RESEARCH STATEMENT

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My research lies at the intersection of complex analysis, geometry, and dynamical systems, with a particular focus on one-dimensional holomorphic dynamics and renormalization theory. I study the topological, geometric, and measure-theoretic properties of dynamical sets associated to holomorphic maps, with an emphasis on rigidity and universality phenomena arising from renormalization.

0.1. **Core problems.** A central problem in one-dimensional complex dynamics is to understand the behavior of rational maps that are not hyperbolic. A rational map is said to be hyperbolic if its global behavior is dominated by attracting periodic dynamics. In contrast, non-hyperbolic rational maps can exhibit complicated dynamical behavior. The density of hyperbolicity conjecture (DoH) states that hyperbolic maps are dense in the space of rational maps. For the family of quadratic polynomials

$$z \mapsto z^2 + c, \qquad c \in \mathbb{C},$$

a fundamental obstacle is the conjecture on the local connectivity of the Mandelbrot set (MLC). MLC implies DoH for quadratic polynomials and provides a complete topological description of the Mandelbrot set through its pinched-disk model. It is equivalent to the combinatorial rigidity conjecture: roughly speaking, a non-hyperbolic quadratic polynomial should be uniquely determined by its combinatorial data. This problem shares deep roots with the celebrated rigidity theorem of Mostow as well as the ending lamination theorem in hyperbolic geometry.

0.2. **Renormalization.** One of the principal frameworks for studying these questions is renormalization theory. The basic idea is to rescale a suitable first-return map and obtain a new dynamical system of the same type. Iterating this procedure often reveals self-similarity and can convert geometric questions about individual dynamical systems into questions about the dynamics of an operator on a space of maps. In the 1970s, Feigenbaum, and independently Coullet and Tresser, used this framework to explain the universal scaling laws of period-doubling cascades, suggesting that such universality arises from hyperbolicity of the renormalization operator. This discovery led to Feigenbaum receiving the 1986 Wolf Prize in Physics. Subsequent work of Sullivan, McMullen, Lyubich, and others developed this picture rigorously in several important settings.

0.3. **Contributions.** My research contributes to this program in several directions. In my PhD thesis and subsequent work, I proved a realization theorem for Herman curves, established rigidity and hyperbolicity results for critical quasicircle maps, studied typical orbits of neutral quadratic polynomials, and developed a unified framework for neutral renormalization theory. These are connected by common themes: geometric bounds, rigidity, and the extension of real renormalization theory to genuinely complex settings. My recent work can be organized around two main directions.

1. Herman curves

The first direction grew out of my PhD thesis, in which I studied the geometry of Herman rings and their degeneration. A *Herman ring* is a periodic annular rotation domain of a rational map. Unlike a Siegel disk (a rotation disk), a Herman ring contains no periodic point and has a non-trivial deformation space, making it particularly more difficult to control geometrically.

1.1. **A priori bounds and degeneration.** I studied the degeneration of Herman rings to *Herman curves*: invariant Jordan curves on which the dynamics are analytically conjugate to irrational rotation but which do not arise as part of the closure of a rotation domain. One of the main difficulties is obtaining geometric estimates that remain uniform in a near-degenerate setting.

In my Inventiones paper [Lim25], I established *a priori bounds* for Herman rings in the bounded type setting.

Theorem A ([Lim25])**.** *Under a simple configuration hypothesis, the boundaries of a Herman ring of a rational map with a bounded type irrational rotation number are quasicircles with dilatation depending only on the degree and the rotation number.*

The proof uses near-degenerate conformal-geometric machinery developed by Kahn, Lyubich, and Dudko [KL05, Kah06, DL22]. These estimates lead to a realization theorem for Herman curves, answering a question posed by Eremenko [Ere20].

Theorem B ([Lim25])**.** *One can construct rational maps admitting Herman curves with any prescribed bounded type combinatorial data.*

Here, the combinatorial data consist of the bounded type irrational rotation number and the combinatorial location of inner and outer criticalities.

1.2. **Rigidity of rational maps.** A rational map f is said to admit an *invariant line field* on its Julia set if there is an f-invariant measurable Beltrami differential $\mu(z)\frac{d\bar{z}}{dz}$ such that $|\mu| = 1$ on a positive measure subset of the Julia set and $\mu = 0$ elsewhere. The absence of invariant line fields implies the lack of non-trivial deformation space supported on the Julia set. McMullen and Sullivan [McM94, MS98] conjectured that every rational map that is not exceptional does not admit any invariant line field. This conjecture implies the DoH conjecture. In [Lim26a], I made a contribution to this problem.

Theorem C ([Lim26a])**.** *A rational map that is geometrically finite away from finitely many bounded type rotation domains and Herman curves supports no invariant line field on its Julia set. In the absence of Herman curves, the Julia set has zero Lebesgue measure.*

This provides a broad extension of earlier rigidity results for geometrically finite maps [Sul83, McM00]. It gives applications to the combinatorial rigidity of rational maps constructed in Theorem A and bounded type non-renormalizable Siegel polynomials [DY25].

1.3. **Critical quasicircle maps.** These results led naturally to the study of critical quasicircle maps. A *(uni-)critical quasicircle map* $f : X \to X$ is a holomorphic dynamical system on an invariant quasicircle X with a single critical point. Such systems form a complex counterpart of the well-developed theory of real-analytic critical circle maps [Lan88, dF99, dFdM99, Yam01, Yam02, Yam03]. Theorem B implies the existence of critical quasicircle maps with arbitrary bounded type combinatorial data, namely the rotation number θ, the inner criticality $d_0 \geq 2$, and the outer criticality $d_\infty \geq 2$. See Figure 1 for an example. In [Lim26a], I proved the rigidity of such maps.

Theorem D ([Lim26a])**.** *Any two critical quasicircle maps with the same bounded type combinatorics are quasiconformally conjugate on a neighborhood of their invariant quasicircles. Moreover, the conjugacy is uniformly $C^{1+\alpha}$-conformal along the quasicircle.*

The proof relies on complex *butterfly bounds* for renormalizations. This theorem yields universality of Hausdorff dimension of such invariant quasicircles, self-similarity for quadratic irrational rotation numbers, and exponential convergence of renormalizations.

Building on this theory, in [Lim26b], I established hyperbolicity of the renormalization operator for critical quasicircle maps with periodic-type rotation number.

Theorem E ([Lim26b])**.** *For each prescribed periodic-type rotation number and pair of criticalities, the corresponding renormalization operator is analytic and admits a hyperbolic fixed point with exactly one unstable direction. Its local stable manifold is the local conjugacy class of the fixed point.*

As a consequence, critical quasicircle maps with fixed criticalities and quadratic irrational rotation number form a codimension ≤ 1 analytic submanifold within their natural Banach space. The most difficult part of the proof is proving the exact dimension of the local unstable manifold. To do so, I proved a rigidity result on the transcendental structure of maps on the unstable manifold; an analog of this for transcendental entire functions was established in a seminal work of Rempe [Rem09].

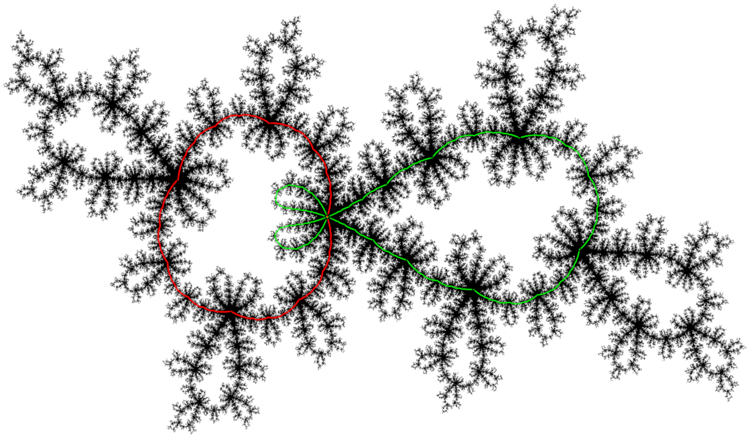

FIGURE 1. The Julia set of a degree 4 rational map that is a critical quasicircle map with golden mean rotation number and imbalanced criticalities $(d_0, d_\infty) = (3, 2)$. The Herman curve is colored red and its strict preimage is colored green.

This theory suggests a broader program, that is, to understand which features of real renormalization theory persist in the genuinely complex setting where combinatorial asymmetry occurs.

2. Neutral Renormalization Theory

My current research focuses on a second, closely related setting: quadratic polynomials with a neutral fixed point,

$$f_\theta(z) := e^{2\pi i\theta} z + z^2.$$

These maps exhibit some of the most delicate behavior in one-dimensional complex dynamics. Depending on the arithmetic properties of θ, the neutral fixed point at 0 may be linearizable or non-linearizable, and the associated dynamical sets can have very different geometric properties.

2.1. **Sector renormalization.** A key development in this direction has been the theory of sector renormalization. For a neutral holomorphic germ $f(z) = e^{2\pi i\theta} z + O(z^2)$, one can construct a fundamental sector near the fixed point at 0 and glue its sides using the dynamics; see Figure 2. The first-return map then produces a new neutral holomorphic germ $\mathcal{R}_{\sec} f$, and the procedure can be iterated. Yoccoz [Yoc95] carefully studied this procedure to prove his seminal result on the optimality of the Brjuno condition for linearizability.

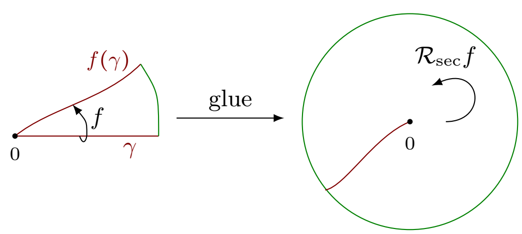

FIGURE 2. Sector renormalization

More recently, Dudko and Lyubich [DL22, DL26a] applied the near-degenerate regime to establish uniform *a priori bounds* for this renormalization scheme for neutral quadratic polynomials f_θ for all irrational θ. Notably, this revolves around the construction of uniformly quasiconformal *pseudo-Siegel disks* $\hat{Z}_\theta$ that contain both the neutral fixed point and the postcritical set (the closure of the critical orbit of f_θ). One main application is the existence of the *Mother Hedgehog* H_θ, a full planar continuum that contains the neutral fixed point and the critical orbit of f_θ such that $f_\theta : H_\theta \to H_\theta$ is a homeomorphism. In [Lim26d], I used these pseudo-Siegel disks to study typical orbits of f_θ.

Theorem F ([Lim26d])**.** *For every $\theta \in \mathbb{R}/\mathbb{Z}$, the postcritical set of f_θ has zero Lebesgue measure.*

Previously, zero-area results were known only under additional arithmetic assumptions [Dou87, PZ04, Che13, Che19]; my theorem gives a uniform statement across all rotation numbers. The proof combines the pseudo-Siegel geometry with distortion estimates for the sector-gluing maps and a porosity argument for typical points. This theorem has numerous applications. For example, almost every point in the Julia set of f_θ is non-recurrent.

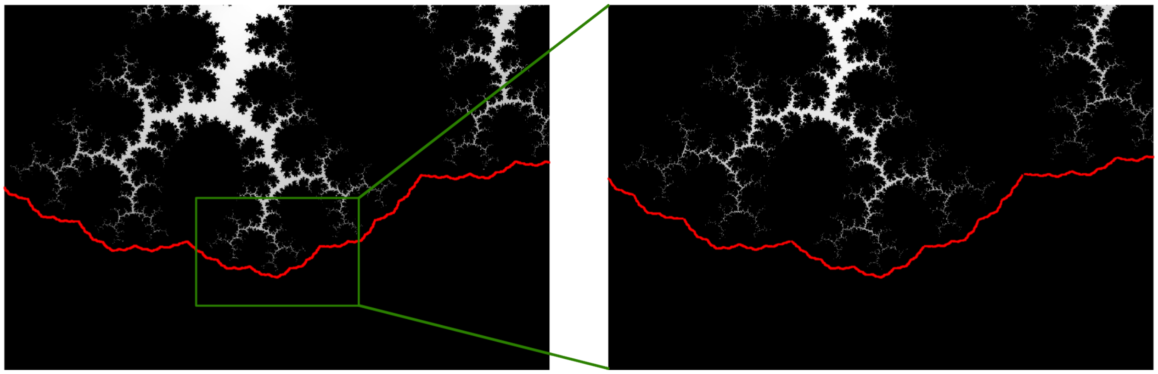

FIGURE 3. Self-similarity of the dynamical plane of f_θ near its critical value in the golden mean case. The postcritical set is colored red.

2.2. **The full renormalization horseshoe.** These pseudo-Siegel bounds lead to a broader renormalization theory. Iterated sector renormalizations form an attractor consisting of bi-infinite towers of neutral holomorphic maps. In joint work with Dudko and Lyubich [DLL26], we established combinatorial rigidity for this attractor.

Theorem G ([DLL26])**.** *Any two combinatorially equivalent bi-infinite neutral renormalization towers are conformally conjugate near their associated invariant Mother Hedgehogs.*

This result yields a complete dynamical universality picture, extending McMullen's famous self-similarity result [McM98]; see Figure 3. After quotienting by conformal equivalence, the full neutral-renormalization attractor is modeled by a shift on its combinatorial data. The underlying combinatorial and arithmetic structure is studied in detail in a separate note of mine [Lim26c].

Our theorem unifies two previously well-developed regimes of neutral dynamics: bounded type renormalization [McM98, DLS20] and near-parabolic renormalization [IS06]. In the bounded type setting, neutral holomorphic dynamics are closely related, via quasiconformal surgery, to the classical renormalization theory of critical circle maps and thus fit naturally into the broader real renormalization paradigm. The unbounded, Cremer regime, however, reveals a genuinely new phenomenon where the associated Mother Hedgehogs are not locally connected and have no clear analog in the classical real renormalization theory. In [DLL26], we have successfully developed a unified framework that accommodates such wild objects and retains similar rigidity and universality phenomena.

An alternative formulation of Theorem G is in terms of *neutral cascades*. These are global transcendental dynamical systems that arise as rescaling limits of f_θ near the critical value. (See Figure 3 again.) We proved that any two combinatorially equivalent neutral cascades are affinely equivalent.

A crucial ingredient in the proof is uniform *butterfly bounds* which provide the uniform control of the dynamics exterior to the Mother Hedgehog at all scales. Such bounds follow from a separate result obtained jointly with Dudko in [DL26b].

Theorem H ([DL26b])**.** *Iterated preimages of the pseudo-Siegel disk of f_θ are uniformly quasiconformal and have uniformly controlled size.*

To our surprise, the proof has to again be carried out in the near-degenerate regime.

3. Ongoing and future projects

The results I mentioned above lead to several natural directions for future research. Here are the three main projects I am currently working on.

(1) **Local connectivity of the Mandelbrot set**
A major long-term direction is the MLC. The remaining problem is on infinitely renormalizable parameters with unbounded combinatorics. With Jeremy Kahn, I am currently studying primitive near-neutral combinatorics, where we aim to establish the required *a priori bounds* by combining techniques from the near-degenerate regime [KL05, Kah06, DL22] with regularization methods [DL22]. This project is a natural continuation of my works [Lim25] and [DL26b].

(2) **Uniform hyperbolicity of neutral renormalization**
With Dzmitry Dudko and Mikhail Lyubich, I am currently working to prove uniform hyperbolicity of the full neutral renormalization horseshoe with exactly one unstable direction. Theorem G establishes the combinatorial side of the theory. Hyperbolicity would provide the corresponding analytic structure. Such a result is expected to have consequences for the fine geometric and measure-theoretic properties of the boundary of the Mandelbrot set.

(3) **Topology of Mother Hedgehogs**
Another direction concerns the topology of the postcritical set of f_θ. Cheraghi [Che23, Che25] constructed a straight toy model predicting a trichotomy of possible topological types under suitable arithmetic conditions: a Jordan curve, a one-sided hairy Jordan curve, or a Cantor bouquet. I am currently working toward extending this picture to all irrational rotation numbers using pseudo-Siegel bounds, the estimates from [Lim26d], and a variant of Cheraghi's toy model. This will imply the longstanding conjecture that every quadratic Siegel disk is a Jordan domain.

Lastly, below is a list of three projects I aim to pursue in the future. The last two should be tractable for strong graduate students.

(4) **Pseudo-Herman rings**
To understand the degeneration of Herman rings across all irrational rotation numbers, one should be able to apply the near-degenerate regime in [DL22, Lim25] and construct uniform *pseudo-Herman rings*, an annular analog of pseudo-Siegel disks, for cubic Blaschke products
$$g_{t,a}(z) = e^{2\pi i t} z^2 \frac{z-a}{1-az}$$
where $a > 3$ and $t \in \mathbb{R}$. Such bounds would imply the existence of a doubly-connected analog of Mother Hedgehogs and concretely show that Cremer phenomena can occur in real dynamics. This project may also help us study the regularity of irrational Arnol'd tongues.

(5) **Biaccessibility of Cremer Julia sets**
One open problem in neutral dynamics is on the accessibility of 0 from the exterior of the Julia set of f_θ when θ is non-Brjuno [McM94]. A solution to this problem is expected to follow from the *butterfly bounds* developed in the proof of Theorem G. This would be a major step towards understanding the topology of the Julia set as outlined in [BO06].

(6) **Rigidity of multicritical quasicircle maps**
It is highly expected for Theorem D to hold in the general multicritical case. The special case where the invariant quasicircle is a round circle has been successfully studied by Gorbovickis and Yampolsky [GY25]. Methods from near-degenerate regime (e.g. from [DL26b]) may help us establish *butterfly bounds* and ultimately the rigidity of multicritical quasicircle maps.

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