Holomorphic Dynamics and Several Complex Variables: A Brief Summary in Preparation for the Oral Exam

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Syllabus

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Chapter 1

Holomorphic Dynamics

Most of the content in this chapter is based on the books of McMullen[1] and Lyubich[2].

1.1 Quasiconformal maps

Definition 1. A K*-quasiconformal* ($K \geq 1$) map $\phi : U \to V$ is an orientation preserving homeomorphism between domains $U, V \subset \hat{\mathbb{C}}$ satisfying any of the following equivalent criteria:

1. for any curve family Γ in U, $\frac{1}{K}\text{mod}(\Gamma) \leq \text{mod}(\phi(\Gamma)) \leq K\text{mod}(\Gamma)$;
2. ϕ is absolutely continuous on lines (ACL), i.e. locally absolutely continuous on almost every horizontal and vertical lines, and $|\phi_{\bar{z}}| \leq \frac{K-1}{K+1}|\phi_z|$ almost everywhere on U;
3. ϕ has locally integrable distributional derivatives and $|\phi_{\bar{z}}| \leq \frac{K-1}{K+1}|\phi_z|$ as distributions.

Proposition 1.1.1. *The space of normalised K-quasiconformal maps $f : U \to V$ is compact in the compact-open topology.*

Definition 2. The *complex dilatation* of a quasiconformal map $\phi : U \to V$ is the Beltrami form $\mu_\phi = \frac{\bar{\partial}\phi}{\partial\phi} = \phi^*\sigma$, where σ is the zero Beltrami form.

Theorem 1.1.2 (Ahlfors-Bers' Measurable Riemann Mapping Theorem)**.** *For every measurable $\mu : U \subset \hat{\mathbb{C}} \to \mathbb{D}$ with supremum norm $\|\mu\|_\infty = k < 1$, there is a quasiconformal map ϕ on U with complex dilatation $\mu_\phi = \mu$ (almost everywhere on U). Moreover, ϕ is unique up to post-composition with a conformal isomorphism and it depends holomorphically on μ.*

[1] C. McMullen. Complex Dynamics and Renormalization. Princeton University Press, 1994.

[2] M. Lyubich. Conformal Geometry and Dynamics of Quadratic Polynomials, Vol I-II. Book in preparation, http://www.math.stonybrook.edu/ mlyubich/book.pdf, 2020.

1.2 Dynamics of Rational Maps

Definition 3. The *Fatou set* $F(f)$ of a holomorphic map $f : X \to X$ on a Riemann surface X is the open set of points $z \in X$ at which the restriction of the forward iterates $(f^n|_U)_{n\in\mathbb{N}}$ to some small neighbourhood U of z forms a normal family. The *Julia set* $J(f)$ of f is the complement $X \backslash F(f)$.

By Montel's theorem, whenever X is a hyperbolic Riemann surface, $F(f) = X$ and $J(f) = \emptyset$. We will restrict ourselves to the case where $X = \hat{\mathbb{C}}$ and f is a degree $d > 1$ rational map. Below is a list of basic properties of Fatou and Julia sets.

1. $F(f)$ is open, $J(f)$ is non-empty and perfect, and both are invariant under f;
2. The number of components of $F(f)$ is either 0, 1, 2, or ∞.
3. f is topologically transitive on $J(f)$;
4. For almost all $z \in J(f)$, the set of all iterated preimages of z is dense in $J(f)$;
5. If $J(f) \neq \hat{\mathbb{C}}$, then $J(f)$ is nowhere dense in $\hat{\mathbb{C}}$;
6. All parabolic periodic cycles are contained in $J(f)$;
7. The set of all repelling periodic points is dense in $J(f)$.

One example of a rational map with $J(f) = \hat{\mathbb{C}}$ is a *Lattès* map. This is obtained by taking the quotient of an affine map on a complex torus via the Weierstrass $\mathcal{P}$-function.

The local dynamics near a p-periodic point z_0 depend on the multiplier λ.

1. Attracting / repelling case:
   When $|\lambda| \neq 0, 1$, there is a conformal map ϕ near z_0 such that $\phi(z_0) = 0$ and $\phi \circ f^p(z) = \lambda\phi(z)$;
2. Superattracting case:
   When $\lambda = 0$, there is a conformal map ϕ near z_0 such that $\phi(z_0) = 0$ and $\phi \circ f^p(z) = \phi(z)^2$.
3. Parabolic case:
   When $\lambda = e^{2\pi i p'/q}$ for some coprime positive integers $p' < q$, if $f^q \not\equiv \mathrm{Id}$, there are ν disjoint petals rooted at z_0 and cyclically labelled as $P_1, \dots P_\nu$ such that:
   - f rotates the flower $\bigcup_{m=1}^{\nu} P_m$ with rotation number $\frac{p'}{q}$;
   - $f^n(z) \to z_0$ for all $z \in P_j$;
   - There are conformal maps ϕ_j on P_j such that $\phi_j \circ f(z) = \phi_j(z) + 1$;

- μ is a multiple of q and the germ of f near z_0 looks like $e^{2\pi i p'/q} z + az^{\nu+1} + \ldots$.

4. Siegel case:
When $\lambda = e^{2\pi i\theta}$ for some irrational θ and $z_0 \in F(f)$, then there is a conformal map ϕ near z_0 such that $\phi(z_0) = 0$ and $\phi \circ f^p(z) = \lambda\phi(z)$.

5. Cremer case:
When $\lambda = e^{2\pi i\theta}$ for some irrational θ and $z_0 \in J(f)$, there is no such linearisation.

In cases 1 (attracting) and 3, the interior of the set of all points z such that $f^{np}(z) \to z_0$ as $n \to \infty$ is called the immediate basin of attraction of z_0. Its union with all the iterated preimages is called the basin of attraction of z_0. In case 4, the largest neighbourhood of z_0 on which ϕ is well-defined is called the *Siegel disk* at z_0.

By Riemann-Hurwitz formula, every rational map f of degree d contains $2d-2$ critical points. Denote the set of critical points by $C(f)$ and the *postcritical set*, i.e. closure of all forward iterates of critical values by $P(f)$.

Theorem 1.2.1. *Let f be a rational map of degree $d \geq 2$.*

1. *When $|P(f)| \leq 2$, then f is conjugate to $z^{\pm d}$.*

2. *When $|P(f)| \geq 3$, then $\hat{\mathbb{C}} \backslash P(f)$ admits a hyperbolic metric $\|\cdot\|$.*

   (a) If $f(z) \notin P(f)$, then $\|f'(z)\| \geq 1$;
   (b) If $z \in J(f)$ and if the forward orbit $\mathcal{O}_f^+(z)$ is disjoint from $P(f)$, then $\|(f^n)'(z)\| \to \infty$ as $n \to \infty$.

Theorem 1.2.2. *The postcritical set $P(f)$ contains all attracting, parabolic and Cremer periodic cycles of f, and the boundary of Siegel disks and Herman rings.*

Theorem 1.2.3 (Ergodic or Attracting)**.** *Either $J(f) = \hat{\mathbb{C}}$ and f acts ergodically on $\hat{\mathbb{C}}$, or dist$(f^n(z), P(f)) \to 0$ in spherical distance as $n \to \infty$ for almost all $z \in J(f)$.*

Thus, $P(f)$ is the measure-theoretic attractor of $f|_{J(f)}$ when $J(f)$ has positive area.

Theorem 1.2.4 (Classification + No wandering domains)**.** *Every component U of the Fatou set of a rational map f is preperiodic. If it is periodic of period p, then one of the following cases holds:*

1. *U is an immediate basin of attraction of an attracting periodic point lying inside U;*

2. *U is an immediate basin of attraction of a parabolic periodic point lying on ∂U;*

3. *U is a Siegel disk;*

4. *U is a Herman ring, i.e. an annular domain on which f^p is conjugate to an irrational rotation.*

The theorem above relies on Sullivan's proof of the No Wandering Domains theorem (via quasiconformal defomations), the study of dynamics of holomorphic self-maps on hyperbolic Riemann surfaces, and the following Lemma.

Lemma 1.2.5 (Necklace Lemma)**.** *Let f be a holomorphic germ with fixed point 0 of multiplier ρ. If $|\rho| = 1$ and there is a domain Ω near but not containing 0 such that $f(\Omega) \cap \Omega \neq \emptyset$ and $f^n(\Omega) \to 0$, then $\rho = 1$.*

Theorem 1.2.6. *Let f be a rational map. The following are equivalent characterisations of* hyperbolicity *of f.*

1. *$J(f) \cap P(f) = \emptyset$;*

2. *$J(f)$ contains no critical points or parabolic cycles;*

3. *Every critical point lies in the basin of an attracting cycle;*

4. *There is a smooth conformal metric $\|\cdot\|_\rho$ on a neighbourhood of $J(f)$ and some $C > 1$ such that $\|f'(z)\| \geq C$ for all $z \in J(f)$;*

5. *For some $n \in \mathbb{N}$, f^n is strictly expanding w.r.t. spherical metric on $J(f)$.*

Proposition 1.2.7. *If f is a hyperbolic rational map and $J(f)$ is connected, then*

- *Each Fatou component is a Jordan disk;*

- *For any $\epsilon > 0$, there are only finitely many Fatou components of diameter $> \epsilon$;*

- *$J(f)$ is locally connected.*

Theorem 1.2.8. *Let f be a rational map. The following are equivalent characterisations of* subhyperbolicity *of f:*

1. *There is a conformal orbifold metric $\|\cdot\|_\rho$ on a neighbourhood of $J(f)$ and some $C > 1$ such that $\|f'(z)\| \geq C$ for all $z \in J(f)$;*

2. *Every critical point is either preperiodic or lies in the basin of an attracting cycle.*

Corollary 1.2.9. *The Julia set $J(f)$ of a subhyperbolic rational map f has zero area unless it is a Lattès map.*

Proposition 1.2.10. *If f is a subhyperbolic rational map and $J(f)$ is connected, then $J(f)$ is locally connected.*

In contrast, there are rational maps f with connected but not locally connected Julia set. One example of which is the Julia set of a quadratic polynomial which has a Cremer periodic cycle.

Definition 4. A *line field* on a subset E of a Riemann surface X is a family of tangent lines through the origin on the tangent space T_zX for each $z \in E$.

Each tangent line can be described by either a unit vector field $v = \pm v(z)\partial_z$ or a Beltrami form $\mu = \mu(z)\frac{d\bar{z}}{dz}$, where $|\mu| = 1$, acting on T_zX in the following way:

$$\mu\left(a(z)\partial_z\right) = \mu(z)\frac{\overline{a(z)}}{a(z)}.$$

v and μ are related by the equation $\mu(v) = 1$. The line field μ is holomorphic / meromorphic if locally, $\mu = \frac{\bar{\phi}}{|\phi|}$ for some holomorphic / meromorphic quadratic differential $\phi = \phi(z)^2 dz^2$ on E. Such a ϕ is unique up to multiplication by a positive number; we say that μ is dual to ϕ.

Definition 5. A rational map f admits an *invariant line field* if there is a measurable line field μ on a subset $E \subset J(f)$ of positive measure (take $\mu = 0$ outside of E) such that $f^*\mu = \mu$ almost everywhere.

Example 1. Let $X = \mathbb{C}\backslash\mathbb{Z} + \tau\mathbb{Z}$, $\tau \in \mathbb{H}$, be a complex torus and define the linear map $L(z) = nz$ on X where $n \in \mathbb{N}_{>1}$. The constant line field ∂_z on X is invariant under L. Therefore, the induced Lattès map $f : \mathbb{C} \to \mathbb{C}$, called an *integral torus endomorphism*, admits an f-invariant line field v on the whole Julia set, which is the whole $\hat{\mathbb{C}}$.

Lemma 1.2.11. *A rational map is an integral torus endomorphism if it admits an invariant line field μ which is holomorphic on an open subset of $J(f)$.*

Theorem 1.2.12. *A rational map f is an integral torus endomorphism if it admits an invariant line field on $J(f)$ and $P(f)$ is not the measure-theoretic attractor on $J(f)$.*

1.3 Dynamics of Quadratic Maps

Let f be a monic polynomial of degree d. The dynamics of f satisfies the following properties:

- The Fatou set $F(f)$ contains no Herman rings;
- ∞ is a superattracting fixed point and a critical point of multiplicity $d-1$.
- There are $d-1$ finite non-repelling periodic cycles counting multiplicity.
- There is a neighbourhood A of ∞, some $r \geq 1$ and a conformal isomorphism called the *Böttcher map* $B : A \to \hat{\mathbb{C}} \backslash \bar{\mathbb{D}}_r$ fixing ∞ such that $B \circ f(z) = B(z)^d$. B is unique up to post-composition with rotation by $(d-1)^{\text{th}}$ roots of unity.
- No finite critical points lie on the basin of infinity $\mathcal{A}_f(\infty)$ if and only if $A = \mathcal{A}_f(\infty)$ and $r = 1$ if and only if the Julia set $J(f)$ is connected.

Define the *filled Julia set* of f as $K(f) = \hat{\mathbb{C}} \backslash \mathcal{A}_f(\infty)$. When $K(f)$ is connected, any Böttcher map B induces foliations induced by *equipotentials* $E^t = B^{-1}(\{|z| = e^t\})$ of levels $t > 0$ and *external rays* $R^\theta = B^{-1}(\{\arg(z) = 2\pi\theta\})$ of angles $\theta \in \mathbb{R}\backslash\mathbb{Z}$.

Let $R^\theta(t) = B^{-1}(e^{t+d\pi i\theta})$ for $t > 0$, $\theta \in \mathbb{R}\backslash\mathbb{Z}$. A ray R^θ *lands* at a point $w \in J(f)$ if and only if $B^{-1}(re^{i\theta}) \to w$ as r decreases.

Lemma 1.3.1. *The set of angles such that R^θ has a landing point has full measure in S^1.*

Lemma 1.3.2. *For any θ, $R^\theta[t, dt] \to 0$ as $t \to 0$.*

Theorem 1.3.3. *Let f be a polynomial of degree $d \geq 2$ with connected Julia set $J(f)$.*

- *For any odd rational θ, R^θ is a periodic ray of some period p' and it lands at a parabolic / repelling periodic point of some period p dividing p';*
- *Every parabolic / repelling periodic point is the landing point of at least one but at most finitely many odd rational external rays.*

By taking pullbacks, the same theorem holds for external rays with even rational angles and pre-periodic points on $J(f)$. If a ray R^θ lands at a Cremer periodic point w, then θ must be irrational and $K(f)\backslash\{w\}$ has infinitely many components.

From here on, we shall focus on the case where $d = 2$. By conjugation via affine maps, it is sufficient to look at the quadratic family $f_c(z) = z^2 + c$, $c \in \mathbb{C}$.

Theorem 1.3.4 (Dichotomy Theorem)**.** *For a quadratic $f = f_c$, either*

- *$0 \in K(f)$ and $K(f)$ is connected, or*
- *$0 \notin K(f)$ and $K(f)$ is a Cantor set.*

In the Cantor case, we can still define foliations on $\mathcal{A}_f(\infty)$, but there will be singularities on the set of precritical points $\mathcal{O}_f^-(0)$. The vertical leaves will be external rays from ∞ to either $K(f)$ or $\mathcal{O}_f^-(0)$, and separatrices, i.e. arcs from $K(f)$ to $\mathcal{O}_f^-(0)$.

Theorem 1.3.5. *Let f be a hyperbolic quadratic map with connected $J(f)$.*

1. *The inverse Böttcher map $B^{-1} : \hat{\mathbb{C}} \backslash \bar{\mathbb{D}} \to \hat{\mathbb{C}} \backslash K(f)$ admits a Hölder continuous extension to $\partial \mathbb{D}$;*
2. *On the immediate basin D_0 of an attracting p-periodic point with multiplier ρ, f^p is conformally conjugate to the Blaschke product $\frac{z(z+\rho)}{1+\bar{\rho}z}$ and has a unique fixed point β_0 on ∂D_0 called the* root *of D_0;*
3. *There is some $\epsilon > 0$ such that for all $z \in J(f)$, all inverse branches of f^{-n} are well defined in $\mathbb{D}(z, \epsilon)$ and have absolutely bounded distortion;*
4. *There's some uniform K such that all finite components of $F(f)$ are K-quasidisks.*

Let f be a hyperbolic quadratic map with connected $J(f)$ and a strictly attracting p-periodic cycle $\alpha_0 \ldots \alpha_{p-1}$ with immediate basin components $D_0, \ldots D_{p-1}$ labelled such that D_0 contains 0. We can perform attracting-superattracting surgery as follows.

1. By looking at the action of f^p on ∂D_0, we can take a quasiconformal map $h_0 : D_0 \to \mathbb{D}$ which quasisymmetrically conjugates $f^p|_{\partial D_0}$ and $f_0|_{\partial \mathbb{D}}$.
2. The Riemann mapping $h_n : (D_n, \alpha_n, \beta_n) \to (\mathbb{D}, 0, 1)$ for $n = 1, \ldots p-1$ satisfies $h_n = h_{n+1} \circ f$ (let $h_p = h_0$).
3. As $h_1 \circ f = f_0 \circ h_0$ on ∂D_0, we have a quasiregular map

$$F = \begin{cases} h_1^{-1} \circ f_0 \circ h_0, & \text{on } D_0, \\ f, & \text{on } \hat{\mathbb{C}} \backslash D_0. \end{cases}$$

Theorem 1.3.6. *F is conjugate to a unique superattracting quadratic f_{c_0} via some quasiconformal map H that is conformal on $\mathcal{A}_f(\infty)$.*

Corollary 1.3.7. *The Julia set $J(f)$ and $J(f_{c_0})$ are quasiconformally equivalent.*

The *Hubbard tree* $\mathcal{T}$ of a postcritically finite quadratic map f_c is the allowable hull of $\mathcal{O}_{f_c}^+(0) = \{c_k := f_c^k(0)\}_{k \geq 0}$ in $K(f)$. If f is hyperbolic but not superattracting, we can define the Hubbard tree similarly by the surgery procedure above. When 0 is not a fixed point, the Hubbard tree has the following properties:

1. The Hubbard tree $\mathcal{T}$ together with its vertices $\{c_k\}$ and branch points $\{b_k\}$ are forward invariant;

2. 0 is not a branch point of $\mathcal{T}$;
3. There is some $l \geq 1$ such that $c_1, \ldots c_l$ are tips of $\mathcal{T}$.
4. The segment $[0, c]$ always contains the α-fixed point.

Suppose $f = f_c$ is superattracting, We also have the following properties on the attracting basin.

1. When superattracting, the basin of attraction of 0 is dense in $\mathcal{T}$.
2. The closures of two immediate basin components are either disjoint or meet at their root points.

We say that f is *primitive* if the closures of immediate basin components are all pairwise disjoint. Otherwise, f is *satellite*.

Let p be the period of the superattracting cycle, D_1 be the valuable immediate basin of attraction of 0 and let β_1 be its root. The system of external rays $\{R^{\theta_j}\}_{j=1,\ldots m}$ landing at β_1 split $\mathbb{C}$ into m sectors. Let the *characteristic sector* S_{ch} be the valuable one; its boundary consists of two characteristic rays $R^{\theta_\pm}$.

The characteristic geodesics γ_{ch} in $\mathbb{D}$ joining $e^{2\pi i\theta_\pm}$ generates a unique geodesic lamination in $\mathbb{D}$ completely invariant under the doubling map $T(z) = z^2$. The quotient of this lamination gives a topological model of $J(f)$.

Theorem 1.3.8. *The characteristic angles $\theta_\pm$ determine the Hubbard tree $\mathcal{T}$ and vice versa.*

For a general quadratic map $f = f_c$ with connected Julia set, we can also consider the combinatorial model using geodesic laminations. For every dividing preperiodic or periodic point $a \in J(f)$, the set of angles of external rays landing at a induces an ideal geodesic/polygon in $\mathbb{D}$. The *combinatorial lamination* of f is defined to be the quotient of $\mathbb{D}$ under the closure of all such geodesics/polygons.

Definition 6. The *combinatorial class* of a quadratic map f is the set of all quadratic maps *combinatorially equivalent* to f, i.e. have the same combinatorial lamination. If the combinatorial class is trivial, we say that f is *combinatorially rigid.*

Lemma 1.3.9. *The Julia sets of hyperbolic quadratic maps are quasiconformally removable.*

Theorem 1.3.10. *Superattracting quadratic maps are combinatorially rigid.*

Definition 7. A map $f : U \to V$ is a *polynomial-like* map of degree d if $U \Subset V$ are open domains of $\mathbb{C}$ and f is a holomorphic branched covering of degree d. The *filled Julia set* and the Julia set of f are the compact set $K(f) = \bigcap_{n\geq 0} f^{-n}(U)$ and $J(f) = \partial K(f)$.

Definition 8. Two polynomial like-maps $f_j : U_j \to V_j$ are *hybrid equivalent* if there is a quasiconformal map ϕ from a neighbourhood of $K(f_1)$ to a neighbourhood of $K(f_2)$ such that $\phi \circ f_1 = f_2 \circ \phi$, and ϕ has zero complex dilatation almost everywhere on $K(f)$.

Theorem 1.3.11 (Straightening Theorem)**.** *Every degree d polynomial like map $f : U \to V$ is hybrid equivalent to some degree d polynomial g. If $K(f)$ is connected, then g is unique up to affine conjugacy.*

Hence, most of the topological and dynamical properties of the filled Julia sets of polynomial-like maps can be deduced from the usual polynomial case. Assume from now on that $d = 2$, 0 is the unique critical point, and quadratic-like maps are even.

Definition 9. A quadratic-like map $f : U \to V$ is n-renormalisable if there are domains $U_n \Subset V_n$ contained in V such that the restriction $f^n : U_n \to V_n$ is quadratic-like with connected filled Julia set K_n.

In fact, K_n of the n-renormalisation is independent of the choice of domains U_n and V_n. The corresponding small filled Julia sets are denoted by $K_n(i) := f^i(K_n)$ for $i = 0, \dots n-1$.

Proposition 1.3.12. *Two small filled Julia sets are either disjoint or intersect on a singleton, which is necessarily a repelling fixed point α of f^n.*

The periodic point α must be universally of the same type (dividing or not). We will assume that it is always non-dividing. (else, we call the renormalisation *crossed.*) This gives us a *satellite* renormalisation. If all small filled Julia sets are disjoint, the renormalisation is *primitive.*

Proposition 1.3.13. *If a quadratic-like map f is n-renormalisable, any non-repelling periodic points have period divisible by n.*

Renormalisability can be expressed in terms of Yoccoz puzzles made up of dynamical external rays and equipotentials. More precisely, the critical orbit is must be recurrent, i.e. $f^{kn}(0)$ must lie in the critical impression of the puzzles for all $k \geq 0$. Every renormalisation of a quadratic map f comes with a unique combinatorics, which can be described by the characteristic rays of the Yoccoz puzzles inducing the renormalisation or the associated Hubbard tree.

Proposition 1.3.14. *If a quadratic-like map f is infinitely renormalisable, then*

1. *For each $p \in \mathbb{N}_{>0}$, there are at most finitely many renormalisation levels n such that the small filled Julia set K_n contains a p-periodic point;*

2. *f is periodically repelling;*

3. *The postcritical set $P(f)$ and the impression of small filled Julia sets $\bigcap_n \cup_{j=0}^{n-1} f^j(K_n)$ contain no periodic points.*

Proposition 1.3.15. *Let f be an n-renormalisable quadratic map. Almost all $z \in K(f)$ eventually land on the small filled Julia set K_n.*

Theorem 1.3.16 (Lyubich-Shishikura)**.** *Let $f = f_c$ be a quadratic map and suppose $J(f)$ is connected with positive area.*

- *If both fixed points are repelling, then f is renormalisable;*
- *If f is periodically repelling, then f is infinitely renormalisable.*

1.4 Parameter spaces

Definition 10. A holomorphic motion of a subset $X \subset \hat{\mathbb{C}}$ parametrised by a pointed complex manifold (Λ, λ_0) is a function $\phi : \Lambda \times X \to \hat{\mathbb{C}}$ such that

- for every $z \in X$, $\phi(\lambda, z)$ is holomorphic in λ;
- for every $\lambda \in \lambda$, $\phi(\lambda, z)$ is injective in z;
- $\phi(\lambda_0, \cdot) = \mathrm{Id}_X$.

Lemma 1.4.1 (λ-Lemma)**.** *Every holomorphic motion $\phi(\lambda, z)$ of X parametrised by (Λ, λ_0) extends to a unique holomorphic motion of the closure $\bar{X}$. The extension is continuous and quasiconformal in z onto its image.*

Theorem 1.4.2 (Slodkowski)**.** *Every holomorphic motion ϕ on X parametrised by the unit disk $\mathbb{D}$ extends to a holomorphic motion on $\hat{\mathbb{C}}$.*

Theorem 1.4.3. *Let $\{f_\lambda\}_{\lambda \in (\Lambda, \lambda_0)}$ be a holomorphic family of rational maps. The following are equivalent characterisations of J-stability:*

1. *The number of attracting cycles of f_λ is locally constant at λ_0;*
2. *The maximum period of attracting cycles of f_λ is locally bounded at λ_0;*
3. *$J(f_\lambda)$ moves holomorphically near λ_0;*
4. *Every indifferent periodic point of f_λ for λ near λ_0 is persistent;*
5. *$J(f_\lambda)$ is continuous in λ near λ_0 in Hausdorff metric topology.*

Moreover, if all critical points are holomorphically parametrised as $c_i(\lambda)$, all of the above are equivalent to the following:

6. *$\{\lambda \mapsto f_\lambda^n(c_i(\lambda))\}_{n \geq 1}$ form a normal family near λ_0;*
7. *For all λ near λ_0, $c_i(\lambda) \in J(f_\lambda)$ if and only if $c_i(\lambda_0) \in J(f_{\lambda_0})$.*

Corollary 1.4.4. *The set Λ^s of J-stable parameters is open and dense in Λ. The set Λ^h of hyperbolic parameters is a clopen subset of Λ^s.*

1.5 The Mandelbrot Set

Proposition 1.5.1. *The Mandelbrot set $\mathbb{M} \subset \mathbb{C}$ can be defined in the following equivalent ways:*

1. $\mathbb{M} = \{c \in \mathbb{C} : f_c^n(0) \not\to \infty \text{ as } n \to \infty\}$;
2. $\mathbb{M} = \{c \in \mathbb{C} : |f_c^n(0)| \leq 2 \text{ for all } n \geq 0\}$;
3. $\mathbb{M} = \{c \in \mathbb{C} : K(f_c) \text{ is connected}\}$.

Let B_c be the unique Böttcher map for the quadratic f_c, defined on a neighbourhood of ∞. When $c \in \mathbb{M}$, B_c defines a foliation of dynamical rays R_c^θ and equipotentials E_c^t on the basin of infinity $\mathcal{A}_{f_c}(\infty)$.

Theorem 1.5.2. *The map $\Psi : \mathbb{C}\backslash\mathbb{M} \to \mathbb{C}\backslash\bar{\mathbb{D}}, c \mapsto B_c(c)$ is the unique conformal isomorphism tangent to the identity at ∞.*

Corollary 1.5.3. *The Mandelbrot set $\mathbb{M}$ is a non-empty bounded connected compact full subset of $\mathbb{C}$. All components of int$\mathbb{M}$ are simply connected.*

Corollary 1.5.4. *The map Ψ gives rise to a foliation of parameter external rays R_{par}^θ and equipotentials E_{par}^t on $\mathbb{C}\backslash\mathbb{M}$. Moreover,*

- $c \in R_{par}^\theta$ *if and only if* $c \in R_c^\theta$,
- $c \in E_{par}^t$ *if and only if* $c \in E_c^t$

Corollary 1.5.5. *Every dynamical ray R_c^θ crashes at some precritical point if and only if $c \in \mathbb{C}\backslash\mathbb{M}$ and $T^n(\theta) = \arg\Phi(c)$ in $\mathbb{R}\backslash\mathbb{Z}$ for some $n \geq 1$.*

On $\Omega := \{(c, z) \in \mathbb{C}^2 : z \in \mathcal{A}_{f_c}(\infty) \text{ if } c \in \mathbb{M}, |B_c(z)| > |B_c(0)| \text{ if } c \notin \mathbb{M}\}$, the map $\mathbb{B} : \Omega \to \mathbb{C}\backslash\bar{\mathbb{D}}$ is a holomorphic submersion. Its fibers $L_b = \mathbb{B}^{-1}(b)$ for $|b| > 1$ are local holomorphic graphs forming a holomorphic foliation $\mathcal{B}$ called the *Böttcher fibration.*

Proposition 1.5.6. *Suppose Σ_{par}^r is the disk bounded by $E_{par}^{\ln r}$. The Böttcher fibration $\mathcal{B}$ satisfies the following properties.*

- *$\mathcal{B}$ is invariant under the fibered dynamics $F(c, z) = (c, f_c(z))$;*
- *Each fiber L_b is the graph of a holomorphic map $z = \phi_b(c)$ for $c \in \Sigma_{par}^{|b|^2}$;*
- *When $|b| > \sqrt{r} > 1$, the fibers L_b restrict to an equivariant biholomorphic motion $B_c^{-1} : \mathbb{C}\backslash\bar{\mathbb{D}}_{\sqrt{r}} \to B_c^{-1}(\mathbb{C}\backslash\bar{\mathbb{D}}_{\sqrt{r}})$ over $c \in \Sigma_{par}^r$.*
- *The diagonal Γ of $(\mathbb{C}\backslash\mathbb{M})^2$ is a global transversal to the foliation $\mathcal{B}$ intersecting each fiber L_b exactly once.*

- *$\mathcal{B}$ extends holomorphically to $\{(c,z)\, z \in \mathcal{A}_{f_c}(\infty)\}$. The extended leaves L_b are graphs of branched coverings over the parameter plane with simple branched points at the precritical locus $\{(c,z)\, z \in \mathcal{O}^-_{f_c}(0)\}$.*

Lemma 1.5.7. *Let $(\Lambda, c_0) \subset \mathbb{C}$ be a pointed domain and θ be an angle such that $T^n(\theta) \neq \arg\Phi(c)$ for all $c \in \Lambda\backslash\mathbb{M}$.*

1. *The Böttcher map $B_c^{-1}B_{c_0}$ induces a holomorphic motion of external rays R_c^θ over (Λ, c_0);*
2. *If $R_{c_0}^\theta$ at some $a_{c_0} \in J(f_{c_0})$, then each ray R_c^θ also lands at some point $a_c \in J(f_c)$ depending holomorphically on $c \in \Lambda$.*

In particular, the domain Λ can be taken to be any component of $\mathbb{C}\backslash\overline{\bigcup_{n\geq 0} R_{par}^{T^n(\theta)}}$.

Theorem 1.5.8. *The complement $\mathbb{C}\backslash\partial\mathbb{M}$ can be characterised in the following equivalent ways:*

- *The set of all parameters c_0 such that the polynomials $\{c \mapsto f_c^n(0)\}_{n\geq 0}$ form a normal family near c_0;*
- *The set of J-stable parameters for the quadratic family;*
- *The largest open subset of $\mathbb{C}$ over which the Böttcher map $B_c^{-1}B_{c_0}$ induces a holomorphic motion of Julia sets $J(f_c)$;*
- *The largest open subset not containing any neutral parameters.*

A parameter $c \in \mathbb{M}$ is called *Misiurewicz* if 0 is strictly preperiodic under the quadratic f_c. In this case, f_c is periodically repelling and subhyperbolic.

Corollary 1.5.9. *The boundary of $\mathbb{M}$ can be approximated by any of the following set of parameters:*

- *Parabolic parameters;*
- *Superattracting parameters;*
- *Misiurewicz parameters, i.e. c such that 0 is strictly preperiodic under f_c.*

A component H of the interior of $\mathbb{M}$ is called *hyperbolic* if it contains a hyperbolic parameter, *queer* if otherwise. Every hyperbolic component H has an associated attracting cycle moving holomorphically throughout H. The *period* of H would refer to the period of this attracting cycle.

Proposition 1.5.10. *For every hyperbolic component H of period n and every non-zero odd rational $p/q \in \mathbb{Q}\backslash\mathbb{Z}$,*

- *there is a unique parabolic parameter c_0 on the boundary of H of rotation number;*
- *there is a hyperbolic component H' of period nq attached to H at c_0.*

Over every pointed hyperbolic component (H, c_0), there is a unique equivariant holomorphic motion $B_c^{-1} \circ B_{c_0} : \mathcal{A}_{f_c}(\infty) \to \mathcal{A}_{f_{c_0}}(\infty)$ which, by λ-lemma, extends to the Julia sets as well.

Theorem 1.5.11 (Multiplier Theorem)**.** *For any hyperbolic component H, the multiplier $\rho(c)$ of the associated attracting periodic cycle of f_c induces a conformal isomorphism $\rho : H \to \mathbb{D}$.*

Corollary 1.5.12. *Every hyperbolic component H has a unique superattracting parameter c_H called the* center *of H, and thus H admits a unique Hubbard tree. There are exactly 2^{p-1} hyperbolic components of period dividing p.*

Lemma 1.5.13. *Let $f = f_c$ be any quadratic map.*

- *f acts on any invariant line field μ on $J(f)$ ergodically on the support of μ;*
- *f can have at most one invariant line field on $J(f)$ up to rotation.*

Theorem 1.5.14. *A parameter c_0 lies in a queer component Q if and only if f_{c_0} has an invariant line field on $J(f_{c_0})$.*

In one direction, if μ_0 is an invariant line field on $J(f_{c_0})$, then for $\lambda \in \mathbb{D}$, $\lambda\mu_0$ is an f_{c_0}-invariant Beltrami coefficient with unique quasiconformal solution $h_\lambda : \mathbb{C} \to \mathbb{C}$ fixing 0 and tangent to Id at ∞. Then, $h_\lambda f_{c_0} h_\lambda^{-1} = z^2 + \sigma(\lambda)$ for some holomorphic function $\sigma : \mathbb{D} \to Q$.

Theorem 1.5.15 (Queer Theorem)**.** *$\sigma : (\mathbb{D}, 0) \to (Q, c_0)$ is a Riemann mapping of the queer component Q.*

A parameter c in the quadratic family $\{f_c\}$ is *structurally stable* if $f_{c'}$ is topologically conjugate to f_c for all c' sufficiently close to c.

Theorem 1.5.16. *The set of structurally stable parameters is the union of $\mathbb{C}\backslash\mathbb{M}$, all queer components, and all hyperbolic components with their centers removed. Moreover, the parameter plane can be decomposed into quasiconformal conjugacy classes as follows:*

- $\mathbb{C}\backslash\mathbb{M}$*;*
- *queer components;*
- *hyperbolic components with their centers removed;*
- *centers of hyperbolic components;*

- *singletons on* $\mathbb{M}$.

The theorem follows from extending the Böttcher holomorphic motion (or a modified version of it in the Cantor case) to the Julia set by λ-lemma. In the hyperbolic case, we can also find an equivariant holomorphic motion of the attracting basins by modifying the Koenigs linearisation on a fundamental annulus and pull it back to spread to the whole attracting basin.

On the boundary of a hyperbolic component H, the attracting cycle associated to H undergoes two types of bifurcation:

- Saddle node bifurcation at the cusps of primitive components, i.e. when an attracting cycle and a repelling cycle collide to a parabolic one of multiplier 1 and then produce two repelling cycles;
- Satellite bifurcation at the roots of hyperbolic components, i.e. when an attracting cycle and a repelling cycle collide and swap.

Proposition 1.5.17. *Let $\tilde{c}$ be the root of a hyperbolic component H with associated parabolic repelling point β_{c_0}. For all $c \in H$,*

- *If $R^{\theta}_{c_0}$ lands at β_{c_0}, then R^{θ}_{c} lands at the repelling root β_c of f_c;*
- *$f_{\tilde{c}}$ and f_c has the same characteristic ray portrait;*
- *The Böttcher conjugacy $B_c^{-1} \circ B_{\tilde{c}}$ extends to a conjugacy $J(f_{\tilde{c}}) \to J(f_c)$.*

Lemma 1.5.18 (Stability Lemma)**.** *Let R be a periodic ray of a polynomial f landing at a repelling periodic point a. For any polynomial $\tilde{f}$ sufficiently close to f, the corresponding ray $\tilde{R}$ for $\tilde{f}$ lands at the perturbed repelling periodic point $\tilde{a}$ of $\tilde{f}$.*

Corollary 1.5.19. *If θ is a rational angle with odd denominator of some period p, the parameter ray R^{θ}_{par} lands at a parabolic parameter r with period divisible by p.*

Theorem 1.5.20 (Wake Theorem)**.** *Let H be a hyperbolic component that is not the main cardioid. Let $r \neq \frac{1}{4}$ be the root of H with associated parabolic cycle α_r with characteristic angles $\theta_{\pm}$. Let Θ be the orbit of $\theta_{\pm}$. The component W_r of $\mathbb{C} \setminus \bigcup_{\theta \in \Theta} \overline{R^{\theta}_{par}}$ containing H is called the* wake *rooted at r and it satisfies the following properties.*

1. *The rays $R^{\theta_{\pm}}_{c}$ and their landing point α_c persist and move holomorphically over the whole wake $c \in W_r$;*
2. *If f_c has a parabolic or repelling cycle with associated periodic angles Θ, then c lies in the wake $W_r \cup \{r\}$;*
3. *The multiplier map ρ on H extends to the whole W_r and, if H is satellite, to a neighbourhood of r.*

By counting the number of p-periodic rational angles with odd denominator and p-periodic hyperbolic components, we observe that exactly two parameter rays land at these parabolic parameters. The closure $\mathcal{L}_r$ of $\mathbb{M} \cap W_r$ is called the *limb* of $\mathbb{M}$ rooted at r. The *center* of $\mathcal{L}_r$ is the center of the hyperbolic component with root r.

Corollary 1.5.21. *Let H be a hyperbolic component rooted at r such that parameter rays $R_{par}^{\theta_\pm}$ land at r. (If $r = \frac{1}{4}$, take $\theta_- = 0$ and $\theta_+ = 1$.)*

- *The set of angles S_H landing on the boundary of H is nowhere dense and has zero measure in $[\theta_-, \theta_+]$;*
- *For every angle $\theta \notin S$, R_{par}^{θ} lands at a unique irrationally indifferent parameter $c \in \partial H_0$;*
- *Every irrationally indifferent parameter is the landing point of exactly one parameter ray.*

When the parabolic parameter $r \neq \frac{1}{4}$ is attached to the main cardioid H_0 and has rotation number $\frac{p}{q}$, we label the corresponding wake $W_{p/q}$ and the limb $\mathcal{L}_{p/q}$. When r is on the boundary of an arbitrary hyperbolic component H, denote the wake and the limb rooted at r by $W_{p/q}(H)$ and $\mathcal{L}_{p/q}(H)$.

Lemma 1.5.22. *For any hyperbolic component H, any Hausdorff limit of an infinite sequence of distinct limbs attached to H is a singleton.*

The rate of shrinking can also be estimated by Yoccoz inequality.

Theorem 1.5.23 (Yoccoz Inequality)**.** *For any hyperbolic component H, there is some $C = C(H) > 0$ such that for any limb $\mathcal{L}_{p/q}(H)$ attached to H,*

$$diam \mathcal{L}_{p/q}(H) \leq \frac{C}{q}.$$

Corollary 1.5.24. *The Mandelbrot set is locally connected at the boundary of hyperbolic components.*

Theorem 1.5.25. *Let $\theta \in \mathbb{R} \backslash \mathbb{Z}$ be some angle.*

- *For every Misiurewicz parameter $c_0 \in \partial\mathbb{M}$, the dynamical ray $R_{c_0}^{\theta}$ lands on the Julia set of f_{c_0} at c_0 if and only if the parameter ray R_{par}^{θ} lands at c_0;*
- *Every rational angle θ with even denominator induces a valuable ray $R_{c_0}^{\theta}$ landing at c_0 for some Misiurewicz parameter $c_0 \in \partial\mathbb{M}$.*

Corollary 1.5.26. *Misiurewicz quadratic maps are uniquely determined by their characteristic angles and are therefore combinatorially rigid.*

The parameter rays landing at a Misiurewicz parameter c_0 split the parameter plane the Mandelbrot set $\mathbb{M}$ into *Misiurewicz wakes* and *Misiurewicz decorations / limbs* rooted at c_0.

Proposition 1.5.27. *The Mandelbrot set $\mathbb{M}$ is well-branched at any Misiurewicz parameter c_0, i.e. every wake contains only one limb component.*

Renormalisability determines the local connectivity of Julia sets and the Mandelbrot set as follows.

Theorem 1.5.28 (Yoccoz)**.** *Let $f = f_c$ be a quadratic map with connected filled Julia set. If f has no indifferent cycles and is at most finitely renormalisable, then*

- *the Julia set $J(f)$ is locally connected;*
- *if f is not hyperbolic, $c \in \partial\mathbb{M}$ and the Mandelbrot set $\mathbb{M}$ is locally connected at c.*

The theorem above can be reduced to the case where f is non-renormalisable. The proof relies on the study of combinatorics of nested annuli around the critical point, resulting in weak local connectivity. The local connectivity can be promoted to the whole Julia set. Phase-parameter relations can help us transfer this property to the parameter plane. We can apply the theorem above to the following:

- if c is a Misiurewicz parameter, the Julia set $J(f_c)$ is locally connected and the Mandelbrot set is locally connected at c;
- if c is a queer parameter, f_c must be infinitely renormalisable.

Below is a number of important conjectures in the study of the dynamics of quadratic maps.

Conjecture 1 (Density of Hyperbolicity)**.** *Hyperbolic parameters are dense in $\mathbb{M}$, i.e. queer components do not exist.*

Conjecture 2 (MLC)**.** *The Mandelbrot set is locally connected.*

Conjecture 3 (No Invariant Line Fields)**.** *No quadratic map $f = f_c$ admits an invariant line field on its Julia set.*

Conjecture 4 (Rigidity)**.** *All periodically repelling parameters are combinatorially rigid.*

The conjectures are known to be related by the following diagram.

$$\text{MLC} \Longleftrightarrow \text{Rigidity} \Longrightarrow \text{Density of Hyperbolicity} \Longleftrightarrow \text{No Invariant Line Fields}$$

The first double arrow can be obtained through the following observation. (The other arrows are obvious.) Rational rays landing at dividing preperiodic or periodic points on $J(f)$ persist on a parameter domain not intersecting any rational or Misiurewicz parameter rays. Therefore...

Proposition 1.5.29. *Let f_{c_0} be periodically repelling and $c_0 \in \partial\mathbb{M}$ have combinatorial class $\mathcal{C}$.*

1. *$\mathcal{C}$ consists of parameters that can't be separated from c_0 by a rational or Misiurewicz cut-line;*
2. *$\mathcal{C}$ is closed and $\partial\mathcal{C} \subset \partial\mathbb{M}$;*
3. *$\mathcal{C}$ is the impression of the rational parapuzzles containing c_0.*

1.6 Review Questions

1. How do you define quasiconformal maps?
2. State the measurable Riemann mapping theorem.
3. Can you tell me about the dependence of the solution of the Beltrami equation on the Beltrami coefficient? What about the inverse? Any example?
4. Explain the general procedure of quasiconformal surgery.
5. How do you define Fatou and Julia sets?
6. Which periodic cycles can be found on the Julia set?
7. Which rational maps have smooth Julia sets?
8. What are the possible Fatou components of a rational map?
9. How many Fatou components can a rational map have?
10. Sketch the proof of the no wandering domains theorem?
11. Which rational maps are hyperbolic or subhyperbolic? Can you tell me about some properties of their Fatou and Julia sets?
12. Which rational maps admit invariant line fields on their Julia sets?
13. Explain how rational or irrational external rays land on the Julia set.
14. Is any periodic point on the Julia set always a landing point of an external ray?
15. How many rays would land on the boundary of a given finite Fatou component of an attracting or parabolic quadratic map?
16. What is the measure of the set of tips of Julia sets?
17. Can you draw the Hubbard tree of a given subhyperbolic quadratic map?
18. What are the complications in defining the Hubbard tree of a higher degree polynomial?
19. Can you draw the associated combinatorial lamination of the Julia set of a given quadratic map? How are they related to their Hubbard trees?
20. Define polynomial-like maps.
21. State the straightening theorem. Can you give a sketch of the proof?

22. Which quadratic maps are renormalisable? Can you draw some Yoccoz puzzles associated to a given quadratic map?

23. Which quadratic maps are infinitely renormalisable?

24. Which quadratic maps have locally connected Julia sets?

25. State the lambda lemma.

26. How do you define structural stability of a holomorphic family of maps? What about J-stability?

27. Define the Mandelbrot set. Can you give a sketch?

28. Explain the phase-parameter relation. How does it arise?

29. How do you uniformise components of the interior of the Mandelbrot set?

30. Which parameters c are structurally stable? Why?

31. Which parameters c are quasiconformally equivalent?

32. Given a parameter on the Mandelbrot set, sketch the corresponding Julia set.

33. How do you define limbs and wakes of the Mandelbrot set?

34. How many parameter rays land on a given hyperbolic component?

35. State the MLC conjecture. What are other equivalent ways to formulate this conjecture and why?

Chapter 2

Several Complex Variables

Most of this chapter follows from the book of Bers[1]. Notation:

- Polydisks in $\mathbb{C}^n$ are of the form $\mathbb{D}^n(z,r) = \mathbb{D}(z_1,r_1) \times \ldots \times \mathbb{D}(z_n,r_n)$. I will denote the essential boundary of each polydisk by $\Gamma(z,r) := \partial\mathbb{D}(z_1,r_1) \times \ldots \times \partial\mathbb{D}(z_n,r_n)$.
- For any compact subset $K \subset \mathbb{C}^n$ and continuous map f on K, $\|f\|_K := \sup_{z\in K} |f(z)|$.

Definition 11. A function $f : U \to \mathbb{C}$ on an open $U \subset \mathbb{C}^n$ is *holomorphic* if it satisfies any of the following equivalent criteria:

1. f is holomorphic in each variable z_j, $j = 1, \ldots n$,
2. f admits a local Taylor series development $\sum_{\alpha\geq 0} c_\alpha (z_1 - a_1)^{\alpha_1} \ldots (z_n - a_n)^{\alpha_n}$ at each point $a \in U$.

Hartogs proved that the first implies the second using Osgood's lemma. In higher dimensions, many classical properties of complex analysis hold:

1. Cauchy's integral formula:
$$f(w) = \frac{1}{(2\pi i)^n} \int_{\Gamma(w,r)} \frac{f(w)}{\prod_{k=1}^n (z_k - w_k)} dz_1 \ldots dz_k.$$
2. Cauchy's estimates
3. Maximum modulus principle
4. Identity theorem
5. Holomorphic functions are closed in the compact-open topology.

[1] L. Bers, Introduction to several complex variables, New York Univ. Press, New York, 1964.

2.1 Domains of Holomorphy

Definition 12. A pair of open subsets $D \subset E \subset \mathbb{C}^n$ is a *Hartogs pair* / exhibits the *Hartogs' phenomenon* if every $f \in \mathcal{O}(D)$ admits a holomorphic extension $F \in \mathcal{O}(E)$, i.e. the restriction map $\mathcal{O}(E) \to \mathcal{O}(D)$ is a ring isomorphism.

Cauchy integrals and Laurent series often give a natural holomorphic extension. The following theorem can be proved using Cauchy-Green.

Theorem 2.1.1 (Hartogs Extension Theorem)**.** *Let K be a non-separating compact subset of a domain $D \subset \mathbb{C}^n$. Then, $(D \backslash K, D)$ is a Hartogs pair.*

Definition 13. Let D be a non-empty open set $D \subset \mathbb{C}^n$.

- D a *region of holomorphy* (if connected, *domain of holomorphy*) if there is some $f \in \mathcal{O}(D)$ which cannot be holomorphically extended to any neighbourhood of any boundary point of D;
- $\zeta \in \partial D$ is *essential* if there is some $f \in \mathcal{O}(D)$ which cannot be holomorphically extended to any neighbourhood of ζ.
- D is $\mathcal{F}$-*convex* if for every compact $K \Subset D$, the $\mathcal{F}$-*convex hull* of K,

  $$\hat{K}_{\mathcal{F}} = \{z \in D : |f(z)| \leq \|f\|_K \text{ for all } f \in \mathcal{F}\},$$

  is also compact in D.

When $\mathcal{F}$ is the space of affine maps, the $\mathcal{F}$-convex hull is the usual convex hull. We are interested in the case when $\mathcal{F}$ is the family of polynomials $\mathcal{P}$or holomorphic maps $\mathcal{O}(D)$.

Theorem 2.1.2 (Cartan-Thullen)**.** *For a domain $D \subset \mathbb{C}^n$, the following are equivalent:*

1. *D is holomorphically convex,*
2. *All boundary points are essential,*
3. *D is a domain of holomorphy,*
4. *for any compact $K \Subset D$, dist$(\hat{K}_{\mathcal{O}(D)}, \partial D) =$ dist$(K, \partial D)$.*

Example 2. Every finite Cartesian product or intersection of regions of holomorphy is a region of holomorphy.

Example 3. Every domain in $\mathbb{C}$ is a domain of holomorphy. This follows from either the theorem above or directly using Weierstrass' theorem.

Example 4. Every convex domain $D \subset \mathbb{C}^n$ is a domain of holomorphy.

Consider the projection map $\rho : \mathbb{C}^n \to [0,\infty)^n, z \mapsto (|z_1|, \dots |z_n|)$. For every domain $D \subset \mathbb{C}^n$, we denote by D^* the image $\rho(D)$.

Definition 14. A domain $D \subset \mathbb{C}^n$ is

1. *Reinhardt* if for every $z \in D$, D contains the whole fiber $\rho^{-1}(\rho(z))$;
2. *complete Reinhardt* if for every $z \in D$, D contains the polydisk $\mathbb{D}(0, \rho(z))$;

A set $E \subset [0,\infty)^n$ is *logarithmically convex* if $\log E := \{(\log x_1, \dots \log x_n) : x \in E\}$ is convex in $\mathbb{R}^n$.

Theorem 2.1.3 (Abel)**.** *Let D be the domain of convergence of a Taylor series $f(z) = \sum_\alpha c_\alpha z^\alpha$.*

- *If the sequence $\{c_\alpha w^\alpha\}_\alpha$ is bounded for some $w \in \mathbb{C}^n$, then D contains the polydisk $\mathbb{D}(0, (|w_1|, \dots |w_n|)$;*
- *D is a complete Reinhardt domain and D^* is logarithmically convex.*

When f is only a Laurent series, completeness can be dropped.

Theorem 2.1.4. *Let $D \subset \mathbb{C}^n$ be a complete Reinhardt domain containing 0. The following are equivalent:*

1. *D is the domain of (normal) convergence of a Taylor series about 0 of a holomorphic function $f(z) = \sum_{\alpha \geq 0} c_\alpha z^\alpha$;*
2. *D is logarithmically convex;*
3. *D is a domain of holomorphy.*

2.2 Pseudoconvexity

Definition 15. A function $f : U \subset \mathbb{R}^n \to [-\infty, \infty)$ is *subharmonic* (SH) if it is upper semi-continuous and it satisfies any of the following equivalent criteria:

1. for every compact $K \subset U$ and continuous map h on K, if h is harmonic on $\text{int}(K)$ and $u \leq h$ on ∂K, then $u \leq h$ on K;
2. for every ball $\mathbb{B}(x, R) \Subset U$, the mean $M_u(x, r)$ of u on $\partial\mathbb{B}(x, r)$ is increasing in $r \in [0, R]$.
3. for every ball $B = \mathbb{B}(x, R) \Subset U$, $u(x) \leq \frac{1}{\text{Vol}(B)} \int_B u(y)dy$.

A function $f : U \subset \mathbb{C}^n \to [-\infty, \infty)$ is *plurisubharmonic* (PSH) if it is upper semi-continuous and subharmonic on every complex line on U.

Below is a list of key properties of SH and PSH functions:

1. PSH functions are SH;
2. SH functions satisfy the maximum principle;
3. (P)SH is a local property;
4. The sum / pointwise maximum of two (P)SH functions is (P)SH;
5. The limit of a decreasing sequence of (P)SH functions is (P)SH;
6. A C^2 function u is SH if and only if the Laplacian Δu is non-negative, PSH if and only if the complex Hessian $(u_{z_i\bar{z}_j})_{1\leq i,j\leq n}$ is positive semi-definite.
7. PSH functions are still PSH under holomorphic change of coordinates.

Theorem 2.2.1 (Kontinuitatsatz)**.** *Let $D \subset \mathbb{C}^n$ be a domain and define $\Delta : D \to [0, \infty), \delta(z) = dist(z, \partial D)$. The following are equivalent characterisations of* pseudoconvexity.

1. *$-\log \Delta(z)$ is plurisubharmonic on D;*
2. *D admits a plurisubharmonic exhaustion $\psi : D \to [-\infty, \infty)$ i.e. $\psi^{-1}[-\infty, c]$ is always compact in U for all $c \in \mathbb{R}$;*
3. *For every embedded complex disk $\Sigma \Subset U$, $\Delta(\Sigma) = \Delta(\partial\Sigma)$;*
4. *D is plurisubharmonically convex;*
5. *For every countable collection of complex disks $\{\Sigma_j\}$ in D, if $\cup_j \partial\Sigma_j \Subset D$, then $\cup_j \Sigma_j \Subset D$.*

Corollary 2.2.2. *Pseudoconvexity is invariant under holomorphic change of coordinates, countable intersections, and a countable union of increasing domains.*

Corollary 2.2.3. *Domains of holomorphy are pseudoconvex.*

Corollary 2.2.4. *A domain D is pseudoconvex if and only if it is locally pseudoconvex along the boundary, i.e. at each $\zeta \in \partial D$, there is some neighbourhood U of ζ such that $D \cap U$ is pseudoconvex.*

Domains of holomorphy and pseudoconvex domains are essentially the same.

Theorem 2.2.5. *A domain $D \subset \mathbb{C}^n$ is a domain of holomorphy if and only if it is pseudoconvex.*

The $\Rightarrow$ direction follows from maximum principle. The $\Leftarrow$ is much more difficult and is commonly known as the Levi problem.

2.3 Cousin Problems

Let X be a topological space, $\mathcal{F}$ be a sheaf of abelian groups on X, and $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X indexed by an index set I. Denote each intersection $\cap_{k=1}^{m} U_{i_k}$ by $U_{i_1 i_2 \dots i_k}$.

For every natural number $r \geq 0$, an *r-cochain* f is defined as a collection of local sections $f_{i_0 i_1 \dots i_r}$ of $\mathcal{F}(U_{i_0 i_1 \dots i_r})$ for each $(r+1)$-tuple $(i_0, i_1 \dots i_1) \in I^{r+1}$ such that

- $f_{i_0 \dots i_r} \equiv 0$ if $U_{i_0 \dots i_r} = \emptyset$;
- $f_{\sigma(i_0) \dots \sigma(i_r)} = \operatorname{sgn}(\sigma) f_{i_0 \dots i_r}$ for any permutation σ of $\{i_0, \dots i_r\}$.

The set of r-cochains $C^r = C^r(X, \mathcal{U}, \mathcal{F})$ forms an abelian group under addition. The coboundary operator

$$\delta : C^r \to C^{r+1}, \quad (\delta f)_{i_0 \dots i_r} = \sum_{j=0}^{r+1} (-1)^j f_{i_0 \dots \hat{i_j} \dots i_r}$$

induces a cochain complex $(C^\bullet, \delta)$.

The *r^{th}-Čech cohomology* $H^r(X, \mathcal{U}, \mathcal{F})$ of $\mathcal{U}$ with coefficients in $\mathcal{F}$ is defined by the quotient of the subgroup of r-cocycles $Z^r = \operatorname{Ker}(\delta : C^r \to C^{r+1})$ modulo the subgroup of r-coboundaries $B^r = \delta(C^{r-1})$. *r^{th}-Čech cohomology* $H^r(X, \mathcal{F})$ of X with coefficients in $\mathcal{F}$ is defined by the following direct limit:

$$H^r(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} H^r(X, \mathcal{U}, \mathcal{F}).$$

A cover $\mathcal{U} = \{U_i\}_{i\in I}$ is a *Leray* cover with respect to $\mathcal{F}$ if every finite intersection V of open sets in $\mathcal{U}$ satisfies $H^r(V, \mathcal{F}) = 0$ for all $r > 0$. Given a Leray cover $\mathcal{U}$, $H^r(X, \mathcal{F}) = H^r(X, \mathcal{U}, \mathcal{F})$. Moreover, assuming X is Hausdorff and paracompact, Čech cohomology is isomorphic to sheaf cohomology.

Let X be a complex manifold of dimension $n \geq 1$. The set of *germs* $\mathcal{O}_p$ at $p \in X$ is the set of Taylor series at p which is convergent on a small neighbourhood of p in X. The *sheaf* $\mathcal{O}$ of holomorphic functions on X is the disjoint union $\bigcup_{p\in X} \mathcal{O}_p$, topologised such that the natural projection $\mathcal{O} \to X$ is continuous.

The field of quotients of $\mathcal{O}$ is the sheaf $\mathcal{M}$ of meromorphic functions; each germ is represented by a local Laurent series. *Meromorphic* functions g on an open subset U can be defined as sections of $\mathcal{M}$ on U, denoted by $g \in \mathcal{M}(U)$.

The quotient sheaf $\mathcal{M}/\mathcal{O}$ can be defined as the sheaf where the germs are principal parts of Laurent series, i.e. we have the following exact sequence of sheaves:

$$0 \to \mathcal{O} \hookrightarrow \mathcal{M} \to \mathcal{M}/\mathcal{O} \to 0.$$

The additive Cousin problem can be formulated in two different, yet equivalent ways:

1st Cousin Problem:

- Given an open cover $\mathcal{U} = \{U_i\}$ of X and meromorphic functions $F_j \in \mathcal{M}(U_j)$ such that $F_j - F_k \in \mathcal{O}(U_{jk})$, can we find a global meromorphic function $F \in \mathcal{M}(X)$ such that $F - F_j \in \mathcal{O}(U_j)$ for all j?
- Is the induced projection map $H^0(X, \mathcal{M}) \to H^0(X, \mathcal{M}/\mathcal{O})$ surjective ?
- Is $H^1(X, \mathcal{O}) = 0$?

By the use of partitions of unity, the sheaf C^∞ of smooth functions on X is a fine sheaf. We conclude that the 1st Cousin problem is solvable if the following $\bar{\partial}$-problem is solvable:

- Given $a_1, \dots a_n \in C^\infty(X)$ where $(a_j)_{\bar{z}_k} = (a_k)_{\bar{z}_j}$ for all $1 \le j, k \le n$, can we find some $A \in C^\infty(X)$ such that $A_{\bar{z}_j} = a_j$ for all j?

Theorem 2.3.1. *The $\bar{\partial}$-problem is solvable when $X = D_1 \times \dots D_n$ is a product of n simply connected domains $D_j \subset \mathbb{C}$.*

The connection between the $\bar{\partial}$-problem, i.e. determining which $\bar{\partial}$-closed (0,1)-forms are $\bar{\partial}$-exact, and determining which 1st Cousin data is solvable is generalised by Dolbeault's theorem, which shall be mentioned in the next section.

Theorem 2.3.2 (Oka Extension Theorem)**.** *If the additive Cousin problem is always solvable on X and $Y = \{\psi = 0\}$ is a regular hypersurface for some $\psi \in \mathcal{O}(X)$, then every $f \in \mathcal{O}(Y)$ admits an extension $F \in \mathcal{O}(X)$.*

A stronger formulation of the theorem above essentially says that the restriction map $H^0(X, \mathcal{O}) \to H^0(Y, \mathcal{O})$ is surjective if $H^1(X, \mathcal{O}) = 0$. In some cases, we may still allow the presence of singularities.

Theorem 2.3.3. *Let X be a complex manifold such that for some $r \ge 0$, $H^r(X, \mathcal{O}) = H^{r+1}(X, \mathcal{O}) = 0$. For any regular hypersurface $Y = \{\phi = 0\}$ on X, $H^r(Y, \mathcal{O}) = 0$.*

Corollary 2.3.4 (H. Cartan)**.** *If the additive Cousin problem is always solvable on a domain $D \subset \mathbb{C}^2$, then D is a domain of holomorphy.*

Corollary 2.3.5. *If a domain $D \subset \mathbb{C}^n$ satisfies $H^r(D, \mathcal{O}) = 0$ for $1 \le r \le n-1$, then D is a domain of holomorphy.*

We may actually extend the results of Oka and Cartan by induction.

Lemma 2.3.6 (Lemma A)**.** *Let X be an n-dimensional complex manifold such that $H^q(X, \mathcal{O}) = 0$ for $q > 0$ and let $f_1, \dots f_r \in \mathcal{O}(X)$ be such that the matrix $\big((f_i)_{z_j}\big)_{i=1\dots r, j=1\dots n}$ has full rank on the zero set $Z = \{z \in X | f_1(z) = \dots f_r(z) = 0\}$. Then,*

- *Z is a regularly embedded submanifold with $H^q(Z, \mathcal{O}) = 0$ for all $q > 0$;*
- *Each $f \in \mathcal{O}(Z)$ is the restricting of some $F \in \mathcal{O}(X)$.*

Define $\mathcal{O}^*$ to be the sheaf of non-vanishing holomorphic functions and define $\mathcal{M}^*$ to be the sheaf of meromorphic functions which are not identically 0. The quotient sheaf $\mathcal{M}^*/\mathcal{O}^*$ is again defined by the following exact sequence of sheaves:

$$0 \to \mathcal{O}^* \hookrightarrow \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0.$$

Global sections of $\mathcal{M}^*/\mathcal{O}^*$ are called *(Cartier) divisors*. A divisor α is *integral* if it has no poles, and it is *principal* if there is a meromorphic $f \in \mathcal{M}^*(X)$ such that $(f) = \alpha$.

The multiplicative Cousin problem can be formulated in four different, yet equivalent ways:

2^{nd} Cousin Problem:

- Given an open cover $\mathcal{U} = \{U_i\}$ of X and meromorphic functions $F_j \in \mathcal{M}(U_j)$ such that $F_j/F_k \in \mathcal{O}(U_{jk})$, can we find a global meromorphic function $F \in \mathcal{M}^*(X)$ such that $F/F_j \in \mathcal{O}^*(U_j)$ for all j?
- Is the induced projection map $H^0(X, \mathcal{M}^*) \to H^0(X, \mathcal{M}^*/\mathcal{O}^*)$ surjective ?
- Is $H^1(X, \mathcal{O}^*) = 0$?
- Is every divisor principal?

Lemma 2.3.7. *If every integral divisor on X is principal, then the multiplicative Cousin problem is always solvable.*

Solvability of the 2^{nd} Cousin problem has important consequences.

Theorem 2.3.8. *Suppose the multiplicative Cousin problem is always solvable.*

1. *For every $F \in \mathcal{M}(X)$, then $F = f/g$ for some $f, g \in \mathcal{O}(X)$;*
2. *Every regular hypersurface admits a global defining function $\phi \in \mathcal{O}(X)$.*

The exponential map induces the following exact sequence:

$$0 \to \mathbb{Z} \hookrightarrow \mathcal{O} \to \mathcal{O}^* \to 0.$$

Theorem 2.3.9 (Serre)**.** *The multiplicative Cousin problem is solvable if $H^1(X, \mathcal{O}) = H^2(X, \mathbb{Z}) = 0$.*

Example 5. The multiplicative Cousin problem is solvable when $X = D_1 \times \ldots D_n$ is a product of n simply connected domains $D_j \subset \mathbb{C}$.

The multiplicative Cousin problem may not be solvable even on domains of holomorphy. This is demonstrated by Poincaré's example where the space is

$$X = \{(z, w) \in \mathbb{C}^2 : 0.75 < |z|, |w| < 1.25\}.$$

2.4 Differential Forms

Lemma 2.4.1 (Poincaré Lemmas)**.**

1. *Every d-closed smooth r-form on the unit open square $(-1,1)^n \subset \mathbb{R}^n$ is d-exact;*
2. *Every d-closed holomorphic r-form on the unit polydisk $\mathbb{D}^n \subset \mathbb{C}^n$ is d-exact;*
3. *Every $\bar{\partial}$-closed smooth (p,q)-form on the unit polydisk is $\bar{\partial}$-exact.*

Corollary 2.4.2. *Let X be an n-dimensional complex manifold, $\mathcal{A}^k$, Ω^k, and $\mathcal{A}^{p,q}$ be the space of all smooth k-forms, holomorphic k-forms, and smooth (p,q)-forms respectively. We have the following exact sequence of sheaves.*

$$0 \hookrightarrow \mathbb{C} \hookrightarrow \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \dots \mathcal{A}^n \to 0$$
$$0 \hookrightarrow \mathbb{C} \hookrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \dots \Omega^n \to 0$$
$$0 \hookrightarrow \mathcal{O} \hookrightarrow \mathcal{A}^{0,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{0,2} \dots \mathcal{A}^{0,n} \to 0$$
$$0 \hookrightarrow \Omega^p \hookrightarrow \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,2} \dots \mathcal{A}^{p,n} \to 0.$$

Theorem 2.4.3 (de Rham's Theorem)**.** *Let a sheaf S on a space X admit a resolution, i.e. an exact sequence $0 \to S \xrightarrow{\phi_{-1}} A_0 \xrightarrow{\phi_0} A_1 \dots$ such that $H^q(X, A_j) = 0$ for all $j \geq 0$ and $q > 0$. Then, for all $p > 0$, $H^p(X,S)$ is isomorphic to* $\mathrm{Ker}(\phi_p^*)/\mathrm{Im}(\phi_{p-1}^*)$.

By the use of partition of unity, we see that the sheaves $\mathcal{A}^k$ and $\mathcal{A}^{p,q}$ are all fine, so $H^k(X, \mathcal{A}^k) = H^q(X, \mathcal{A}^{p,q}) = 0$ for all $k, q > 0$.

Corollary 2.4.4. *For any smooth manifold X, $H^p_{dR}(X,\mathbb{R})$ is isomorphic to $H^p(X,\mathbb{R})$.*

Corollary 2.4.5 (Dolbeault's Theorem)**.** *For any complex manifold X, we have the following isomorphisms*

- $H^q(X,\mathcal{O}) = \mathrm{Ker}(\bar{\partial} : \mathcal{A}^{0,q} \to \mathcal{A}^{0,q+1})/\bar{\partial}(\mathcal{A}^{p,q-1})$;
- $H^q(X,\Omega^p) = \mathrm{Ker}(\bar{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1})/\bar{\partial}(\mathcal{A}^{p,q-1})$.

2.5 Polynomial and Analytic Polyhedra

An open subset X of $\mathbb{C}^n$ is a *Runge region* if it is a region of holomorphy such that every $f \in \mathcal{O}(X)$ can be normally approximated by polynomials.

Theorem 2.5.1 (Oka-Weil 1). *Let X be a polynomial polyhedron, i.e. of the form $\bigcap_{j=1}^{r} p_j^{-1}(\mathbb{D})$ for some polynomials $p_1, \dots p_j$ in $\mathbb{C}^n$. Then,*

- *$H^q(X, \mathcal{O}) = 0$ for all $q > 0$;*
- *X is a Runge region.*

The theorem above uses Lemma A on the regular submanifold

$$\{(z, \zeta) \in \mathbb{D}^{n+r} \mid \zeta_j = p_j(z) \text{ for all } j = 1 \dots r\}$$

after assuming that X is contained in the unit polydisk $\mathbb{D}^n$.

An open subset Y of a complex manifold X is *Runge (rel X)* if each $f \in \mathcal{O}(Y)$ can be normally approximated by functions in $\mathcal{O}(X)$. Using techniques similar to the $\bar{\partial}$-Poincaré lemma, we have the following.

Lemma 2.5.2 (Lemma B). *Let $X \in \mathbb{C}^n$ be an open subset with exhaustion $X_1 \Subset X_2 \Subset X_3 \dots$ such that $H^q(X_j, \mathcal{O}) = 0$ for all $q > 0$ and each X_j is Runge rel X_{j+1}. Then, $H^q(X, \mathcal{O}) = 0$ for all $q > 0$ and each X_j is Runge rel X.*

Polynomially convex domains in $\mathbb{C}$ are those whose complement has no bounded components. Indeed, by the maximum principle, the polynomial hull of any compact subset $K \subset \mathbb{C}$ is the union of K and all the bounded components of $\mathbb{C} \backslash K$. In higher dimensions, polynomial convexity is much more subtle.

Theorem 2.5.3. *Let X be an open subset of $\mathbb{C}^n$. The following are equivalent:*

1. *X is polynomially convex;*
2. *X can be exhausted by polynomial polyhedra in D;*
3. *X is a Runge region.*

An open set X is an *analytic polyhedron* in a larger open set $D \subset \mathbb{C}^n$ if there are some $f_1, \dots f_r \in \mathcal{O}(D)$ such that $A = \bigcap_{j=1}^{r} f_j^{-1}(\mathbb{D}) \Subset D$.

Proposition 2.5.4. *Every analytic polyhedron is a region of holomorphy. Conversely, every region of holomorphy D can be exhausted by analytic polyhedra in D.*

Lemma 2.5.5 (Fundamental Lemma). *Let X be an analytic polyhedron in D defined by f_1, f_r. The Oka image $\Sigma = \{(z, f_1(z), \dots f_r(z)) \in \mathbb{C}^{n+r} \mid z \in \bar{X}\}$ is polynomially convex.*

By the fundamental lemma and lemma A,

Theorem 2.5.6 (Oka-Weil 2). *Let X be an analytic polyhedron in $D \subset \mathbb{C}^n$ defined by $f_1, \dots f_r$. Then,*

- *$H^q(X, \mathcal{O}) = 0$ for all $q > 0$;*
- *Every $f \in \mathcal{O}(X)$ can be normally approximated by polynomials in $z, f_1(z), \dots f_r(z)$. In particular, X is Runge rel D.*

Combining lemma B, Oka-Weil, and Cartan's theorem, we have a slick characterisation of regions of holomorphy using sheaf cohomology.

Corollary 2.5.7. *D is a region of holomorphy if and only if $H^q(D, \mathcal{O}) = 0$ for all $q > 0$.*

We also have an analog of Theorem 2.5.3.

Theorem 2.5.8. *Let $X \subset D \subset \mathbb{C}^n$ be open sets. The following are equivalent:*

1. *X is $\mathcal{O}(D)$-convex;*
2. *X can be exhausted by analytic polyhedra in X;*
3. *X is a region of holomorphy and it is Runge rel D.*

2.6 Review Questions

1. What is a domain of holomorphy? What are other equivalent ways of defining it?
2. What is a subharmonic function? What are other equivalent ways of defining it?
3. Given a Taylor series about 0, what can you say about the domain of convergence?
4. Given a Laurent series centered at 0, what can you say about the domain of convergence?
5. Given a domain in $\mathbb{C}^n$, is it pseudoconvex or strictly pseudoconvex? If so, construct an appropriate plurisubharmonic exhaustion.
6. Given an additive Cousin data, does there exist a solution?
7. If $H^1(X, \mathcal{O}) \neq 0$ for some domain X, is it finite dimensional?
8. On a given domain X, does every $\bar{\partial}$-problem have a solution?
9. What is an example of an unsolvable multiplicative Cousin data on a domain of holomorphy?