Critical Quasicircle Maps: Realization and Rigidity

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Realization

A **(uni-)critical quasicircle map** is a homeomorphism $f : \mathbf{H} \to \mathbf{H}$ of a quasicircle $\mathbf{H}$ admitting a holomorphic extension on a neighborhood of $\mathbf{H}$ with exactly one critical point on $\mathbf{H}$.

The behaviour at the unique critical point on $\mathbf{H}$ can be encoded by two positive integers, namely the inner criticality $d_0(f)$ and the outer criticality $d_\infty(f)$. The total local degree of f at the critical point is $d_0+d_\infty-1$ and it must be at least 2.

We say that a critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$ is a **(uni-)critical circle map** if $\mathbf{H}$ is a Euclidean circle and so $\boldsymbol{d_0 = d_\infty}$. Aside from unimodal maps, critical circle maps provide one of the two classical examples of one-dimensional dynamical systems exhibiting remarkable universality phenomena. We argue that critical quasicircle maps also exhibit universality.

Petersen [P04] proved a generalization of the Herman-Świątek theorem: a critical quasicircle map is quasisymmetrically conjugate to irrational rotation if and only if its rotation number is of bounded type. In the bounded type regime, there are 2 possibilities:

- (B) $\mathbf{H}$ is the boundary of a rotation domain of f, or equivalently, $\min\{d_0, d_\infty\} = 1$;
- (H) $\mathbf{H}$ is a **Herman quasicircle**, i.e. not (B).

The prototype example of case (B) is the boundary of a Siegel disk of $e^{2\pi i\theta}z + z^d$. Previously known examples of case (H) are critical circle maps. Recently, we proved the existence of case (H) of arbitrary inner and outer criticalities.

Realization Theorem [L23a]

Given a bounded type θ and integers $d_0, d_\infty \geq 2$, there exists a unique rational map f such that

- (i) f has exactly 3 critical points 0, ∞, 1 of local degrees d_0, d_∞, and $d_0 + d_\infty - 1$ respectively;
- (ii) $f(0) = 0$ and $f(\infty) = \infty$;
- (iii) f has a Herman quasicircle $\mathbf{H}$ of rotation number θ containing 1 and separating 0 and ∞.

See Figure X for examples of such rational maps.

Degeneration of Herman Rings

In the proof of Realization Theorem, we consider degree $d_0 + d_\infty - 1$ rational maps g such that

- (I) g has superattracting fixed points at 0 and ∞ of local degrees d_0, d_∞ respectively;
- (II) g has a Herman ring $\mathbb{H}$ of rotation number θ separating 0 and ∞;
- (III) All free critical points lie on $\partial\mathbb{H}$.

In [L23a], we establish *a priori bounds* for the Herman ring $\mathbb{H}$ of g that are independent of the modulus: components of $\partial\mathbb{H}$ are quasicircles with dilatation depending only on d_0, d_∞, and θ.

We can deform the complex structure of the grand orbit of $\mathbb{H}$ so that mod$(\mathbb{H})$ becomes arbitrarily close to 0. With *a priori bounds*, we know that in the limit, g and $\mathbb{H}$ converge to a degree $d_0+d_\infty-1$ rational map f and a Herman quasicircle $\mathbf{H}$ of f with the same rotation number θ. This yields a multicritical generalization of the Realization Theorem.

The proof of *a priori bounds* uses renormalization and near-degenerate regime: non-intersecting principle, quasi-additivity law, covering lemma, and canonical weighted arc diagrams. Many of our steps are inspired by [KL05, K06, DL22].

The uniqueness part of Realization Theorem follows from showing that such a rational map f admits no invariant line field on its Julia set.

Rigidity

Beyond the realm of rational maps, we have the following rigidity result, a generalization of [dFdM].

Rigidity Theorem [L23b]

Any two critical quasicircle maps $f_1 : \mathbf{H}_1 \to \mathbf{H}_1$ and $f_2 : \mathbf{H}_2 \to \mathbf{H}_2$ of the same criticalities and bounded type rotation number are quasiconformally conjugate on a neighborhood of $\mathbf{H}_1$ and $\mathbf{H}_2$. The conjugacy is uniformly $C^{1+\alpha}$-conformal on $\mathbf{H}_1$.

For each $n \in \mathbf{N}$, denote by I_n the shortest interval in $\mathbf{H}$ connecting the critical point c and $f^{q_n}(c)$. The n^th pre-renormalization of f is the commuting pair

$$\left(f^{q_n}|_{I_{n+1}},\ f^{q_{n+1}}|_{I_n}\right)$$

and the **n^th renormalization** $\mathcal{R}^n f$ of f is the normalized commuting pair obtained by rescaling the pair to unit size.

We use Avila-Lyubich's quasicritical circle maps [AL22] to obtain *complex bounds*, i.e. uniformly nice domains for $\mathcal{R}^n f$ for sufficiently high n. By the pullback argument, *complex bounds* imply qc rigidity.

To prove that the qc conjugacy is $C^{1+\alpha}$, we apply McMullen's Dynamic Inflexibility Theorem [McM96] and show that the Julia set of the map f in Realization Theorem is uniformly twisting and points on $\mathbf{H}$ are uniformly deep points of $J(f)$.

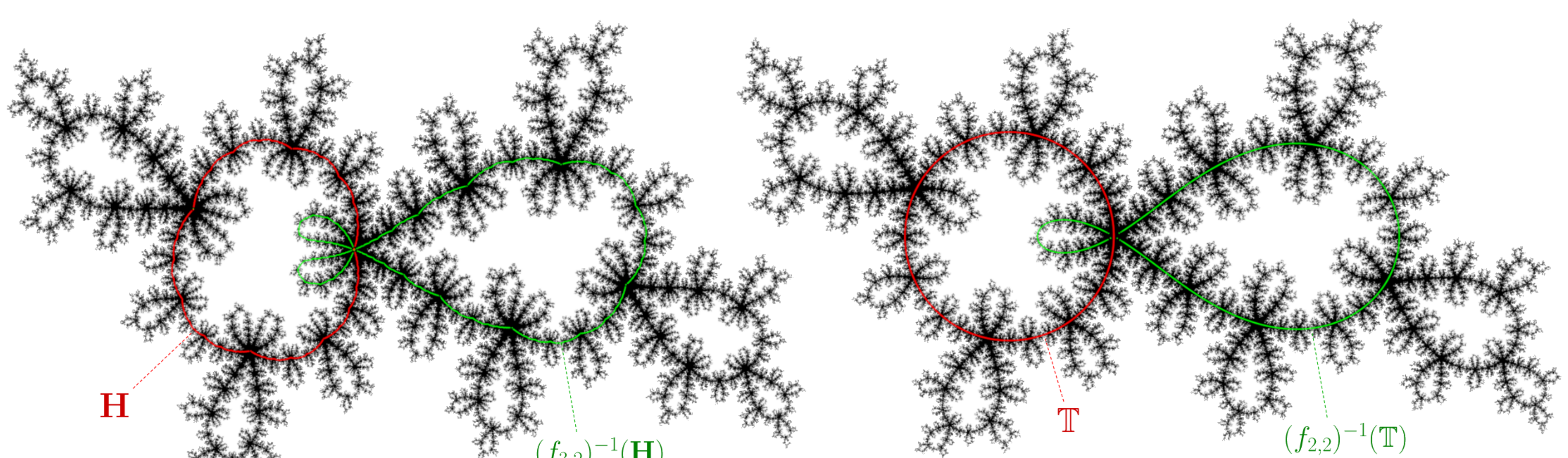

Figure X: The Julia sets of $f_{3,2}(z) = bz^3\dfrac{4-z}{1-4z+6z^2}$ (left) and $f_{2,2}(z) = cz^2\dfrac{z-3}{1-3z}$ (right). The constants $b \approx -1.144208 - 0.964454i$ and $c \approx -0.755700 - 0.654917i$ are picked such that $f : \mathbf{H} \to \mathbf{H}$ is a critical quasicircle map on some quasicircle $\mathbf{H}$ with $d_0(f_{3,2}) = 3$ and $d_\infty(f_{3,2}) = 2$, whereas $f_{2,2} : \mathbb{T} \to \mathbb{T}$ is a critical circle map with $d_0(f_{2,2}) = d_\infty(f_{2,2}) = 2$, both with golden mean rotation number.

Universality

Here are some applications of the Rigidity Theorem.

- ▷ The Hausdorff dimension HD$(\mathbf{H})$ of $\mathbf{H}$ is universal, i.e. depends only on d_0, d_∞, and θ.
- ▷ $\mathbf{H}$ is C^1 smooth iff HD$(\mathbf{H}) = 1$ iff $d_0 = d_\infty$.
- ▷ The limiting scaling ratios $\dfrac{f^{q_{n+1}}(c) - c}{f^{q_n}(c) - c}$ are universal.
- ▷ If θ is a quadratic irrational, then $\mathbf{H}$ is self-similar at the critical point with a universal scaling factor.
- ▷ $\mathcal{R}^n f$ converges exponentially fast to an $\mathcal{R}$-invariant horseshoe attractor.

Hyperbolicity (in progress)

In our current project, we will establish hyperbolicity of renormalization by adapting *Pacman Renormalization Theory* for Siegel disks [DLS20, DL23] and rigidity results for the escaping dynamics of transcendental entire functions [R09].

"Given a periodic type θ and fixed criticalities (d_0, d_∞), there is a compact analytic renormalization operator $\mathcal{R}$ with a hyperbolic fixed point $f_\star$. The stable manifold has codimension 1 consisting of critical quasicircle maps qc conjugate to $f_\star$. Maps on the unstable manifold admit maximal σ-proper extension onto $\mathbb{C}$."

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