to

$$\mu(z) = \mu(F(z))\frac{\overline{F_z(z)}}{F_z(z)}, \tag{4.2}$$

so then $\|\mu\|_\infty = \|\mu_F\|_\infty < 1$. By MRMT, we have a unique quasiconformal homeomorphism $G : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ fixing 0, 1, and ∞ such that $G_{\bar{z}} = \mu G_z$. Let $g = G \circ F \circ G^{-1}$. As μ is F-invariant, g must be a rational map. By construction, $g^{-1}(\infty) = \infty$, so g must be a polynomial of degree d. We have then created a hybrid conjugation $G \circ \phi$ from f to g.

From Theorem 4.2, having a connected $K(f)$ implies that g must be unique up to affine conjugacy. If $d = 2$, by Proposition 3.12, g must be affine conjugate to a unique f_c. As $K(f_c)$ is connected, the parameter c must also lie in the Mandelbrot set. ☐

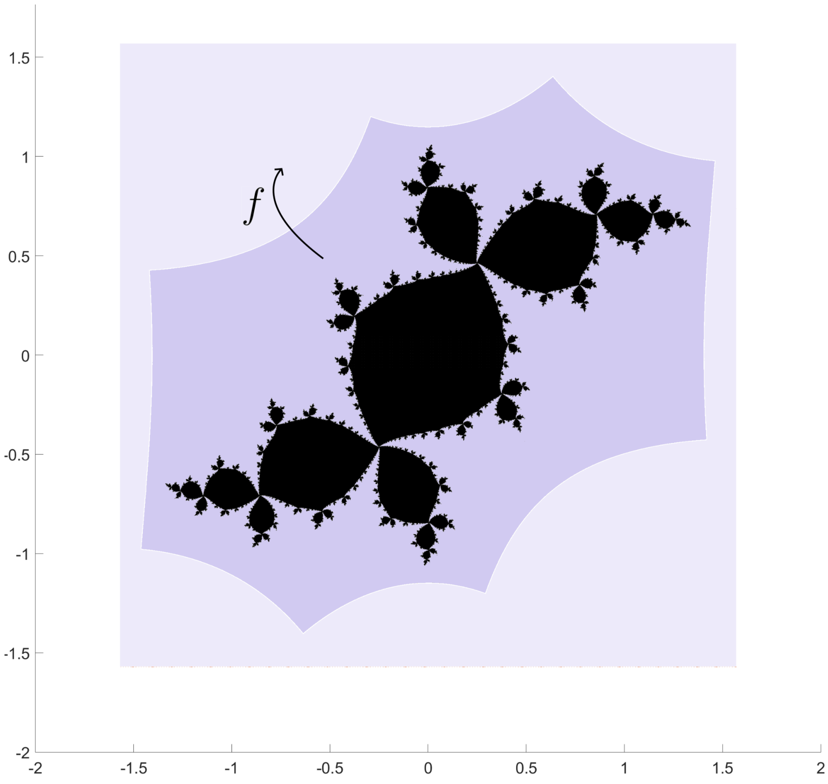

Figure 4.1: The map $f(z) = 2\cos(z) - 1.9 + 0.7i$ has a quadratic-like restriction from $f^{-1}(V) \cap V$ onto V where V is a square of side π centred at 0. The filled Julia set, shown in black, is a Douady rabbit.

Definition 4.4. A map $\phi : V\backslash\overline{U} \to \mathbb{A}_{r,r^2}$ from the proof of the theorem above is the *tubing* of f. We define the *straightening operator* $\chi : f \mapsto f_c$ to be the map from the space of quadratic-like maps with connected filled Julia set to the space of quadratic maps f_c where $c \in \mathbb{M}$, sending a quadratic-like map f to the unique quadratic map f_c.

The straightening theorem allows us to conveniently transfer our knowledge of the dynamics of polynomials to the dynamics of polynomial-like maps.

Corollary 4.4. *Let $f : U \to V$ be a polynomial-like map of degree d. Then:*

(A) repelling periodic points of f are dense in $J(f)$;
(B) for any $z \in J(f)$, the backward orbit $O_f^-(z)$ is dense in $J(f)$;
(C) for any $z \in J(f)$ and open neighbourhood $W \subset U$ of z, $J(f) \subset f^N(U)$ for sufficiently large $N \in \mathbb{N}$;
(D) any immediate attracting or parabolic basin contains a critical point.

Moreover, if f is quadratic-like,

(E) f has at most one non-repelling periodic cycle;
(F) $K(f)$ is either a connected set or a Cantor set, and $K(f)$ is connected if and only if $K(f)$ contains the critical point 0.

It is natural to ask whether the filled Julia set of a polynomial-like map f depends on the choice of domains $U \Subset V$.

Proposition 4.5. *Let $f_i : U_i \to V_i$ be two polynomial-like maps of degree d_i for $i = 1, 2$ such that $f_1 = f_2 = f$ on $U_1 \cap U_2$. If U is a connected component of $U_1 \cap U_2$ containing 0 and if $V := f_1(U)$, then $f : U \to V$ is a polynomial-like map of degree $d \leq d_i$ for $i = 1, 2$ with filled Julia set $K(f) = U \cap K(f_1) \cap K(f_2)$. Additionally, if $d = d_1 = d_2$, then $K(f) = K(f_1) = K(f_2)$.*

Proof. The map $f : U \to V$ is a proper covering map as it is a restriction of polynomial-like maps f_1 and f_2. The number of critical points of f is bounded above by that of f_1 and f_2, thus f has degree $d \leq \min\{d_1, d_2\}$. Moreover, we can express $K(f)$ as $\bigcap_n (f_1^{-n}(U_1) \cap f_2^{-n}(U_2) \cap U)$.

Suppose $d = d_1 = d_2$, then for any $x \in J(f)$, the backward orbit $O_f^-(x)$ will coincide with $O_{f_i}^-(x)$ for $i = 1, 2$. By the density of the backward orbit of x, we then have $J(f) = J(f_1) = J(f_2)$. □

Definition 4.5. The *intersection* of two polynomial-like maps $f_1 : U_1 \to V_1$ and $f_2 : U_2 \to V_2$ is the polynomial-like map $f : U \to V$ constructed in the previous proposition.

Lemma 4.6. *Let f be a polynomial with connected $K(f)$ and U be a topological disk in $\mathbb{C}\backslash K(f)$ such that its boundary is a simple closed curve intersecting $J(f)$ on a closed non-degenerate arc. Then, $f^n(U)$ separates $K(f)$ from ∞ for sufficiently large n.*

Proof. As $K(f)$ is connected, the Böttcher map extends univalently to a Riemann map $B_f : \mathbb{C}\backslash K(f) \to \mathbb{C}\backslash\overline{D}$ conjugating f and $f_0(z) = z^d$, where d is the degree of f. Then, $f_0^n(B_f(U))$ eventually separates $K(f_0) = \overline{\mathbb{D}}$ from ∞ (and is in fact an annulus). Pull it back via B_f to obtain the result. □

Theorem 4.7 (Connectedness Principle)**.** *Let f be a polynomial with connected filled Julia set $K(f)$ such that there is a pair of Jordan domains U and V and some $n \in \mathbb{N}$ such that $f^n : U \to V$ is a polynomial-like map with connected filled Julia set K_n. Then, $L \cap K_n$ is connected for any closed connected subset $L \subset K(f)$.*

Proof. We know that $J_n := \partial K_n \subset J(f)$. Suppose K_n is a proper subset of $K(f)$ (otherwise the result is trivial), and suppose $L \cap K_n$ is non-empty and not connected. We can check that there is a bounded component W of $\mathbb{C}\backslash(L \cup K_n)$ where $L \cap \partial W \subsetneq \partial W$. By maximum modulus principle, as $\partial W \subset K(f)$, W lies in $K(f)$ too.

There is a simple arc γ in $\overline{W}$ connecting two distinct points in J_n on its ends. From Lemma 4.6, the open region U bounded by $\gamma \cup K_n$ eventually surrounds K_n and separates it from ∞, but then as $U \subset K(f)$, for all sufficiently large n, $f^n(U) \subset K(f)$ implies that K_n must lie in the interior of $K(f)$. This contradicts the fact that $\partial K_n \subset \partial K(f)$. □

4.2 Renormalisation

Definition 4.6. A quadratic map $f \equiv f_c$ with connected filled Julia set $K(f)$ is *renormalisable* if there exist an integer $n > 1$ and open domains U_n and V_n containing the critical point 0 such that $f^n : U_n \to V_n$ is a quadratic-like map with connected filled Julia set. We will call this restriction of f^n the *n-renormalisation* of f.

Remark. We can also say that a quadratic-like map $g : U \to V$ is n-renormalisable if there exists a natural number $n > 1$ and open domains U_n, $V_n \subset V$ containing 0 such that $g^n : U_n \to V_n$ is quadratic-like with connected filled Julia set. With this definition, we have that a quadratic-like map g is n-renormalisable if and only if the straightening of g, i.e. some quadratic map f_c is n-renormalisable. As such, we can again transfer all of our results we are going to discuss on renormalisation of quadratic maps to that of quadratic-like maps.

Proposition 4.8. *Any two n-renormalisations of a quadratic map f have the same filled Julia set.*

Proof. Let $g_1 : U_1 \to V_1$ and $g_2 : U_2 \to V_2$ be two renormalisations of f with period n. $K(g_1)$ is a connected closed set in $K(f)$, so by connectedness principle, $K(g_1) \cap K(g_2)$ is connected. We then obtain $K(g_1) = K(g_2)$ immediately by applying Proposition 4.5 to the restriction of g_i on a connected component of $U_1 \cap U_2$ containing $K(g_1) \cap K(g_2)$. □

Suppose f is n-renormalisable with renormalisation $f^n : U_n \to V_n$. Following McMullen's notation, we will denote by K_n the unique filled Julia set of the n-renormalisation of f. Call the images $K_n(i) := f^i(K_n)$, for all $i = 1, 2, \dots n$, the *small filled Julia sets*. Note that these are cyclically permuted by f and $K_n(n) = K_n$.

We will also let $V_n(i) = f^i(U_n)$ and $U_n(i)$ the component of $f^{i-n}(U_n)$ such that $U_n(i) \Subset V_n(i)$, for all $i = 1, 2 \dots n$. We will summarise its properties as follows:

- $f : U_n \to V_n(1)$ is proper branched double covering map;
- for all $i < n$, $f : V_n(i) \to V_n(i+1)$ is univalent;
- for all i, $f^n : U_n(i) \to V_n(i)$ is quadratic-like with critical point $f^i(0)$ and filled Julia set $K_n(i)$.

Lemma 4.9. *Let f be an n-renormalisable quadratic map. For any non-empty subset $I \subset \{1, 2, \dots n\}$, the union $\bigcup_{i \in I} K_n(i)$ is full.*

Proof. The filled Julia set $K(f)$ as well as all the small filled Julia sets $K_n(i)$ are full. Suppose for a contradiction that the set $\mathbb{C} \setminus \bigcup_{i \in I} K_n(i)$ has a bounded open component W, and so by fullness, $W \subset K(f)$. Then, pick a path $\gamma : [0, 1] \to \overline{W}$ dividing W into two such that their endpoints are distinct and lying on ∂W. The intersection $\gamma([0, 1]) \cap \bigcup_{i \in I} K_n(i)$ is not connected, thus contradicting Theorem 4.7. ☐

Theorem 4.10. *Let f be an n-renormalisable quadratic map. The intersection between any two distinct small filled Julia sets $K_n(i)$ and $K_n(j)$, where $i \neq j$, is either empty or a singleton consisting of a repelling fixed point of f^n. In the latter case, all intersections of small filled Julia sets are fixed points of the same type (α or β).*

Proof. Assume the $K_n(i) \cap K_n(j)$ is non-empty, then by Theorem 4.7, it is connected. Let W be the component of $U_n(i) \cap U_n(j)$ containing $K_n(i) \cap K_n(j)$. Since the critical points $f^i(0)$ and $f^j(0)$ do not lie in $K_n(i) \cap K_n(j)$, $f^n : W \to f^n(W)$ must be univalent and $W \Subset f^n(W)$. By Lemma 2.3, $f^{-n} : f^n(W) \to W$ must be a contraction with respect to the hyperbolic metric on $f^n(W)$ and $K_n(i) \cap K_n(j)$ must be a singleton, specifically a repelling fixed point of f^n.

Now, suppose a pair of small filled Julia sets intersect at an α fixed point of f^n while another pair intersect at a β fixed point of f^n. The two fixed points will be permuted by f across all small filled Julia sets such that each $K_n(i)$ will intersect some other small filled Julia sets at its α and β fixed points, namely a_n and β_n. Let α_n and β_n have periods m and k respectively. We then have a graph of $m + k$ vertices formed by each α_i and β_i and n edges representing each small filled Julia set. As $m + k \leq n$, this graph has a cycle, but then $\bigcup_{i=1}^{n} K_n(i)$ is full by Lemma 4.9. This is a contradiction. ☐

Definition 4.7. From our theorem above, we have 3 different types of renormalisation depending on the way small filled Julia sets intersect. A renormalisation of f is *crossed* if it is of α type and *simple* if it is of β or disjoint type. We will also call the β type renormalisation *satellite* and the disjoint type *primitive.*

From now on, we shall disregard the possibility that a renormalisation can be crossed and say that a quadratic-like map is renormalisable when it is simply renormalisable, unless otherwise stated.

Proposition 4.11. *If f is an n-renormalisable quadratic map, the small filled Julia set K_n does not contain the β fixed point of f.*

Proof. Suppose for a contradiction that $\beta \in K_n$, then from Lemma 3.21, K_n must contain $-\alpha$. Then, each little filled Julia set $f^i(K_n)$ contains α and β. This contradicts Theorem 4.10. ☐

Definition 4.8. Let f be a quadratic map. A positive integer $n > 1$ is a *renormalisation level* of f if f is n-renormalisable. The set of renormalisation levels is denoted by $\mathcal{R}(f)$. The *first renormalisation level* of f is the minimum value of $\mathcal{R}(f)$, if non-empty. A quadratic map f is *infinitely renormalisable* if $\mathcal{R}(f)$ is infinite.

Proposition 4.12. *Let f be a quadratic map $f(z) = z^2 + c$. For any $m, n \in \mathcal{R}(f)$, if $m < n$, then m divides n and $K_m \subset K_n$.*

Proof. Take renormalisation representatives $f^m : U_m \to V_m$ and $f^n : U_n \to V_n$. Suppose $l = \operatorname{lcm}(m, n)$, then for $j \in \{m, n\}$, define the sets $\tilde{U}_j$ and $\tilde{V}_j$ by

$$\tilde{U}_j := \bigcup_{i=1}^{l/j-1} f^{-ij}(U_j), \qquad \tilde{V}_j := f^l(\tilde{U}_j)$$

We then have polynomial-like maps $f^l : \tilde{U}_j \to \tilde{V}_j$ for $j \in \{m, n\}$. By the connectedness principle, $K_m \cap K_n$ must be connected. By Theorem 4.5, the intersection of $f^l : \tilde{U}_m \to \tilde{V}_m$ and $f^l : \tilde{U}_n \to \tilde{V}_n$ is a polynomial-like map $f^l : U_l \to V_l$ with filled Julia set $K_l := K_m \cap K_n$. This map is in fact quadratic-like since f^l only has one critical point 0 on K_l.

If m divides n, then $l = n$ and in particular we have that $K_m \subset K_n$. Supppose m does not divide n, then let $h = \operatorname{hcf}(m, n)$. K_m meets $K_n(h)$ since $am + bn = h$ for some integers a, b. Meanwhile, $K_n(h)$ meets $K_m(h)$ since K_m clearly meets K_n. The set $L := K_m \cup K_n(h) \cup K_m(h)$ is connected due to the connectedness principle. As $K_n(h) \cap K_n$ is either empty or a singleton, we have $L \cap K_n = \big(K_m \cup K_m(h)\big) \cap K_n$ and since this intersection is connected, then $K_m \cap K_m(h) \cap K_n$ must be non-empty. Thus, the β-fixed point of $f^m : U_m \to V_m$ is in $K_l = K_m \cap K_n$, which is a contradiction to Proposition 4.11. ☐

Proposition 4.13. *Suppose f is an m-renormalisable. Then, any m-renormalisation of f is n-renormalisable if and only if f is mn-renormalisable.*

Proof. Suppose $f^m : U_m \to V_m$ is a renormalisation of f. If it has an n-renormalisation $f^{mn} : U_{mn} \to V_{mn}$, then f^{mn} is obviously an mn-renormalisation of f.

Conversely, let f is mn-renormalisable. From Proposition 4.12, $K_{mn} \cap K_m = K_{mn}$. By Theorem 4.5, the intersection of the polynomial-like map $f^{mn} : f^{-m(n-1)}(U_m) \to V_m$ and some mn-renormalisation $f^{mn} : U_{mn} \to V_{mn}$ is a quadratic-like map $f^{mn} : U \to f^{mn}(U)$ with connected filled Julia set K_{mn}. This is in fact an n-renormalisation of f^m. □

Proposition 4.14. *Let f be an n-renormalisable quadratic map. Any non-repelling periodic cycle of f has a period divisible by n.*

Proof. Let w be a non-repelling periodic point of period p and let $f^n : U_n \to V_n$ be the renormalisation of f with period n. If w is attracting, parabolic or Cremer, w is a limit point of the critical orbit, hence contained in $P(f)$. As w is not repelling, $w \in K_n(i)$ for some unique i.

Suppose w is a Siegel point, then by Proposition 3.8, the boundary of the corresponding Siegel disk W is contained in $P(f)$ and thus in $\cup_{i=1}^{n} K_n(i)$. f^p is an irrational rotation in W, therefore W is a connected component of the interior of $K_n(i)$ for some unique i.

In any of the cases mentioned, as $f^p(w) = w \in K_n(i)$, p must be divisible by n. □

From the proposition, we can tell from the periodic cycles whether a quadratic map is non-renormalisable, renormalisable, or infinitely renormalisable.

Corollary 4.15. *Let f be a quadratic map with connected filled Julia set $K(f)$. Then,*

- *(A) if f has a non-repelling fixed point, f is non-renormalisable;*
- *(B) if f has a non-repelling cycle, f is at most finitely renormalisable;*
- *(C) if f infinitely renormalisable, f has no non-repelling cycles and thus its filled Julia set $K(f)$ has empty interior, i.e. $K(f) = J(f)$.*

Example 4.3. The quadratic map $f_{1/4}$ has a parabolic fixed point at $z = 1/2$. Thus, $f_{1/4}$ cannot be renormalisable.

The following is a theorem by McMullen. Details of the proof can be found in [McM94a, §7.2.].

Theorem 4.16 (High Periods). *Let f be an infinitely renormalisable quadratic map. For every $p > 1$, there are at most finitely many renormalisation levels $n \in \mathcal{R}(f)$ such that the small filled Julia set K_n contains a periodic point of period p.*

Corollary 4.17. *Let f be an infinitely renormalisable quadratic map. Them, the set $\mathcal{O}_f := \bigcap_{n \in \mathcal{R}(f)} \bigcup_{i=1}^{n} K_n(i)$ and the postcritical set $P(f)$ do not contain any periodic point.*

Proof. Let x be a periodic point of f of some period p. By Theorem 4.16, there are only finitely many n in $\mathcal{R}(f)$ such that $w \in K_n(i_n)$ for some $i_n \leq n$. As such, x is not in $\bigcup_{i=1}^{n} K_n(i)$ for sufficiently large n. Hence, $x \notin \mathcal{O}_f$. Moreover, as $P(f) \subset \mathcal{O}_f$, x does not lie in $P(f)$ either. $\square$

Example 4.4. If f is a postcritically finite quadratic map, the critical point is either periodic or pre-periodic (Misiurewicz). Thus, f is at most finitely renormalisable.

Example 4.5. Suppose a quadratic map f_c has a superattracting periodic point at 0 of period $p > 1$. Let D_0 be the connected component containing 0 of the Fatou set of f_c. On D_0, the Böttcher map is a conformal conjugation between f_c and the doubling $f_0 : \mathbb{D} \to \mathbb{D}, z \mapsto z^2$, so the set of preimages $S = \{z \in \overline{D_0} \mid f_c^k(z) = 0 \text{ for some } 1 \leq k \leq p-1\}$ is finite and disjoint from $\overline{D_0}$.

Let U be a 0-symmetric open neighbourhood of $\overline{D_0}$. We can pick U such that $U \Subset V := f_c^p(U)$. Moreover, we can assume that U is disjoint from S so that for each $1 \leq k \leq p-1$, $0 \notin f_c^k(U)$ and $f_c : f_c^k(U) \to f_c^{k+1}(U)$ is univalent. As such, $f_c^p : U \to V$ is a quadratic-like map with connected filled Julia set $\overline{D_0}$.

Removing a point from a closed disk $\overline{D_0}$ will not change its connectivity. Thus, $f_c^p : U \to V$ is a p-renormalisation of f_c and it is hybrid conjugate to $f_0(z) = z^2$. Conversely, we can also prove that if a quadratic map f_c has a renormalisation which is hybrid conjugate to the doubling map, then c must be a non-zero superstable parameter.

4.3 Baby Mandelbrot Sets

One of the most prominent applications of renormalisation theory is that it explains the presence of little copies of the Mandelbrot set $\mathbb{M}$ in itself. A non-empty proper subset M of the Mandelbrot set $\mathbb{M}$ is a *baby Mandelbrot set* if M is homeomorphic to $\mathbb{M}$. The following theorem by Douady-Hubbard [DH85] explains two types of baby Mandelbrot sets.

Theorem 4.18. *Suppose for some $c \in \mathbb{M}$ that f_c is n-renormalisable. Then:*

- *(A) there is a proper subset $M \subset \mathbb{M}$ called a baby Mandelbrot set containing c and a homeomorphism $\sigma : M \to \mathbb{M}$ where for any $\tilde{c} \in M$, $f_{\sigma(\tilde{c})}$ is the straightening of any n-renormalisation of $f_{\tilde{c}}$;*
- *(B) if n is the first renormalisation level, then M is maximal, i.e. not contained in any other baby Mandelbrot set.*

Note that the homeomorphism σ is well-defined since the straightening of a quadratic-like map f is independent of its domains.

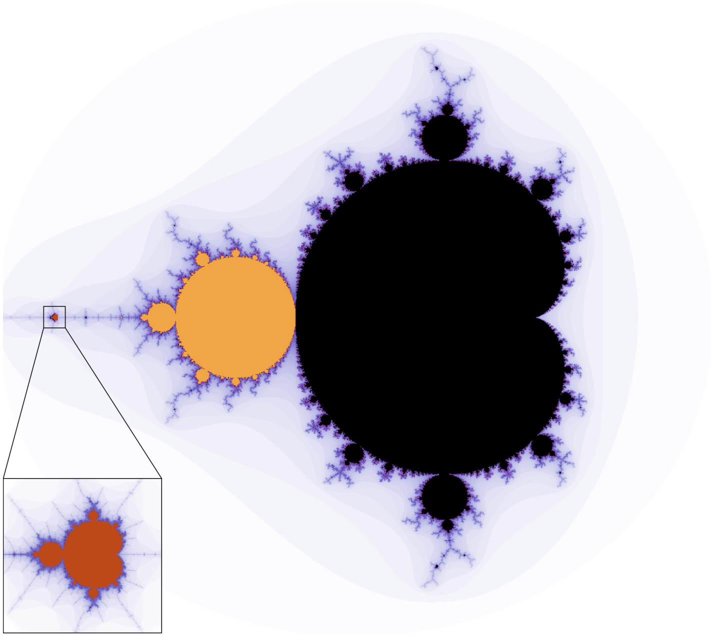

Figure 4.2: Primitive and satellite baby Mandelbrot sets
are shown in brown and yellow respectively. Both copies are maximal.

If the first renormalisation of f_c is satellite, the subset M will intersect with the boundary of the main cardioid at a parabolic parameter, which is non-renormalisable. Thus the composition of the straightening and the renormalisation is only well-defined onto $\mathbb{M}\backslash\{1/4\}$. However, we can ignore this problem as we can extend the homeomorphism to its closure.

Definition 4.9. The corresponding homeomorphism $\sigma : M \to \mathbb{M}$ of a baby Mandelbrot set M is called the *stretching homeomorphism* of M.

Recall that each hyperbolic component of $\operatorname{int}(\mathbb{M})$ aside from the main cardioid contains a unique superstable centre c of some period $n > 1$, which is n-renormalisable. From Example 4.5, this renormalisation is hybrid conjugate to the doubling map f_0. As such, we have the following result.

Corollary 4.19. *Each hyperbolic component H of the interior of $\mathbb{M}$ aside from the main cardioid H_0 is contained in a baby Mandelbrot set M. Furthermore, M is unique if the stretching homeomorphism σ of M maps H to H_0 and the superstable centre of H to 0.*

As each hyperbolic component has a unique superstable centre, we have a natural bijection between the set of non-zero superstable parameters c and the set of all baby Mandelbrot sets. We call c *maximal* if its corresponding baby Mandelbrot set is maximal.

Consider a baby Mandelbrot set $M \subset \mathbb{M}$ which intersects the real axis $\mathbb{R}$. Both M and $\mathbb{M}$ are symmetric about $\mathbb{R}$ and indeed, the homeomorphism $\sigma : M \to \mathbb{M}$ restricts to a homeomorphism $I \to [-2, \frac{1}{4}]$, where $I = M \cap \mathbb{R} \subset [-2, \frac{1}{4}]$. Consequently, σ must have a fixed point $c_0 \in I$. (c_0 is in fact unique, but we will not prove it here. This is Lyubich's self similarity theorem in [Lyu99].) We thus know for sure that c_0 is a parameter corresponding to an infinitely renormalisable quadratic map.

Definition 4.10. Let $f = f_c$ be an infinitely renormalisable quadratic map with renormalisation levels $\mathcal{R}(f) = \{n_k\}_{k\in\mathbb{N}}$ labelled in ascending order. The *tuning invariant* $\tau(f)$ of f is an infinite tuple of maximal superstable parameters $\langle c_1, c_2, \ldots\rangle$ such that for each $k \in \mathbb{N}$, if $f_{\tilde{c}_k}$ is the straightening of the n_k-renormalisation of f, then parameters $\tilde{c}_k$ and c_k lie in the same maximal baby Mandelbrot set. The map f has *bounded combinatorics* if the tuning invariant consists of only finitely many distinct parameters.

Example 4.6. The Feigenbaum map $f_{c_F}(z) = z^2 + c_F$ where $c_F \approx -1.4011551890$ is a real infinitely renormalisable quadratic map characterised as the unique limit of the real period-doubling cascade. In particular, the Feigenbaum parameter c_F is a fixed point of the homeomorphism $\sigma : M \to \mathbb{M}$ where M is the maximal baby Mandelbrot set containing c_F. It has stationary combinatorics and it is the unique quadratic map determined by the tuning invariant $\tau(f_{c_F}) = \langle -1, -1, -1, \ldots\rangle$.

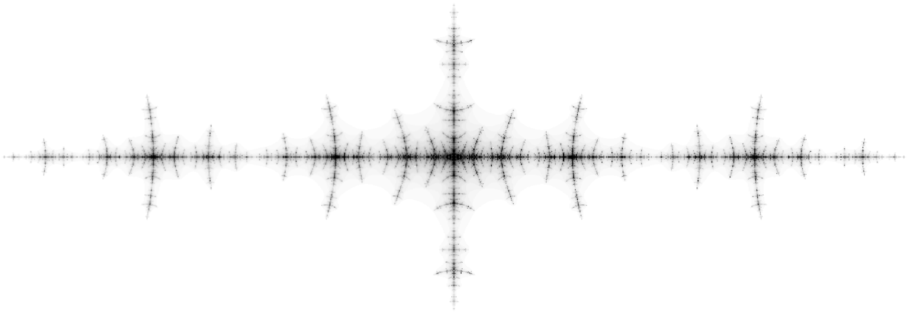

Figure 4.3: The Julia set of the Feigenbaum map f_{c_F}.

Chapter 5

Yoccoz Puzzles

This chapter aims to obtain a requirement for f to be renormalisable and an algorithm to construct renormalisation domains. This is done through a powerful tool called *puzzles* introduced by Yoccoz for quadratic polynomials ([Hub93] and [Mil00]), and Branner and Hubbard for cubic polynomials ([BH92]).

5.1 Yoccoz Puzzles

We know from Corollary 4.15 that a quadratic map with connected filled Julia set are non-renormalisable when it has a non-repelling fixed point. Assume from now on that $f = f_c$ is a quadratic map with both fixed points repelling, and that the dividing repelling fixed point α of f does not lie on the critical orbit $O_f^+(0) = \{c_i \mid i \geq 0\}$ where $c_i := f^i(0)$. The point α is a landing point of $r > 1$ external rays $R_i(\alpha)$, $i = 1, 2 \ldots r$. Pick any real number $t > 1$ and let E_t be the equipotential of $K(f)$ of radius t.

The *puzzle pieces* of depth 0 of f are the closed regions $P_0(c_i)$, $i = 1, 2, \ldots r$ whose interiors are pairwise disjoint bounded components of the complement of $E_t \cup \bigcup_{i=1}^r R_i(\alpha)$ in $\mathbb{C}$. The pieces are labelled accordingly such that $c_i := f^i(0) \in P_0(c_i)$.

The puzzle pieces of depth 1 of f is the collection of components of $f^{-1}(P_0(c_i))$ for all $i = 0, 1 \ldots r$. These are $2r - 1$ components bounded by the equipotential $E_{\sqrt{t}}$ and the external rays landing at α and the preimage $-\alpha$. Inductively, we can define the puzzle pieces of depth $d + 1$ of f as the collection of closed components of $f^{-1}(P_d)$ for all puzzle pieces P_d of depth d.

Definition 5.1. For any $d \in \mathbb{N}$ and $z \in K(f) \backslash O_f^-(\alpha)$, we denote by $P_d(z)$ the *puzzle piece* of depth d containing z. Specifically, we say that $P_d(0)$ is the *critical* puzzle piece of depth d, and $P_d(c)$ is the *valuable* puzzle piece of depth d.

Proposition 5.1 (Markov Property)**.** *For any puzzle pieces P and P' of depths d and d' with $d \leq d'$, then either P and P' have disjoint interiors or $P' \subset P$.*

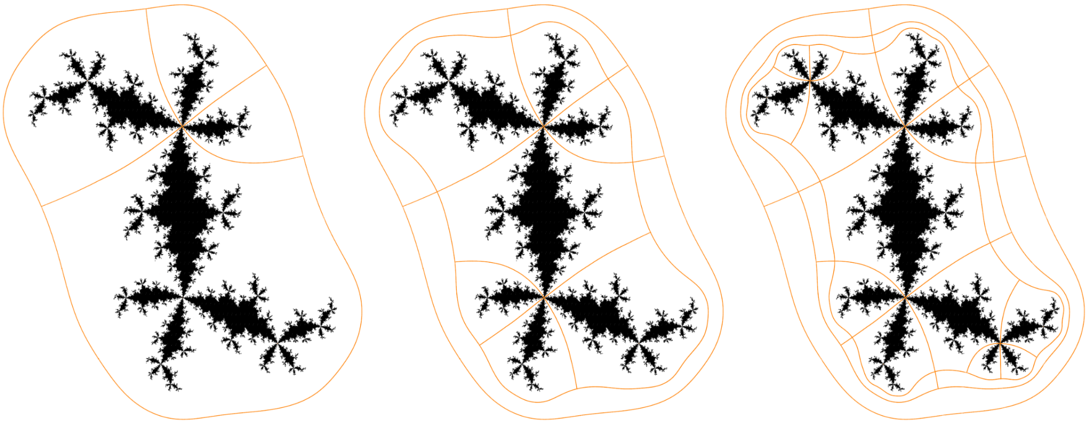

Figure 5.1: Puzzle pieces of depths 0, 1 and 2.

Proposition 5.2. *For any puzzle piece P, $K(f) \cap P$ is connected.*

Proof. For any puzzle piece P_0 of depth 0, $P_0 \cup K(f)$ is connected as it is the union of $\{\alpha\}$ and a connected component of $K(f)\backslash\{\alpha\}$. To prove this for deeper pieces, we will proceed by induction.

Assume that the lemma holds for puzzle pieces of depth $d-1$ for some positive $d \in \mathbb{N}$. For any noncritical puzzle piece P_d, we have a univalent restriction $f : P_d \to P_{d-1}$ for some puzzle piece P_{d-1} of depth $d-1$. We can restrict further to a homeomorphism $f : P_d \cap K(f) \to P_{d-1} \cap K(f)$, so $P_d \cap K(f)$ must be connected. For the critical piece $P_d(0)$, assume for a contradiction that $P_d(0) \cap K(f)$ is not connected and take a component L not containing 0. Then, by connectedness of $P_{d-1}(c)$, there must be a point on $f(L)$ such that every open neighbourhood intersects with $K(f) \cap P_{d_1}(c)\backslash f(L)$. Lifting each neighbourhood back via f gives a contradiction to the assumption that L and $P_d(0) \cap K(f)\backslash L$ are disjoint components. Thus, $P_d(0) \cap K(f)$ is connected. ☐

Definition 5.2. The *critical tableau* of f is the collection of puzzle pieces $\{P_d(c_k)\}_{d,k\in\mathbb{N}}$. We say that the critical tableau is *periodic* of period n if $n > 1$ is the least positive integer where for all depths d, $P_d(c_n) = P_d(0)$.

Theorem 5.3. *Let $f(z) = z^2 + c$ be a quadratic map with both fixed points repelling and connected $K(f)$. If the critical orbit does not contain the α fixed point, then f is renormalisable if and only if the critical tableau of f is periodic. Moreover, the period of the critical tableau is the first renormalisation level of f.*

Proof. Suppose that the critical tableau of f is periodic of period n. For sufficiently large d, the first return time of 0 back to $P_d(0)$ is n, so then $f^n : P_{d+n}(0) \to P_d(0)$ is a proper double covering map. When necessary, we can thicken $P_{d+n}(0)$ and $P_d(0)$ in the following way.

Replace each external ray R_θ on $\partial P_d(0)$, with a nearby ray outside $P_d(0)$ which differs in angle by a sufficiently small $\epsilon > 0$. Moreover, for each preimage of α on $\partial P_d(0) \cap \partial P_{d+n}(0)$, take a small circle of sufficiently small radius $\delta > 0$ centred at this point. Denote by $\hat{P}_d(0)$ the thickened simply connected region bounded by the union of these new rays, circles, and the same equipotential on $\partial P_d(0)$. Let $\hat{P}_{d+n}(0)$ be the connected component of $f^{-n}\big(\hat{P}_d(0)\big)$ containing $P_{d+n}(0)$. Then, it follows from the expanding nature of the doubling map that $\hat{P}_{d+n}(0) \Subset \hat{P}_d(0)$ and $f^n : \hat{P}_{d+n}(0) \to \hat{P}_d(0)$ is a quadratic-like map, and it has a connected Julia set due to periodicity of the critical tableau.

Suppose two small filled Julia sets $K_n(i)$ and $K_n(j)$ meet, then the intersection must consist of a repelling fixed point of f^n, and by our puzzle construction, this is a preimage of α. Thus, $K_n(i) \cap K_n(j) = \{\alpha\}$, and since α is the fixed point of f, all small filled Julia sets intersect each other at α. The point α is then located on the boundary of the piece $P_{d+n}(0)$ so α cannot divide K_n. The iterate $f^n : \hat{P}_{d+n}(0) \to \hat{P}_d(0)$ is indeed a n-renormalisation of f.

Suppose now that f is m-renormalisable. If $\alpha \notin K_m$, then $0 \in K_m \subset P_0(0)$. Otherwise, α is the β-fixed point of f^m since α is in all the small filled Julia sets. Hence, $K_m \subset P_0(0)$.

If $K_m \subset P_{jm}(0)$, then pull K_m back via f^m to a component of $f^{-m}(K_m)$ containing K_m, so then $K_m \subset f^{-m}(K_m) \subset P_{(j+1)m}(0)$. By induction, $K_m \subset P_{mj}(0)$ for all $j \in \mathbb{N}$. As all puzzle pieces containing 0 are nested, $K_m \subset P_d(0)$ for all depths d. As K_m is m-periodic, the critical tableau has period $\leq m$. □

Yoccoz puzzles provides an algorithm to construct the first renormalisation of a renormalisable quadratic map. We will generalise this idea in the next section.

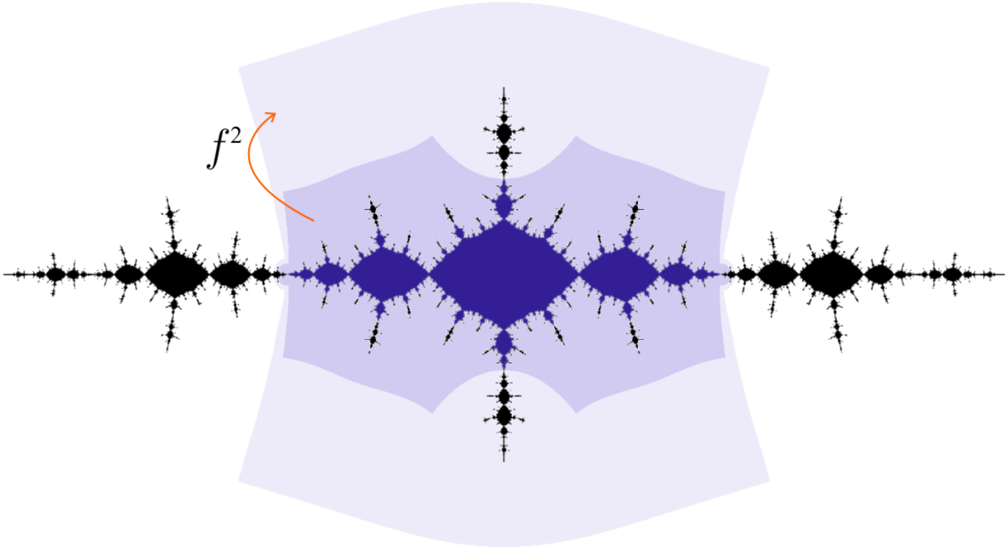

Figure 5.2: The map $f(z) = z^2 - 1.333$ is 2-renormalisable on the thickened critical puzzle piece of depth 3 onto the thickened critical puzzle piece of depth 1. This renormalisation is satellite and K_2 is shown in dark blue.

5.2 Douady-Hubbard Renormalisation

We will construct puzzle pieces using external rays generated from the rays landing at a dividing repelling periodic cycle $\{\alpha_k\}_{k=0,1,\ldots p-1}$ where each point divides $K(f)$ into q components. We assume that the cycle we have picked does not lie in the postcritical set $P(f)$ of f.

Pick any $t > 1$ and equipotential E_t of $K(f)$. Let $\mathcal{R}(\alpha)$ be the union of the cycle $\{\alpha_k\}_{k=0,1,2\ldots p-1}$ and all external rays landing on them. The puzzle pieces of depth 0 induced by $\{\alpha_k\}_{k=0,1,\ldots p-1}$ are the closure of bounded components of $\mathbb{C}\backslash(E_t \cup \mathcal{R}(\alpha))$. Again, we define inductively the puzzle pieces of depth d induced by $\{\alpha_k\}$ as the preimages of those of depth $d-1$.

Similar to the case where $p = 1$, these puzzle pieces satisfy the Markov property. The following lemma will help the explicit construction of renormalisation domains. This procedure is called the *DH renormalisation* of f.

Lemma 5.4. *There is a puzzle piece $P_{pq} \subset P_0(c)$ of depth pq such that the map $f^{pq} : P_{pq} \to P_0(c)$ is a branched double covering map.*

Proof. We can assume that for $k = 0, 1, \ldots p-2$, $f(\alpha_k) = \alpha_{k+1}$, $f(\alpha_{p-1}) = \alpha_0$ and α_1 is attached to $P_0(c)$. Let $L_0 \subset \mathcal{R}(\alpha)$ be the union of the point α_1 and the two external rays attached to $\partial P_0(c)$.

Let $L_k \subset \mathcal{R}(\alpha)$ be the unique pullback of L_0 via f^k for each $k = 1, \ldots pq-1$. The pullback of $P_0(c)$ via f is a single component $P_1(0)$, a critical puzzle piece attached to $L_1 \subset \mathcal{R}(\alpha)$ where $f(L_1) = L_0$. The restriction $f : P_1(0) \to P_0(c)$ is a branched double covering map.

Let P_{pq} be the pullback of $P_1(0)$ via f^{pq-1} such that P_{pq} is attached to $L_{pq} \equiv L_0$ and for each $m = 1, \ldots pq-1$, the puzzle piece $P_{pq-m} := f^m(P_{pq})$ is attached to L_{pq-m}. Suppose there are two nested puzzle pieces $P_{pq-m} \subset P_{pq-m'}$ for some $m < m' \leq pq-1$, then since $L_{pq-m} \neq L_{pq-m'}$, the interior of $P_{pq-m'}$ intersects L_{pq-m}. Thus, $P_0(c)$ would intersect both L_0 and $L_{m'-m}$, i.e. more than 3 external rays on $L_k \subset \mathcal{R}(\alpha)$, which is a contradiction.

Thus, the puzzle pieces P_{pq-m} for $m = 0, 1, \ldots pq-1$ all have disjoint interiors, so then they are all non-critical except for $P_1 \equiv P_1(0)$. It follows that $f : P_{pq-m} \to P_{pq-m-1}$ is univalent for each $m \leq pq-2$, and $f : P_1 \to P_0$ is a branched double covering map. □

Proposition 5.5. *If c is contained in the piece P_{pq} in the previous lemma, then $f^{pq} : \hat{P}_{pq+1}(0) \to \hat{P}_1(0)$ is a quadratic-like map. Additionally, if $c_{kpq} \in P_{pq+1}(0)$ for all $k \in \mathbb{N}$, then $f^{pq} : \hat{P}_{pq+1}(0) \to \hat{P}_1(0)$ is a pq-renormalisation of f.*

Proof. From Lemma 5.4, $f^{pq} : P_{pq+1}(0) \to P_1(0)$ is also a branched double covering and the pieces can be thickened to a quadratic-like map, similar to the first part of the proof of Theorem 5.3. The second assumption ensures that $K(f^{pq})$ is connected. □

Definition 5.3. The map f is *DH-renormalisable* if it satisfies the criterion in Proposition 5.5. Also, f is said to be *immediately renormalisable* if a DH-renormalisation can be constructed with a dividing repelling fixed point, i.e. the case where $p = 1$.

From the proof of Theorem 5.3, the small filled Julia sets of an immediate renormalisation touch at the fixed point α. This then leads to the following.

Proposition 5.6. *Any immediate renormalisation of f is of satellite type.*

It is natural to ask whether any satellite renormalisation can be obtained through the Douady-Hubbard procedure. To tackle this question, we need to introduce the principal nest.

5.3 Principal Nest

Consider the puzzle pieces constructed from a repelling fixed point α having $q > 1$ landing rays. This section aims to analyse the role of the critical puzzle pieces. Following Lyubich's notation, we will first expand our vocabulary further.

Definition 5.4. A critical puzzle piece $P_d(0)$ where $d > 0$ is *protected* if it is compactly contained in the previous piece $P_{d-1}(0)$.

Lemma 5.7. *Let f be a quadratic map with both fixed points repelling. If f is not immediately renormalisable, then the shallowest protected critical puzzle piece W_0 is the piece $P_{kq+1}(0)$, where k is the smallest positive integer such that $c_{kq} \notin P_1(0)$.*

Proof. As f is not immediately renormalisable, then such a k exists due to Proposition 5.5. As $P_1(c_{kq}) \Subset P_0(0)$, then $P_{kq+1}(0) \Subset P_{kq}(0)$, i.e. $P_{kq+1}(0)$ is protected. We claim that α is attached to each critical puzzle piece $P_l(0)$ for all $l \leq kq$.

As $-\alpha \in P_0(0)$, then for $m \leq q$, the preimages $f^{-m-1}(\alpha)$ are disjoint from the interior of $P_1(0)$, thus α must be attached to $P_m(0)$. As $c_{lq} \in P_1(0)$ for all $l < q$, then $f^q(P_{r+q}(0)) = P_r(0)$ for all $r < kq$. We will use this to prove the inductive step.

Suppose for some $m \leq kq$ we have that for all $l < m$, $P_l(0)$ is not compactly contained in $P_{l-1}(0)$. Then, $P_m(0)$ is not compactly contained in $P_{m-1}(0)$ too since otherwise we have a contradiction:

$$P_{m-q}(0) = f^q(P_m(0)) \Subset f^q(P_{m-1}(0)) = P_{m-q-1}(0).$$

Thus, $P_m(0)$ are unprotected for all $m \leq kq$. ☐

Suppose $k_0 > 0$ is the first number such that $c_{k_0} = f^{k_0}(0) \in \operatorname{int}(W_0)$. Let W_1 be the critical puzzle piece which is the pullback of W_0 via f^{k_0}. Then, define $k_1 > 0$ as the first number such that $c_{k_1} \in \operatorname{int}(W_1)$, and let W_2 be the critical pullback of W_1 via f^{k_1}.

Repeat the procedure to obtain sequences of positive integers $\{k_l\}$ and critical puzzle pieces $\{W_l\}$.

Definition 5.5. For $i \in \mathbb{N}$, the critical piece W_i is *principal* of level i. The *principal nest* is the sequence $W_0 \Supset W_1 \Supset W_2 \Supset \dots$. We call f *combinatorially recurrent* if the critical orbit visits all critical puzzle pieces, *non-recurrent* if otherwise, i.e. the critical orbit eventually never returns to some sufficiently deep critical piece.

Using our new vocabulary, we see that f is combinatorially recurrent if and only if its principal nest is infinite. Observe that for each level $l \geq 1$, the principal first return map $g_l := f^{k_l} : W_l \to W_{l-1}$ is a quadratic-like map.

Definition 5.6. The return of 0 to W_l is *central* when $g_{l+1}(0) \in W_{l+1}$. We say that a finite sequence $W_l \Supset \dots \Supset W_{l+N-1}$ is a *central cascade* of length N if the return of 0 to W_{l+j} is central for $j = 0, 1, \dots N-2$. A central cascade is *maximal* if it cannot be extended to a longer central cascade.

Remark. When a principal nest ends up with an infinite central cascade starting from level l, then for any $j > l$, the return time k_j coincides with k_{l+1} and the principal first return map g_j coincides with the restriction $g_{l+1}|_{W_j}$.

Theorem 5.8. *Let f be a quadratic map with connected filled Julia set and both fixed points repelling. Then, exactly one of the following three cases occur:*

(A) f is non-renormalisable,

(B) f is immediately renormalisable,

(C) f is primitively renormalisable.

Moreover, case (C) occurs if and only if f is combinatorially recurrent with an infinite central cascade starting from $l-1$ for some l. In this case, $g_l : W_l \to W_{l-1}$ is a k_l-renormalisation of f.

Proof. Suppose f is not immediately renormalisable and the principal nest of f is finite or does not end with an infinite central cascade. Then, the critical tableau is not periodic and by Theorem 5.3, this leads to case (A).

Suppose for some n that f has a primitive renormalisation $f^n : U_n \to V_n$ with small filled Julia set K_n. As the repelling fixed point α does not lie in K_n, then by complete invariance K_n cannot contain all preimages of α too. Thus, K_n is contained in all principal puzzle pieces W_l. It then follows that the sequence of first return times k_l is non-decreasing and bounded by n, hence it is eventually constant. We then obtain an infinite central cascade.

Suppose f has an infinite central cascade starting from $l-1$ for some l. The principal return map $g_l : W_l \to W_{l-1}$ is a quadratic-like map. The filled Julia set of g_l is connected because centrality ensures that the critical orbit under g_l never escapes. Moreover, the sets $f^j(W_l)$ for $j = 0, 1, \dots k_l - 1$ are pairwise disjoint since otherwise if $f^i(W_l)$ intersects $f^j(W_l)$, then W_l would intersect the boundary of $f^s(W_l)$ for some $s < k_l$, contradicting the fact that W_l is protected. Thus, g_l is a primitive renormalisation of f, and by Theorem 5.3, the renormalisation period coincides with the period of the critical tableau, which is k_l. ☐

Corollary 5.9. *Any first renormalisation of a quadratic-like map is a DH renormalisation.*

Proof. From the theorem above, any satellite renormalisation of f is in fact an immediate renormalisation. For the primitive case, we need to choose the appropriate repelling periodic point α such that the induced DH renormalisation corresponds with the primitive renormalisation $g_l : W_l \to W_{l-1}$ in the previous theorem. The proof (refer to [Lyu19]) will not be discussed here as it requires technical construction of a periodic ray landing at α. ☐

We see that the principal nest provide us with an algorithm to determine whether or not a quadratic map is renormalisable and, if so, the type of renormalisation together with the explicit renormalisation domains. However, the principal nest does not give any information on the second or subsequent renormalisations, although they can generally be obtained through the Douady-Hubbard procedure using a smart choice of dividing repelling cycle.

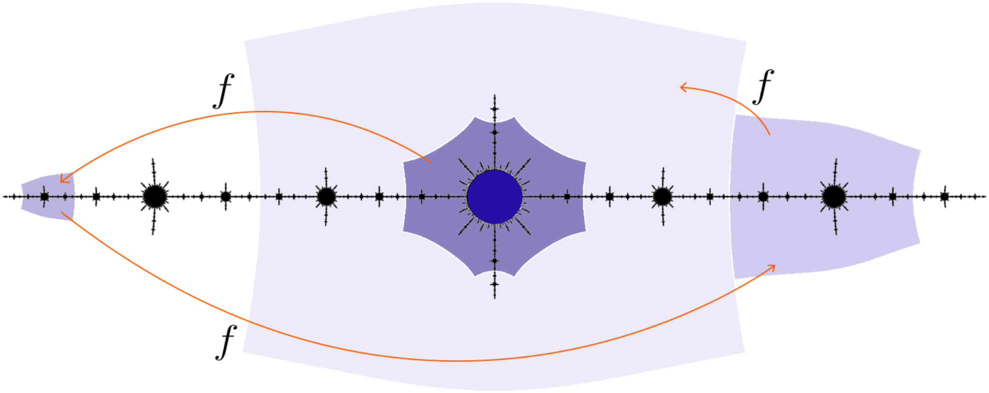

Figure 5.3: The quadratic map $f(z) = z^2 - 1.755$ is 3-renormalisable on the critical puzzle piece of depth 4 onto the critical puzzle piece of depth 1. This renormalisation is primitive and K_3 is the middle disk shown in dark blue.

5.4 Local Connectivity of Julia Sets

Lyubich proved in [Lyu97] that the annular moduli induced by non-central principal pieces grow linearly.

Theorem 5.10. *Let $f : U \to V$ be a quadratic-like map with connected Julia set satisfying $\operatorname{mod}(V \backslash \overline{U}) \geq \mu > 0$. Let the escaping time of 0 from $W_0 \cup \bigcup_{i=1}^{q-1} P_0(c_i)$ be less than N, then there is a constant C depending only on μ and N such that if the increasing sequence of non-central return levels is denoted by $\{l_n\}$, then*

$$\operatorname{mod}(W_{l_n+1} \backslash \overline{W_{l_n+2}}) \geq Cn$$

Yoccoz puzzles are of great importance in helping us prove local connectivity of quadratic Julia sets.

Lemma 5.11. *Let f be a quadratic map with connected Julia set and both fixed points repelling. Suppose the forward orbit of a point $z_0 \in K(f)$ is disjoint from some critical puzzle piece $P_d(0)$, then the puzzle pieces containing z_0 shrink to a point, i.e. $\bigcap P_j(z_0) = \{z_0\}$, and thus $K(f)$ is locally connected at z_0.*

Proof. This is a straightforward adaptation of Lemma 4 on Milnor's [Mil00]. Assume that the forward orbit of z_0 never reaches some thickened critical puzzle piece $\hat{P}_d(0)$. Label all non-valuable thickened pieces of depth $d-1$ as $\hat{P}^k_{d-1}$, where $k = 1 \dots m$ for some $m \in \mathbb{N}$, and endow them with the usual hyperbolic metric. The pullback of $\hat{P}^k_{d-1}$ via a branch of f^{-1} is some thickened piece P_d of depth d compactly contained in $\hat{P}^l_{d-1}$ for some l, so then by Schwarz-Pick lemma, f^{-1} is a contraction.

Suppose that all branches of f^{-1} over all non-critical $\hat{P}^k_{d-1}$ have contraction factor bounded above by $\lambda < 1$ and let D be the maximum hyperbolic diameter over all thickened pieces $\hat{P}_d$ such that $f(\hat{P}_d) = \hat{P}^k_{d-1}$ for some k. The point z_0 lies in $\hat{P}^K_{d-1}$ for some K, then the hyperbolic diameter of puzzle pieces around z_0 satisfies

$$\operatorname{diam} P_{d+n}(z_0) \leq \lambda^n D.$$

Taking $n \to \infty$, then the puzzle pieces around z_0 shrink to $\{z_0\}$. Given any small neighbourhood U of z_0, there is always some puzzle piece $P_N(z_0)$. By Proposition 5.2, we obtain local connectivity at z_0. □

Theorem 5.12 (Yoccoz). *Any non-renormalisable quadratic-like map f with connected Julia set and both fixed points repelling has locally connected Julia set.*

Proof. Suppose f is non-renormalisable. We will split the proof into two cases. Suppose f is combinatorially recurrent, then the principal nest has infinitely many non-central levels. Using Grötzsch inequality and Theorem 5.10, it is clear that the $P_d(0) \cap K(f)$ shrinks to the singleton $\{0\}$, thus $J(f)$ is locally connected at 0. If f is combinatorially non-recurrent, then the orbit of $f^m(0)$ for some $m > 0$ never reaches sufficiently deep criticap puzzle pieces. By Lemma 5.11, we have local connectivity of $J(f)$ at $f^m(0)$, and thus at 0.

We can in fact spread this information by Koebe distortion to the whole critical grand orbit. Let's pick some $z_0 \in J(f)$. Suppose the forward orbit of z_0 must be disjoint from some critical puzzle piece $P_d(0)$, then from Lemma 5.11, $J(f)$ is locally connected at z_0. If otherwise, then the forward orbit intersects $\bigcap_{d\in\mathbb{N}} P_d(0) \cap K(f)$ which is the singleton $\{0\}$, so z_0 lies in the critical grand orbit anyway. ☐

The following is the full theorem by Yoccoz. Details of the proof (refer to [Hub93]) will not be discussed here. The main technique used in Yoccoz's theorem is an analogue of puzzles, called *parapuzzles*, on the parameter plane.

Theorem 5.13 (Yoccoz)**.** *Let $f = f_c$ be an at most finitely renormalisable quadratic map with no non-repelling periodic cycles.*

- *(A) Any infinite nest of puzzle pieces always shrink to a point and in particular $J(f)$ is locally connected,*
- *(B) The parameter c lies in $\partial\mathbb{M}$ and $\mathbb{M}$ is locally connected at c.*

Remark. The MLC statement in (B) above can be extended to all finitely renormalisable maps. If the map f_c has an attracting cycle, c lies in a hyperbolic component and local connectivity is trivial. The case where f_c has an indifferent cycle is non-trivial and is discussed in [Hub93] as well.

Yoccoz's theorem was one of the first major breakthrough to the MLC at the time. It reduces the problem to parameters which are infinitely renormalisable. Infinite renormalisation will be the central theme of the next chapter.

Chapter 6

Renormalisation Fixed point

6.1 The Space of Quadratic-Like Germs

Recall that the set of closed subsets of $\mathbb{C}$ is a metric space with respect to the Hausdorff metric d_H defined by

$$d_H(A, B) := \inf\{\epsilon \geq 0 \mid A \subseteq B_\epsilon \text{ and } B \subseteq A_\epsilon\},$$

where $B_\epsilon := \{x \in \mathbb{C} \mid \text{dist}(x, B) < \epsilon\}$ denotes the ϵ-neighbourhood of B, and the same goes to A_ϵ. The set of all open subsets of $\mathbb{C}$ is also a metric space with Hausdorff metric defined by

$$d_H(A, B) := d_H(\mathbb{C}\backslash A, \mathbb{C}\backslash B).$$

Denote the set of all quadratic-like maps with connected filled Julia set as $\mathcal{QL}$. This set is equipped with *Carathéodory topology* defined by saying that a sequence of quadratic-like maps $f_n : U_n \to V_n$ converges to $f : U \to V$ if and only if $U_n \to U$ in the Hausdorff topology and $f_n \to f$ uniformly on compact subsets of U.

Let $\mathcal{Q} := \{f_c \mid c \in \mathbb{M}\}$ be the space of quadratic maps with connected filled Julia set up to affine conjugacy. This space is endowed with the usual compact-open topology. The straightening operator $\chi : \mathcal{QL} \to \mathcal{Q}$ sends each quadratic-like map f to the unique quadratic map f_c representing its hybrid class. Hybrid conjugacy gives rise to foliations in $\mathcal{QL}$ where each $c \in \mathbb{M}$ defines a unique leaf $\mathcal{QL}_c = \chi^{-1}(f_c)$.

Lemma 6.1. *The straightening operator $\chi : \mathcal{QL} \to \mathcal{Q}$ is a continuous surjection.*

The need to adjust the domain for a quadratic-like map poses many unnecessary problems. We will consider the germs of quadratic-like maps up to affine conjugacy by introducing an equivalence relation $\sim$ on $\mathcal{QL}$ where $f \sim g$ if and only if f and g are affine conjugate around some neighbourhoods of their filled Julia sets.

Definition 6.1. Each equivalence class $[f]$ is called a *quadratic-like germ* and its space is denoted by $\mathcal{G} = \mathcal{QL}/\sim$.

Any quadratic-like germ $[f]$ has a normalised representative f such that the β-fixed point is 1. By Proposition 4.5, all normalised quadratic-like representatives of a germ have the same filled Julia set $K(f)$. Thus, each quadratic-like germ $[f]$ has a well-defined filled Julia set $K(f)$ which is 0-symmetric and has β-fixed point 1.

If a quadratic-like map f is real, i.e. commutes with $z \to \bar{z}$, then it is not difficult to show that $K(f)$ is symmetric about the real axis; if f is normalised, $K(f) \cap \mathbb{R} = [-1, 1]$. This will then restrict to a unimodal map $f : [-1, 1] \to [-1, 1]$ with critical point at 0.

We define a topology on the space of germs $\mathcal{G}$ by saying that $[f_n] \to [f]$ if and only if there are representatives $f_n : U_n \to V_n$ in $\mathcal{QL}$ converging to $f : U \to V$ in the Carathéodory topology.

Since $\chi(f) = \chi(g)$ whenever $f \sim g$ in $\mathcal{QL}$, we also have leaves $\mathcal{G}_c = \mathcal{QL}_c / \sim$ representing hybrid classes in the germ level. Moreover, we can adapt Lemma 6.1 to the topology on $\mathcal{G}$.

Lemma 6.2. *The straightening operator $\chi : \mathcal{G} \to \mathcal{Q}$ is a continuous surjection.*

6.2 A Priori Bounds

This section is focused on a crucial precompactness property called *a priori bounds*. The following is a theorem by McMullen.

Theorem 6.3. *For any $\mu > 0$, the space $\mathcal{QL}(\mu) := \{f : U \to V \in \mathcal{QL} \,|\, \mathrm{mod}(V \backslash \overline{U}) \geq \mu\}$ is precompact up to affine conjugacy.*

Corollary 6.4. *For any $\mu > 0$, the space $\mathcal{G}(\mu) = \{[f] \in \mathcal{G} \mid f \in \mathcal{QL}(\mu)\}$ is precompact.*

Proposition 6.5. *Let $f : U \to V$ be a quadratic-like map in $\mathcal{QL}(\mu)$. There exist topological disks $U' \subset U$ and $V' \subset V$ with smooth boundaries and constants m_μ, d_μ, C_μ, D_μ and E_μ depending only on μ such that we have the following:*

(A) $f : V' \to U'$ is a quadratic-like map in $\mathcal{QL}(m_\mu)$;

(B) U' and V' are C_μ quasidisks with eccentricity bounded by E_μ about 0;

(C) f can be expressed as a composition $h \circ f_0$, where $h : f_0(U') \to V'$ is a biholomorphism with distortion bounded by a constant d_μ;

(D) $f : U' \to V'$ can be straightened by a hybrid conjugation $\psi : \mathbb{C} \to \mathbb{C}$ of quasiconformal dilatation D_μ.

Proof. The first part is merely a modification of Lemma 4.1. Let $g : \mathbb{D} \to V$ be a Riemann map with $g(0) = 0$. From Lemma 2.17, there is some $r_\mu \in (0, 1)$ such that $g^{-1}(\overline{U}) \subset \mathbb{D}_{r_\mu}$. Then, we can define the subset $V' \subset V$ as the topological disk containing U with boundary $\partial V' = g(\mathbb{T}_{\sqrt{r_\mu}})$, and $U' \subset U$ as the preimage of V' under f. By

Grötzsch inequality,

$$\operatorname{mod}(V'\backslash\overline{U}') = \operatorname{mod}\big(\mathbb{D}_{\sqrt{r_\mu}}\backslash g^{-1}(\overline{U}')\big) \geq \operatorname{mod}(\mathbb{A}_{r_\mu,\sqrt{r_\mu}}).$$

Thus, (A) holds immediately with $m_\mu := \operatorname{mod}(\mathbb{A}_{r_\mu,\sqrt{r_\mu}})$.

Notice that $\partial V'$ and $\partial U'$ are the core curves of $V\backslash g(\overline{\mathbb{D}_{r_\mu}})$ and $U\backslash f^{-1}g(\overline{\mathbb{D}_{r_\mu}})$. By Propositions 2.21 and 2.23, we have (B).

The composition $f = h \circ f_0$ follows from 0-symmetry. Applying Koebe distortion theorem, g and $h^{-1} \circ g$ have bounded distortion on $\overline{V}'$ depending on $\sqrt{r_\mu}$, so then h will have distortion bounded by some d_μ. (C) holds.

In the proof of Theorem 4.3, the dilatation of the hybrid conjugation ultimately depends on the dilatation of the tubing $\phi : V'\backslash\overline{U'} \to \mathbb{A}_{r,r^2}$ for some $r > 1$, so we will focus on the construction of ϕ.

Define a map $\phi : \partial V' \to \mathbb{T}_{r^2}, z \mapsto \frac{r^2}{\sqrt{r_\mu}} g^{-1}(z)$. By Koebe distortion, ϕ and ϕ^{-1} have bounded derivatives depending only on μ. Thus, it is bi-Lipschitz and consequently D'_μ-quasisymmetric.

The map $f : \partial U' \to \partial V'$ has a bounded distortion, hence it is Lipschitz. It then follows that ϕ lifts to a D''_μ-quasisymmetric homeomorphism satisfying $\phi \circ f = f_0 \circ \phi$ on $\partial U'$. By interpolation in Corollary 2.13, we have a quasiconformal homeomorphism $\phi : V'\backslash\overline{U'} \to \mathbb{A}_{r,r^2}$ with dilatation depending on D'_μ and D''_μ - both of which depend only on μ. This tubing ϕ thus gives us a D_μ-quasiconformal hybrid conjugation $\psi : V' \to \psi(V')$.

To extend the domain to the whole $\mathbb{C}$, it is sufficient to extend ϕ outside V'. As V' is a quasidisk, we have some quasiconformal homeomorphisms e_1 and e_2 on $\hat{\mathbb{C}}$ where $e_1(\overline{V'}) = e_2(\overline{\mathbb{D}_{r^2}}) = \overline{\mathbb{D}}$. Then, since $e_2 \circ \phi \circ e_1^{-1}$ is quasisymmetric on the unit circle $\mathbb{T}$, we can do partial interpolation to obtain a quasiconformal homeomorphism from $\hat{\mathbb{C}}\backslash\mathbb{D}$ to itself. Lift this extension to a quasiconformal map $\phi : \hat{\mathbb{C}}\backslash\overline{V'} \to \hat{\mathbb{C}}\backslash\overline{\mathbb{D}_{r^2}}$ with dilatation depending only on μ. Thus, (D) holds. □

Definition 6.2. An infinitely renormalisable quadratic-like map $f : U \to V$ is said to have *a priori bounds* when there is a constant $\mu > 0$ such that for each renormalisation level n of f, there is a renormalisation representative $f^n : U_n \to V_n$ in $\mathcal{QL}(\mu)$.

Precompactness ensures that the sequence of renormalisations $\{f^n\}_{n\in\mathcal{R}(f)}$ of any infinitely renormalisable quadratic-like map f with a priori bounds has a limit point. This property has been the central assumption in many results on fixed points of the renormalisation operator. Before delving into renormalisation fixed point, we state some properties which come with a priori bounds.

For an infinitely renormalisable quadratic map f, we will denote by $\mathcal{O}_f$ the set of points contained in some little Julia sets of all renormalisation levels, i.e.

$$\mathcal{O}_f := \bigcap_{n\in\mathcal{R}(f)} \bigcup_{i=1}^{n} K_n(i).$$

Lemma 6.6. *Let $f : U \to V$ be an infinitely renormalisable quadratic-like map. The following are equivalent:*

(A) $\sup_{i \leq n} \operatorname{diam} K_n(i) \to 0$ *as n increases in $\mathcal{R}(f)$;*

(B) $\mathcal{O}_f$ *is a Cantor set.*

Proof. Label the elements of $\mathcal{R}(f)$ as an increasing sequence $n_1 < n_2 < n_3 < \ldots$. Define another sequence of positive integers $\{m_i\}_{i \in \mathbb{N}}$ by setting $m_1 := n_1$ and $m_i := \frac{n_i}{n_{i-1}}$ for $i \geq 2$. Define $D_i = \{0, 1, \ldots, m_i - 1\}$, a finite set endowed with discrete topology. The infinite product $C = \prod_{i=1}^{\infty} D_i$ equipped with the product topology is metrisable by the metric $d\big((x_1, x_2, \ldots), (y_1, y_2, \ldots)\big) := \sum_{i=1}^{\infty} 2^{-i} |x_i - y_i|$. Then, C is non-empty, compact, perfect, totally disconnected, and metrisable, hence, a Cantor set.

Suppose (A) holds. Any sequence $\mathbf{b} = (b_1, b_2, \ldots) \in C$ naturally induces a unique sequence $\mathbf{a} = (a_1, a_2, \ldots)$ such that $K_{n_1}(a_1) \supset K_{n_2}(a_2) \supset K_{n_3}(a_3) \supset \ldots$ by setting $a_1 := b_1$ and $a_{i+1} = a_i + b_{i+1} n_i$, and letting $K_{n_i}(0) = K_{n_i}(n_i)$ and for convenience. As such, we can define an injection $\Phi : C \to \mathcal{O}_f$ where $\Phi(\mathbf{b})$ is the unique point in $\bigcap_{i \in \mathbb{N}} K_n(a_i)$.

The map Φ is a bijection since any point $w \in \mathcal{O}_f$ induces a unique sequence $(a_1, a_2, \ldots)$ satisfying $w \in K_{n_i}(a_i)$ for each i, and consequently a unique sequence $(b_1, b_2, \ldots)$ satisfying $b_1 = a_1$ and $b_i = \frac{a_i - a_{i-1}}{n_{i-1}}$. The basis topology of C is generated by cylinder sets of the form

$$I_{(b_1, \ldots, b_k)} = \{(x_1, x_2, \ldots) \in C \mid x_i = b_i \text{ for } i \leq k\}$$

and $\Phi(I_{(b_1, \ldots, b_k)}) = K_{n_k}(a_k) \cap \mathcal{O}_f$, where a_k is determined from the same recurrence relation as above. Each $K_{n_k}(a_k) \cap \mathcal{O}_f$ is compact and it must be disjoint from its complement $\mathcal{O}_f \backslash K_{n_k}(a_k)$, since otherwise it will intersect at a repelling periodic point and contradict Corollary 4.17. Hence, it is an open subset of $\mathcal{O}_f$. As Φ^{-1} is a continuous bijection, $\mathcal{O}_f$ is compact, and C is Hausdorff, Φ is indeed a homeomorphism, so (B) holds.

Conversely, assume (B) instead. By Corollary 4.17, each $K_{n_i}(m) \cap \mathcal{O}_f$ is an open subset of $\mathcal{O}_f$ disjoint from $K_n(j)$ for $j \neq m$. Thus, connected components of $\mathcal{O}_f$ are of the form $\bigcap_{n \in \mathcal{R}(f)} K_{n_i}(a_i)$ which are singletons due to total disconnectivity. Hence, (A) holds. □

Theorem 6.7. *If $f : U \to V$ be an infinitely renormalisable quadratic-like map with a priori bounds, then any decreasing nest of filled Julia sets $\{J_n(a_n)\}_{n \in \mathcal{R}(f)}$ shrinks to a point, i.e.* $\operatorname{diam} J_n(a_n) \to 0$ *as n increases in $\mathcal{R}(f)$. Moreover, $P(f)$ is a Cantor set.*

Proof. There is some $\mu > 0$ such that for each renormalisation level $n \in \mathcal{R}(f)$, we have a renormalisation $f^n : U_n \to V_n$ of f in $\mathcal{QL}(\mu)$. We can assume by Proposition 6.5 that each U_n and V_n have bounded eccentricity about 0 depending on μ.

Suppose there is an open neighbourhood W of 0 in $\bigcap_{n \in \mathcal{R}(f)} U_n$, then $J(f) \subset f^N(W)$ for sufficiently large $N \in \mathbb{N}$. But since $f^N(W) \subset V_N$, there is a contradiction. Hence, the inner radius of U_n about 0 must converge to 0 as $n \to \infty$ in $\mathcal{R}(f)$.

By bounded eccentricity of U_n, $\mathrm{diam} K_n \leq \mathrm{diam} U_n \to 0$. Consider any arbitrary decreasing nest of small filled Julia sets $K_n(a_n)$. The same argument as above would apply to the inner radius of $U_n(a_n)$ about a point in $\bigcap_{n \in \mathcal{R}(f)} P_n(a_n)$ (Cantor's intersection theorem guarantees the existence of this point as each $P_n(a_n)$ is compact). Thus, any decreasing nest $K_n(a_n)$ and $P_n(a_n)$ shrinks to a point. In particular, $P(f) = \mathcal{O}_f$. By Lemma 6.6, $P(f)$ is a Cantor set. □

Corollary 6.8. *If $f : U \to V$ is an infinitely renormalisable quadratic-like map with a priori bounds, then the Julia set $J(f)$ is locally connected at* 0.

Proof. Let $U \subset \mathbb{C}$ be an arbitrary open neighbourhood of 0. Let $0 \in J_n(a_n)$ for some a_n and all $n \in \mathcal{R}(f)$, then there is sufficiently high level n such that $J_n(a_n) \subset U$. As the n-renormalisation is DH, we can use appropriate puzzle pieces such that $J_n(a_n)$ is the infinite intersection of $P_d(0) \cap J(f)$ over all $d > 0$, so again there is sufficiently deep d where $\mathrm{int} P_d(0) \cap J(f)$, a connected neighbourhood of 0, is contained in U. □

Local connectivity throughout the whole Julia set is currently still an open problem. Hu and Jiang proved local connectivity under the additional assumption that f has bounded combinatorics. This assumption allows the renormalisation to be *unbranched*, i.e. where for all levels n, $V_n \cap P(f) = K_n \cap P(f)$. See [Jia00] and [McM96, Chapter 8].

Theorem 6.9. *Let f be an infinitely renormalisable quadratic map with bounded combinatorics and a priori bounds, then the Julia set $J(f)$ is locally connected.*

The result turns out to be negative for some maps without the bounded combinatorics assumption. This result is due to Douady, and Hubbard, and Sørensen (see [Sø00] [Mil00]). Levin ([Lev11]) improved their results using weaker combinatorial assumptions. We will only state a particular case of Levin's result below.

Theorem 6.10. *There is an infinitely renormalisable quadratic map f_c with unbounded combinatorics such that $J(f_c)$ is not locally connected, yet the Mandelbrot set $\mathbb{M}$ is locally connected at c.*

Typically, for a generic quadratic map f_c, the absolute value of the multiplier of a repelling periodic cycle tends to increase with the period. For instance, for the doubling map f_0, the multiplier of a repelling periodic cycle of period k has absolute value 2^k. Infinitely renormalisable maps having a priori bounds provide a counterexample to this observation.

Theorem 6.11. *Any infinitely renormalisable quadratic map f with a priori bounds with constant μ has an infinite sequence of repelling periodic cycles with multiplier values bounded by some $\lambda_\mu > 0$, a constant depending only on μ.*

Proof. Pick renormalisation representatives $\{f^n : U_n \to V_n\}_{n\in\mathcal{R}(f)}$ of f in $\mathcal{QL}(\mu)$. Take β_n, the β-fixed point of f^n in K_n. By Proposition 4.11, $\{\beta_n\}_{n\in\mathcal{R}(f)}$ is an infinite sequence of distinct repelling periodic points of f.

Pick a level $n \in \mathcal{R}(f)$ and let $\beta = \beta_n$. By Proposition 6.5, we can assume that the associated hybrid conjugation h, where $h \circ f^n = f_c \circ h$ for some unique $c \in \mathbb{M}$, has quasiconformal dilatation bounded by a constant $D_\mu \geq 1$. Let $\lambda := f^n(\beta)$ be the multiplier of β with respect to f^n and let $\lambda' := f_c'(h(\beta))$ be the multiplier of $h(\beta)$ with respect to f_c. By Proposition 3.14, we get an upper bound

$$|\lambda'| = 2|\phi(\beta)| \leq 4. \tag{6.1}$$

Denote by ϕ the Koenigs lineariser of f^n at β such that $\phi(\beta) = 0$ and ϕ is a conformal conjugation between f^n and the linear map $z \mapsto \lambda z$ in some neighbourhood of β. Similarly, let ψ be the Koenigs lineariser of f_c in a neighbourhood of $h(\beta)$.

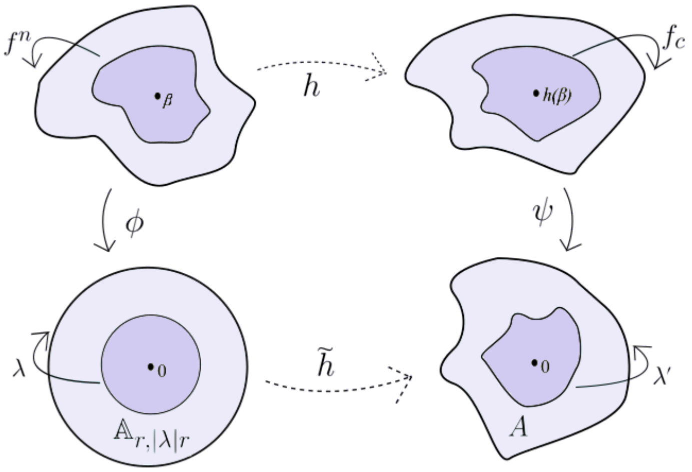

Figure 6.1: Koenigs linearisation and straightening.

We can pick a sufficiently small $r > 0$ such that ϕ^{-1} is univalent on $\mathbb{D}_{|\lambda|r}$ and ψ is univalent on $h(\phi^{-1}(\mathbb{D}_{|\lambda|r}))$. Then, $\tilde{h} = \psi\circ h\circ\phi^{-1}$ is a D_μ-quasiconformal homeomorphism on $\mathbb{A}_{r,|\lambda|r}$ onto its image $A = \tilde{h}(\mathbb{A}_{r,|\lambda|r})$. Then,

$$\operatorname{mod}(A) \geq \frac{1}{D_\mu}\operatorname{mod}(\mathbb{A}_{r,|\lambda|r}) = \frac{1}{2\pi D_\mu}\log(|\lambda|). \tag{6.2}$$

The annulus A is bounded by the image of two circles under $\tilde{h}$, namely γ and Γ where γ separates 0 from Γ. Let $\mathbb{A}_{s,t}$ be the smallest regular annulus containing A centred at 0, then it is clear that $|\lambda'|s \leq t$. By Proposition 2.19, the closed curve Γ has bounded eccentricity about 0, i.e. $t \leq E_\mu s|\lambda'|$, where $E_\mu > 0$ is some constant depending only on D_μ, hence depending only on μ. As such,

$$\operatorname{mod}(A) \leq \operatorname{mod}(\mathbb{A}_{s,t}) = \frac{1}{2\pi}\log\left(\frac{t}{s}\right) \leq \frac{1}{2\pi}\log\left(E_\mu|\lambda'|\right) \tag{6.3}$$

Combining (6.1), (6.2) and (6.3), we will obtain an upper bound: $|\lambda| \leq (4E_\mu)^{D_\mu}$. Consequently, the absolute value of the multiplier of f at any β is bounded by $\Lambda_\mu := (4L_\mu)^{D_\mu}$, which is independent of the renormalisation level $n \in \mathcal{R}(f)$. □

So far we have seen a number of properties of infinitely renormalisable maps having a priori bounds, but perhaps a more fundamental question to ask ourselves is whether or not such a map exists. This is answered rather nicely in Sullivan's paper [Sul88].

Theorem 6.12 (Sullivan's Complex Bounds). *Any infinitely renormalisable real quadratic map* $f(z) = z^2 + c$ *with bounded combinatorics has a priori bounds depending only on the combinatorics. Moreover,* f *is uniquely determined by its tuning invariant.*

Example 6.1. The Feigenbaum map f_{c_F} has stationary combinatorics. By Sullivan's, it is the unique map with tuning invariant $\langle -1, -1, -1, \ldots \rangle$ and it has a priori bounds. The fact that the postcritical set of f_{c_F} is a Cantor set is in sync with the bifurcation diagram in our introduction.

6.3 Renormalisation Fixed Point

Let $\mathcal{QL}^{(p)}$ be the set of all p-renormalisable quadratic-like maps in $\mathcal{QL}$, and let $\mathcal{G}^{(p)} := \mathcal{QL}^{(p)}/\sim$. We can define the renormalisation operator $\mathcal{R}_p : \mathcal{G}^{(p)} \to \mathcal{G}$ mapping a germ of a p-renormalisable map to the quadratic-like germ of its p-renormalisation. This operator is well-defined as any renormalisation representative has the same filled Julia set. If p is not specified, then it is taken to be the first renormalisation level.

In this section, we are primarily interested in the solutions of the Cvitanovic-Feigenbaum equation:

$$f^p(z) = af(a^{-1}z) \tag{6.4}$$

for some normalisation constant $a \in \mathbb{C}^*$ and integer $p \geq 2$. A quadratic-like germ $[f]$ is a fixed point of $\mathcal{R}_p : \mathcal{G}^{(p)} \to \mathcal{G}$ if and only if f satisfies (6.4) on a neighbourhood of its filled Julia set. To find renormalisation fixed points, we will follow McMullen's formulation of quadratic-like *towers* [McM96, Chapter 5].

Definition 6.3. Let S be a subset of $\mathbb{Q}_{>0}$ containing 1 and ordered by division. A *tower* T with *level set* S is a sequence of quadratic-like maps $\{g_s : U_s \to V_s\}_{s \in S}$ such that:

(A) each g_s has a critical point at 0 and connected filled Julia set $K(f_s)$;

(B) whenever $s < t \in S$, g_s is $\frac{t}{s}$-renormalisable, its renormalisation is hybrid conjugate to g_t, and $g_s^{t/s} = g_t$ on $K(g_t)$.

A tower has *a priori bounds* with constant $\mu > 0$ if for all $s \in S$, $g_s \in \mathcal{QL}(\mu)$. We will denote by $Tow(\mu)$ the space of all towers with a priori bounds with constant μ. A tower has *bounded combinatorics* if there is a constant $B > 1$ such that $\frac{t}{s} \leq B$ for any adjacent levels $s < t$ in the level set.

Example 6.2. Any renormalisable quadratic map f generates a natural tower $T = \{f_s : U_n \to V_n\}_{n\in S}$ with level set $S = \mathcal{R}(f) \cup \{1\}$, where $f_1 = f$. If f is infinitely renormalisable, the tower can be chosen to be infinite. Furthermore, if f has a priori bounds, then we can ensure that the tower T has a priori bounds.

Example 6.3. Let $f : U \to V$ be a normalised quadratic-like map in $\mathcal{QL}$ satisfying $\mathcal{R}_p[f] = [f]$. Equivalently, there is some $a \in \mathbb{C}^*$ such that $f^p(z) = af(a^{-1}z)$ for all $z \in K(f)$. Then, f induces a bi-infinite tower $T = \{f_s\}_{s\in S}$, where the level set is $S = \{p^n \mid n \in \mathbb{Z}\}$ and each map is defined by $f_{p^n} : a^n f^{-p^n}(V) \to a^n V, z \mapsto a^n f(a^{-n}z)$.

Proposition 6.13. *Let $T = \{f_s : U_n \to V_n\}_{n\in S}$ be a tower and for some $s < t \in S$, let U and V be components of $U_s \cap U_t$ and $V = V_s \cap V_t$ containing $K(f_t)$. Then, $f_t = f_s^{t/s}$ on U and $f_t : U \to V$ is a $\frac{t}{s}$-renormalisation of f_s.*

Proof. By holomorphic continuation, the region in which $f_t = f_s^{t/s}$ can be extended from a small neighbourhood of $K(f_t)$ to U, then $f_t : U \to V$ is still quadratic-like by Proposition 4.5 and it is indeed a renormalisation of f_s since $K_{t/s}(f_s) = K(f_t)$. ☐

Definition 6.4. A *conjugation* between two towers $T = \{f_s\}$ and $T' = \{g_s\}$ having the same level set S is a sequence of conjugations $\{\phi_s\}$ between f_s and g_s for each s. The towers T and T' are *conjugate* if such a conjugacy exists, and it is *quasiconformal/conformal* if the conjugacies at all levels are quasiconformal/conformal.

The following is a theorem by McMullen.

Theorem 6.14 (Rigidity of Towers). *Any bi-infinite tower $T \in Tow(\mu, B)$ is quasiconformally rigid, i.e. any quasiconformal conjugacy from T to another tower is conformal.*

In dealing with germs, a more useful notion would be one which does not depend on the domains.

Definition 6.5. Two towers $T = \{f_s\}$ and $T' = \{g_s\}$ having the same level set S are:

- *hybrid conjugate* if for each level s, f_s and g_s are hybrid conjugate to each other,
- *isomorphic* if for each level s, f_s and g_s can be restricted to smaller neighbourhoods of their respective filled Julia sets such that they are conformally conjugate.

Theorem 6.15. *Any pair of infinitely high towers T and T' in $Tow(\mu, B)$ are isomorphic if they are hybrid conjugate to each other.*

Recall that each baby Mandelbrot set M has an associated stretching homeomorphism $\sigma : M \to \mathbb{M}$. On the function space, we define analogously the homeomorphism

σ_p onto $\mathcal{Q}$ acting on the family of p-renormalisable quadratic maps f_c where $c \in M$.

Theorem 6.16. *Let f_c be an infinitely renormalisable quadratic map with a priori bounds. If $\sigma_p(f_c) = f_c$ for some $p > 1$, then:*

(A) there is a unique fixed point $[F] \in \mathcal{G}_c$ of the renormalisation operator $\mathcal{R}_p$;

(B) for any $[f] \in \mathcal{G}_c$, $\mathcal{R}_p^n([f]) \to [F]$ as $n \to \infty$.

Proof. Suppose $[F]$ is a fixed point of $\mathcal{R}_p$ in $\mathcal{G}_c$. Pick a normalised quadratic-like map $F : U \to V$ in $\mathcal{QL}$ representing $[F]$, then from example 6.3, $[F]$ induces a bi-infinite tower $T = \{F_s\}_{s \in S}$ where $S = \{p^n \mid n \in \mathbb{Z}\}$. By the assumption, T has a priori bounds and stationary combinatorics.

If $[\tilde{F}]$ is another fixed point of $\mathcal{R}_p$ in $\mathcal{G}_c$, the tower $\tilde{T}$ induced by $[\tilde{F}]$ will be hybrid conjugate to T since all maps in T and $\tilde{T}$ have the same hybrid class. By rigidity in Theorem 6.15, $\tilde{T}$ is isomorphic to T. In particular $[\tilde{F}] = [F]$.

Pick any $[f] \in \mathcal{G}_c$. For any quadratic-like representative $f : U \to V$ in $\mathcal{QL}_c$, there is a hybrid conjugation between f and f_c with dilatation depending on $\text{mod}(V \backslash \overline{U})$. The conjugation thus distorts moduli by a definite factor, so f must have a priori bounds with some constant m. The limit set of the orbit $\{\mathcal{R}_p^n[f]\}_{n \in \mathbb{N}}$ is non-empty since the entire orbit lies in a precompact set $\mathcal{G}(m)$. From the hypothesis, $\mathcal{G}_c$ is an invariant set of the operator $\mathcal{R}_p$, so by continuity of the straightening operator, the limit set is indeed contained in $\mathcal{G}_c$.

Let $[g_1]$ be a limit of the orbit and with a normalised quadratic-like representative g_1. Let T be the infinite tower with level set $S = \{p^n \mid n \in \mathbb{N}\}$ from $[g_1]$ generated by g_1. We can extend T to a bi-infinite tower as follows.

If $[g_1]$ is the limit of some subsequence $\{\mathcal{R}_p^{n_i}(f)\}_{i \in \mathbb{N}}$, then the limit point $g_{p^{-1}}$ of $\{\mathcal{R}_p^{n_i - 1}(f)\}_{i \in \mathbb{N}}$ satisfies $\mathcal{R}_p[g_{p^{-1}}] = [g_1]$. Consequently, we can pick a quadratic-like map $g_{p^{-1}}$ representing $[g_{p^{-1}}]$. Continue extending it inductively to obtain a bi-infinite sequence $\{g_n\}_{n \in \mathbb{Z}}$ satisfying $R_p(g_n) = g_{n+1}$. We then have a bi-infinite tower T, and it has a priori bounds since all maps in T must lie in $\mathcal{G}(m)$.

Let $[\tilde{g}_1]$ be another limit of the orbit $\{\mathcal{R}_p^n[f]\}_{n \in \mathbb{N}}$ with corresponding bi-infinite tower $\tilde{T}$, then T and $\tilde{T}$ are again hybrid conjugate. By Theorem 6.15, T and $\tilde{T}$ are isomorphic, so then $[\tilde{g}_1] = [g_1]$ and in particular, $\mathcal{R}_p^n[f] \to [g_1]$ as $n \to \infty$. As $[g_1]$ is a fixed point of $\mathcal{R}_p$, $[g_1] = [F]$. We have then obtained existence and convergence. ☐

The next goal is to obtain from the fixed point $[F] \in \mathcal{G}_c$ a holomorphic map satisfying the Cvitanovic-Feigenbaum equation (6.4). To do this, we wish to extend the domain of a normalised quadratic-like representative of the germ and obtain some uniqueness property.

Definition 6.6. A holomorphic map $f : W \to \mathbb{C}$ is an *extended quadratic-like* map if $W \subset \mathbb{C}$ is a topological disk containing 0, f has a critical point at 0 and can be restricted to a quadratic-like map in $\mathcal{QL}$. The space of all extended maps is denoted by $\mathcal{H}$.

An extended quadratic-like map $f \in \mathcal{H}$ obviously has a well-defined filled Julia set $K(f)$ since any two quadratic-like restrictions of f have the same filled Julia set due to Proposition 4.5. Moreover, the straightening operator $\chi : \mathcal{H} \to \mathcal{Q}$ can be defined by taking the straightening of a quadratic-like restriction. As such, we can also denote the fibres of χ in $\mathcal{H}$ as the leaves $\mathcal{H}_c$, for $c \in \mathbb{M}$.

The topology of $\mathcal{H}$ is defined by saying that $f_n : W_n \to \mathbb{C}$ converges to $f : W \to \mathbb{C}$ if and only if for any compact subset $K \subset W$, $K \subset W_n$ for all sufficiently large n and $f_n \to f$ uniformly on K.

Similar to $\mathcal{G}$, we can define the renormalisation operator $\mathcal{R}_p$ on $\mathcal{H}$ as follows. If $f : W \to \mathbb{C}$ is an extended quadratic-like map which can be restricted to a p-renormalisable quadratic-like map, then define

$$\mathcal{R}_p f : W' \to \mathbb{C}, \quad z \mapsto a f^p(a^{-1}z),$$

where $a \in \mathbb{C}^*$ is a normalising constant and $W' = aW''$ where W'' is a component of $f^{-p}(\mathbb{C})$ containing 0. We will adapt McMullen's argument in [McM96, §7.3].

Theorem 6.17. *Let $[F]$ be a fixed point of the renormalisation operator $\mathcal{R}_p$ in $\mathcal{G}_c$ for some $c \in \mathbb{M}$. Then:*

(A) any normalised quadratic-like map $F : U \to V$ with germ $[F]$ has a unique maximal analytic continuation $\tilde{F} : W \to \mathbb{C}$ in $\mathcal{H}$;

(B) the map $\tilde{F}$ is the unique fixed point of $\mathcal{R}_p$ in $\mathcal{H}$ with germ $[F]$;

(C) for any $f \in \mathcal{H}_c$, $\mathcal{R}_p^n f \to \tilde{F}$ as $n \to \infty$.

Proof. Let $F_n := \mathcal{R}_p^n F : W_n \to a^n V$ where $W_n = a^n F^{-p^n}(V)$. Since $\mathcal{R}_p[F] = [F]$, we have $F_n = F$ on a neighbourhood of $J(F)$. Each F_n is then a proper analytic continuation of F.

Observe that $a^n V \to \mathbb{C}$ in the Hausdorff topology since V contains an open neighbourhood of 0. Let $W = \bigcup_{n \in \mathbb{N}} W_n$. Pick any compact connected set K together with an analytic continuation $\hat{F}$ of F defined in a small neighbourhood W' of K, then $\hat{F}(K) \subset a^N V$ for some sufficiently large N. Obviously, $\hat{F}$ is an analytic continuation of F_n, and by properness, the open subset of $W' \cup W_N$ on which $\hat{F} = F$ holds is closed, so then by connectedness, $\hat{F} = F$ on the whole W' and $K \subset W' \subset W_N$. In particular, the argument holds for any compact connected set $K \subset W$, so then $W_n \to W$ in the Carathéodory topology. This proves the maximality of the domain W and existence of the limit $\tilde{F} : W \to \mathbb{C}$ of F_n, which is an analytic continuation of F.

To show that $\tilde{F} \in \mathcal{H}$, we need that W is simply connected. Pick any $n \in \mathbb{N}$, then $a^n V$ is eventually contained in $a^N V$ and by the same closed-open argument as above, $W_n \subset W_N$ for all sufficiently large N. Now pick any simple closed loop $\gamma \subset W$. By compactness, γ is covered by a finite number of W_n's, all of which are eventually contained in W_N for some sufficiently large N. As W_N is simply connected, the region bounded by γ lies in W and in particular W must be simply connected.

The domain of $\mathcal{R}_p \tilde{F}$ is W since it cannot be larger than W by maximality and any compact K in the domain must also be contained in W_n for sufficiently large n. As $\mathcal{R}_p\tilde{F} = \mathcal{R}_p \lim \mathcal{R}_p^n F = \lim \mathcal{R}_p^{n+1} F = \tilde{F}$, $\tilde{F}$ is fixed by $\mathcal{R}_p$.

Let $G : Z \to \mathbb{C}$ be another fixed point of $\mathcal{R}_p$ in $\mathcal{H}$ with germ $[F]$. Using similar arguments, its quadratic-like restriction $G|_{U'} : U' \to V'$ for some open topological disks $U' \Subset V'$ will have unique maximal continuation $\tilde{F}$ and $\mathcal{R}_p^n G|_{U'} \to \tilde{F}$. Thus, $W \subset Z$. By maximality of W, it follows that $G = \tilde{F}$.

Pick any $f \in \mathcal{H}_c$ and assume that it is normalised (else, take $\mathcal{R}_p f$ instead). It is sufficient to prove convergence for some quadratic-like restriction of f. The restriction of f is hybrid conjugate to F via some quasiconformal map ϕ. Recall from 6.5 that we can assume ϕ to be defined on the whole $\hat{\mathbb{C}}$. We then also obtain a sequence of quasiconformal maps $\phi_n : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ acting as hybrid conjugacies between $\mathcal{R}_p^n f$ and $\mathcal{R}_p^n F$ on their respective neighbourhood of Julia sets.

From the proof of Theorem 6.16 $[\mathcal{R}_p^n f] \to [F]$, so any limit of ϕ_n, which exists following compactness in Theorem 2.8, 1 and ∞, gives a quasiconformal conjugacy between the bi-infinite tower generated by F and itself. By rigidity in Theorem 6.14, the towers are affinely conjugate and in fact it will be the identity. Thus, $\phi_n \to Id$ and we have $\lim \mathcal{R}_p^n f = \lim \mathcal{R}_p^n F = \tilde{F}$. □

Recall that infinitely renormalisable real quadratic maps f_c where $c \in \mathbb{R}$ always have a priori bounds.

Corollary 6.18. *For any real maximal superstable parameter $\tilde{c} \in \mathbb{M} \cap \mathbb{R}$ of period $p > 1$, the corresponding stretching homeomorphism $\sigma : M \to \mathbb{M}$ has a fixed point $c \in \mathbb{M} \cap \mathbb{R}$ and we have the following:*

- *(A) there is a unique quadratic-like germ $[F] \in \mathcal{G}_c$ fixed by $\mathcal{R}_p$ and for any germ $[f] \in \mathcal{G}_c$, $\mathcal{R}_p^n([f]) \to [F]$;*
- *(B) there is a unique extended quadratic-like map $\tilde{F} \in \mathcal{H}_c$ fixed by $\mathcal{R}_p$ and for any map $f \in \mathcal{H}_c$, $\mathcal{R}_p^n(f) \to \tilde{F}$.*

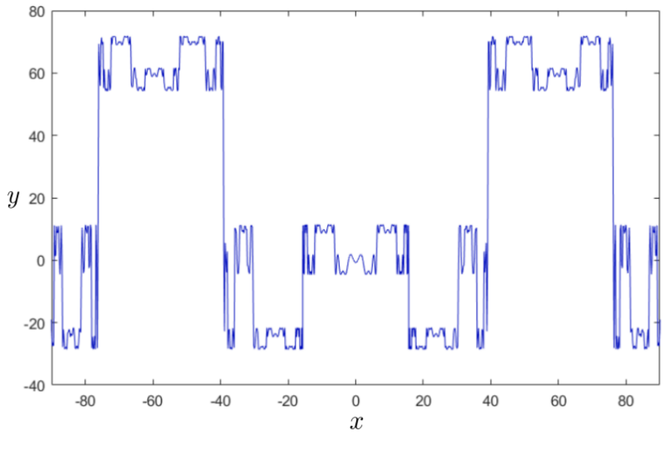

Figure 6.2: The real graph of the analytic continuation of the renormalisation fixed point corresponding to the Feigenbaum map f_{c_F}.

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