Herman Rings: A Priori Bounds and Degeneration

Willie Rush Lim

Advisor: Dzmitry Dudko

Stony Brook University

A Priori Bounds

Let $\theta = [0; a_1, a_2, \ldots]$ be a bounded type irrational number. By the works of Douady [Dou87], Ghys [Ghy84], Herman [Her86], Swiatek [Swi88], Shishikura, and Zhang [Zha08], we know that the boundary of a fixed Siegel disk of a rational map of rotation number θ is a quasicircle containing a critical point with dilatation depending only on the degree and $\eta(\theta) := \max_{i \geq 1} a_i$.

Question 1

Can we say the same for Herman rings?

Shishikura's surgery [Shi87] gives a way to construct a Herman ring out of two Siegel disks. With this procedure, the dilatation of the boundary components of any fixed Herman ring of a rational map generally depends on the degree, $\eta(\theta)$, and the modulus.

Let $\mathcal{H}_{d_0,d_\infty,\theta} \subset \text{Rat}_{d_0+d_\infty-1}$ denote the space of all rational maps f obtained from Shishikura's surgery applied to two polynomials P_0 and P_∞ of degrees d_0 and d_∞ having fixed Siegel disks Z_0 and Z_∞ of rotation numbers θ and $1-\theta$ such that all finite critical points lie on ∂Z_0 and ∂Z_∞ respectively.

In [Lim22], we answer Question 1 for $\mathcal{H}_{d_0,d_\infty,\theta}$:

Theorem A

The boundary components of the Herman ring of every rational map in $\mathcal{H}_{d_0,d_\infty,\theta}$ are quasicircles with dilatation depending only on d_0, d_∞ and $\eta(\theta)$.

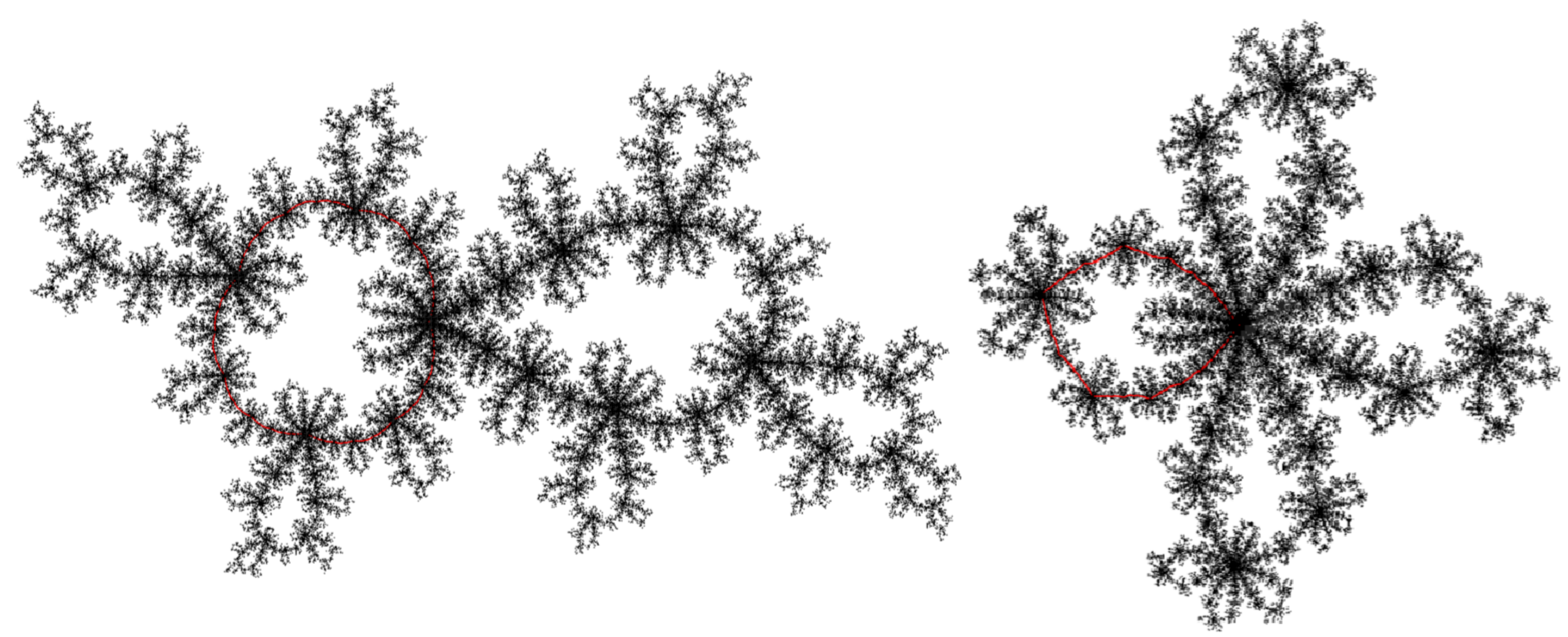
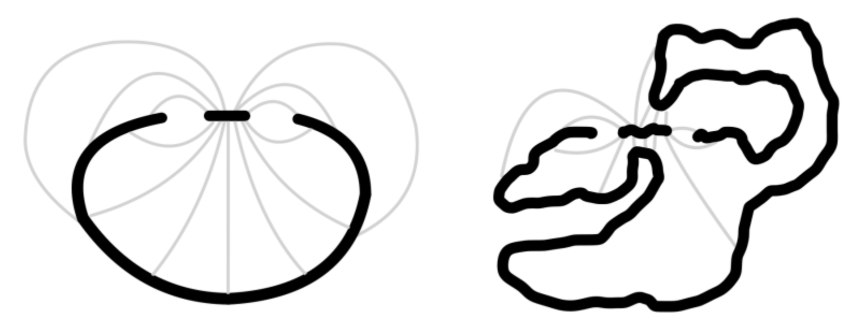


Figure X: Examples of non-trivial Herman quasicircles. The figure shows the Julia sets of $f(z) = \alpha z^3(z-4)/(6z^2-4z+1)$ on the left and $g(z) = \beta z^2(z^3-5z^2+10z-10)/(5z-1)$ on the right. Both have a unique free critical point at $z=1$. The coefficients $\alpha \approx 1.14421 + 0.96445i$ and $\beta \approx 0.38663 + 0.32050i$ are determined numerically such that f and g admit Herman quasicircles (shown in red) passing through 1 of golden mean rotation number $\theta = \frac{\sqrt{5}-1}{2}$.

Near-Degenerate Regime

Let H be a component of $\partial\mathbb{H}$. The conjugacy between $f|_H$ and the rigid rotation $R_\theta|_{S^1}$ gives rise to a combinatorial metric on H.

Let I be an interval in H of (combinatorial) length $|I| < 0.1$. Following [DL22], we denote by $W_{10}(I)$ the extremal width of curves connecting I and $H \backslash 10I$, where $10I$ is the interval of length $10|I|$ having the same midpoint as I.



small $W_{10}(I)$ large $W_{10}(I)$

$W_{10}(I)$ tells us how close H is to degeneracy (as a quasicircle) near I. To prove Theorem A, it is sufficient to show that every interval $I \subset H$ satisfies $W_{10}(I) \leq K$ for some bound K depending only on d_0, d_∞, and $\eta(\theta)$. Our goal is reduced to showing:

"There is some $K = K(d_0, d_\infty, \eta(\theta)) > 0$ such that if there is an interval I with $W_{10}(I) > K$, then there is another interval J such that $W_{10}(J) > 2K$."

The proof of such uses renormalisation and many facets of Kahn's near-degenerate regime: non-intersecting principle, quasi-additivity law, covering lemma, and canonical weighted arc diagrams. Many of our steps are inspired by [KL05, Kah06, DL22].

Herman Curves

An invariant Jordan curve $\mathbf{H} \subset \hat{\mathbb{C}}$ of a rational map f is a **Herman curve** if

- ▷ $f|_{\mathbf{H}}$ is conjugate to a rigid rotation on the S^1, and
- ▷ $\mathbf{H}$ is not contained in the closure of a rotation domain of f.

Additionally, we call $\mathbf{H}$ a **Herman quasicircle** if it is a quasicircle. The combinatorics of $\mathbf{H}$ is encoded by the criticality and relative combinatorial position of the critical points on $\mathbf{H}$.

The following question was posed by Eremenko:

Question 2

Does there exist Herman curves that are non-trivial, i.e. not induced by Blaschke products (in which case $\mathbf{H}$ is the unit circle) nor quasiconformal deformations of such?

Degenerating Herman Rings

A priori bounds gives us pre-compactness to study the dynamics for the limit space

$$\mathcal{G}_{d_0,d_\infty,\theta} := \overline{\mathcal{H}_{d_0,d_\infty,\theta}} \backslash \mathcal{H}_{d_0,d_\infty,\theta}.$$

Take a normalised family $\{f_\mu\}_{0<\mu\leq 1}$ in $\mathcal{H}_{d_0,d_\infty,\theta}$ where f_μ has Herman ring of modulus μ and the same combinatorics. We can let $\mu \to 0$ and obtain a Herman curve of the same combinatorics.

Theorem B

Every $f \in \mathcal{G}_{d_0,d_\infty,\theta}$ admits a Herman quasicircle $\mathbf{H}$ of rotation number θ containing every critical point of f other than 0 and ∞. Its dilatation depends only on d_0, d_∞, and $\eta(\theta)$.

Given any combinatorial data, there exists a Herman quasicircle $\mathbf{H}$ from the space $\mathcal{G}_{d_0,d_\infty,\theta}$ realising such prescribed combinatorics.

This answers Question 2! See **Figure X**.

Open Questions

- ▷ For $f \in \mathcal{G}_{d_0,d_\infty,\theta}$, does $J(f)$ have zero area or Hausdorff dimension < 2?
- ▷ Is every Herman curve the limit of degenerating Herman rings?
- ▷ Can we extend our results to θ of unbounded type?

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