What is... an orbifold?

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Recall...

Theorem

If M is a smooth manifold and a subgroup G of Diff(M) acts freely and properly discontinuously on M, then M/G is a smooth manifold.

What happens if our group G does not act freely?

Examples

- $S^n/\{\pm Id\} = \mathbb{RP}^n$
- $\mathbb{R}^n/\text{lattice} = \mathbb{T}^n$
- $\mathbb{C}\backslash\{0\}/\langle iz \rangle = \text{infinite cylinder}$
- $\mathbb{C}/\langle iz \rangle = \text{not a manifold...}$

What are orbifolds?

Definition

An n-dimensional smooth **orbifold** O is a Hausdorff paracompact space locally homeomorphic to $\mathbb{R}^n$/finite group.

More precisely, O admits

- an open cover $\{U_i\}$,
- finite groups Γ_i acting smoothly on open subsets $V_i \subset \mathbb{R}^n$,
- homeomorphisms $\phi_i : U_i \to V_i/\Gamma_i$,

satisfying certain compatibility conditions.

Main property

Proposition

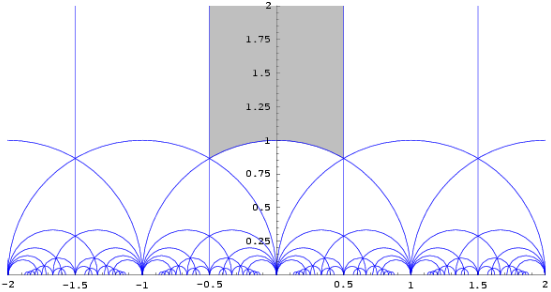
If M is a smooth manifold and a subgroup G of Diff(M) acts properly discontinuously on M, then M/G is a smooth orbifold.

Examples

- Every manifold is trivially an orbifold.
- $\mathbb{C}/\langle e^{2\pi i/n} z \rangle =$ infinite cone of order n.
- $\mathbb{C}/D_{2n} =$ infinite wedge with a corner reflector of order n.
- $\mathbb{T}^d/\mathrm{Sym}_d =$ Mobius strip with mirror boundary if $d = 2$

Examples

- $\mathbb{H}^2/\{\frac{az+b}{cz+d} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\}$ = topological disk with 2 cone points of orders 2 and 3.



Reduction

Proposition

Every orbifold O is locally homeomorphic to $\mathbb{R}^n/\Gamma$ where Γ is some finite subgroup of $O(n)$.

- For $n = 1$, $\Gamma = \{Id\}, \{\pm Id\}$.
- For $n = 2$, $\Gamma = \{Id\}, D_{2n},$ or C_n (Da Vinci's Theorem)

Covering Space Theory

Definition

An **orbifold covering map** $f : O \to P$ is a continuous surjection where every $y \in P$ admits a neighbourhood V such that $f^{-1}(V)$ is a disjoint union of open sets $\{U_i\}$ such that $f : U_i \to V$ is a "quotient map" between two quotients of $\mathbb{R}^n$ by finite groups.

The **orbifold universal covering map** $\tilde{O}$ can be defined by the usual universal property. O is **good** if $\tilde{O}$ is a manifold, **evil** if otherwise.

The **orbifold fundamental group** of O is defined as the deck transformations of $\tilde{O} \to O$.

Euler Characteristic

Definition

Build a finite CW complex structure on a 2-D compact orbifold O such that every singular point/mirror is a cell. The **Euler characteristic** of a compact 2-D orbifold O is defined as

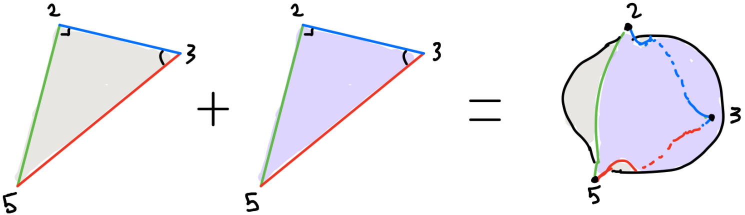
$$\chi(O) = \sum_{\text{cell } c} \frac{(-1)^{\dim(c)}}{|\Gamma_c|}.$$

Riemann-Hurwitz Formula

If $O \to P$ is an orbifold covering map of degree d, $\chi(O) = d\chi(P)$.

Further Reduction

- If the orbifold O has topological boundary, consider the double O_d.
- There's a natural orbifold double covering $O_d \to O$.
- The boundary of O acts as an axis of reflection.



- To classify all 2-D compact orbifolds, we may as well assume O has no topological boundary and the only singularities are cone points.

Classification of 2-D Compact Orbifolds

Type	Universal Cover	Topology	Cone Singularities of order $(m_1, \ldots m_N)$
Evil		S^2	(m), (m_1, m_2) $[m_1 \neq m_2]$
Elliptic $(\chi > 0)$	S^2	S^2	$()$, (m, m), $(2, 2, m)$, $(2, 3, k)$ $[3 \leq k \leq 5]$
		$\mathbb{RP}^2$	$()$, (m)
Parabolic $(\chi = 0)$	$\mathbb{R}^2$	S^2	$(2, 2, 2, 2)$, $(2, 3, 6)$, $(2, 4, 4)$, $(3, 3, 3)$
		$\mathbb{RP}^2$	$(2, 2)$
		$\mathbb{T}^2$	$()$
		Klein B	$()$

If O is neither of the above, O is hyperbolic ($\chi(O) < 0$ and covered by $\mathbb{H}^2$).

Thank you!