Lecture 1

Objects : ① Local : $f(z) = e^{2\pi i\theta} z + O(z^2)$

② Global : $f_\theta(z) = e^{2\pi i\theta} z + z^2 \; : \mathbb{C} \to \mathbb{C}$

Question: How does θ affect the dynamics of ① and ② ?

Plan : Lecture 1 - ①

Lecture 2 - ② for θ being bounded type

Lecture 3 - ② for all irrationals

We say that $f(z) = \lambda z + O(z^2)$ is <u>linearizable</u> at 0 if $\exists$ holomorphic change of variables h near 0 such that $hfh^{-1}(z) = \lambda z$.

- If $|\lambda| \neq 0, 1$, always linearizable ☺
- If $\lambda = e^{2\pi i\theta}$
  - $\theta = \frac{p}{q}$ never linearizable unless it's finite order
  - $\theta \notin \mathbb{Q}$ ....?

<u>Thm</u> [Siegel '42] If θ is Diophantine, f is always linearizable.

<u>Thm</u> [Brjuno '71, Yoccoz '95] If θ is Brjuno, f is always linearizable.

Recall.. θ is Diophantine if $\sup\limits_{n \geq 1} \dfrac{\log q_{n+1}}{\log q_n} < \infty$,

θ is Brjuno if $B(\theta) = \sum\limits_{n=0}^{\infty} \dfrac{\log q_{n+1}}{q_n} < \infty$.

Siegel & Brjuno's proof : There is a unique formal power series $h(z) = \sum h_n z^n$ such that $h \circ f(z) = e^{2\pi i\theta} h(z)$. Brjuno condition $\longrightarrow$ positive radius of conv.

Yoccoz's proof uses a geometric procedure called Sector Renormalization.

It also yields :

<u>Thm</u> [Y '95] If θ is not Brjuno, $f_\theta(z) = e^{2\pi i\theta} z + z^2$ is not linearizable at 0.

<u>Conjecture</u> : If θ is not Brjuno, then any polynomial f with $f(0) = 0$ and $f'(0) = e^{2\pi i\theta}$ is linearizable at 0.

Sector Renormalization

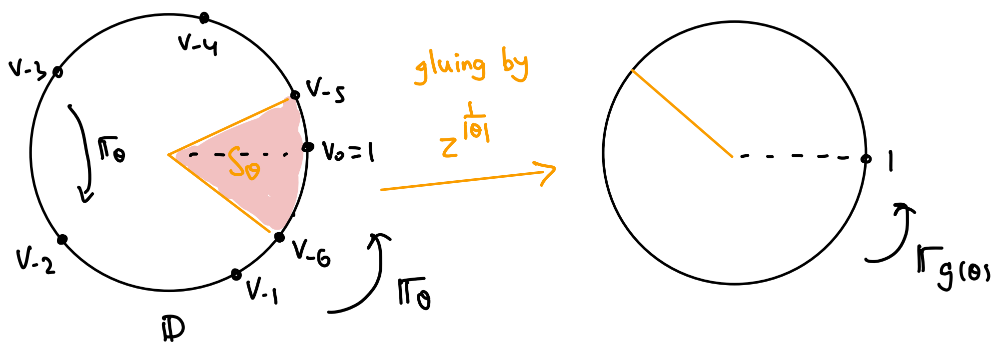
Notation: $\Theta = (-\frac{1}{2}, \frac{1}{2}) \setminus \mathbb{Q}$,

For $\theta \in \Theta$, denote rigid rotation: $\mathcal{R}_\theta(z) = e^{2\pi i \theta} z$, $z \in \mathbb{D}$,

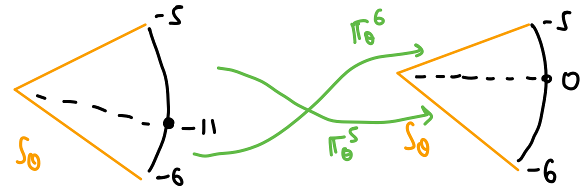
orientation: $\varepsilon(\theta) = \mathrm{sign}(\theta) \in \{-1, +1\}$

1st return time: $\bar{a}(\theta) = \lfloor \frac{1}{|\theta|} \rfloor \geq 2$.

$\boxed{\text{E.g.}}$ $\theta = 0.19755\ldots$, $\varepsilon = +$, $\bar{a} = 5$.



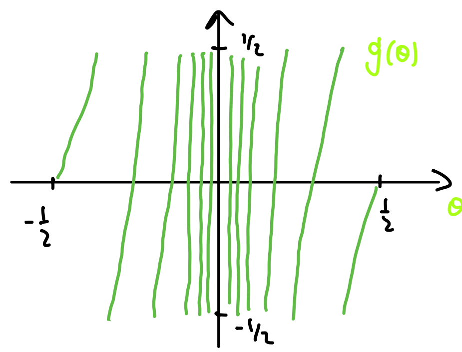
1st return map:



Under $z^{\frac{1}{|\theta|}}$, 1st return map to S_θ projects to a new rotation $\mathcal{R}_{g(\theta)}$ of angle $g(\theta) \in \Theta$. We have

$$g : \Theta \to \Theta, \quad g(\theta) \equiv -\frac{1}{\theta} \mod 1.$$

g extends to a self-map of $\mathbb{T}$, continuous and C^1 everywhere except at 0.



Prop $g : \Theta \to \Theta$ is conjugate to the shift map $\mathcal{S} : (\{-,+\} \times \mathbb{N}_{\geq 2})^{\mathbb{N}} \circlearrowleft$ via:

$$\mathcal{H} : (\varepsilon_n, \bar{a}_n)_{n \geq 1} \longmapsto \cfrac{1}{\varepsilon_1 b_1 - \cfrac{1}{\varepsilon_2 b_2 - \cfrac{1}{\varepsilon_3 b_3 - \ddots}}} \quad \text{where } b_n = \bar{a}_n + \frac{\varepsilon_n \varepsilon_{n+1} + 1}{2}.$$

Convergents: $\frac{p_{[n]}}{q_{[n]}} = 1/(\varepsilon_1 b_1 - 1/(\ldots - 1/\varepsilon_n b_n)..)$

$\frac{p_{[0]}}{q_{[0]}} = \frac{0}{1}, \quad \frac{p_{[1]}}{q_{[1]}} = \frac{\varepsilon_1}{b_1}, \quad \ldots \qquad q_{[n+1]} = b_{n+1} q_{[n]} - \varepsilon_n \varepsilon_{n+1} q_{[n-1]}.$

Some properties:
- $q_{[n]} = 1^{st}$ time after $q_{[n-1]}$ such that $|\mathbb{r}_\theta^{q_{[n]}}(1) - 1| < \frac{1}{2} |\mathbb{r}_\theta^{q_{[n-1]}}(1) - 1|$
- $\mathbb{r}_\theta^{q_{[n]}}$ = rotation by angle $q_{[n]}\theta - p_{[n]} = |\theta_0 \ldots \theta_{n-1}| \theta_n$ where $\theta_k = g^k(\theta),\ k \geq 0$.

$\boxed{\text{E.g.}}$ $\mathfrak{X}\langle(+,2),(-,2),(+,2),(-,2),\_\rangle = \cfrac{1}{2 - \cfrac{1}{-2 - \cfrac{1}{2 - \cfrac{1}{-2 \ddots}}}} = \frac{1}{\sqrt{5}}$

$\boxed{\text{E.g.}}$ $\theta_{gm} = \frac{3-\sqrt{5}}{2} = \cfrac{1}{3 - \cfrac{1}{3 - \cfrac{1}{3 \ddots}}}, \quad \mathfrak{X}^{-1}(\theta_{gm}) = \langle(+,2),(+,2),(+,2),\ldots\rangle$

$-\theta_{gm} = \cfrac{1}{-3 - \cfrac{1}{-3 - \cfrac{1}{-3 \ddots}}}, \quad \mathfrak{X}^{-1}(-\theta_{gm}) = \langle(-,2),(-,2),(-,2).\_\rangle$

$q_{[n]}$'s are $1, 3, 8, 21, 55, \ldots$ We have: $q_{[n+1]} = 3q_{[n]} - q_{[n-1]}$.

<u>STRATEGY</u>

Yoccoz function:

$$Y(\theta) = \log\frac{1}{|\theta_0|} + |\theta_0| \log\frac{1}{|\theta_1|} + |\theta_0\theta_1| \log\frac{1}{|\theta_2|} + |\theta_0\theta_1\theta_2| \log\frac{1}{|\theta_3|} + \ldots, \quad \theta_k = g^k(\theta).$$

Unique function satisfying:

$$Y(\theta) = \log\frac{1}{|\theta|} + |\theta|\, Y(g(\theta)). \qquad (*)$$

<u>Lemma</u> $|Y(\theta) - B(\theta)| = O(1)$. Hence, θ is Brjuno iff $Y(\theta) < \infty$.

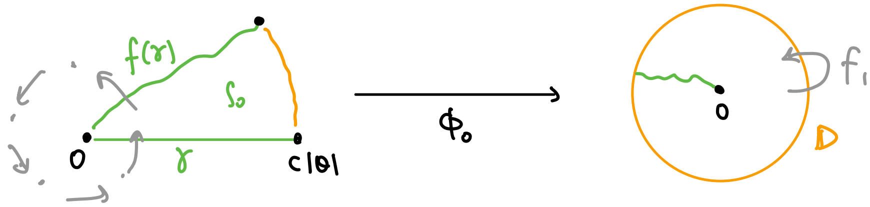
What happens if we perform sector renormalization to $f(z) = e^{2\pi i\theta} z + O(z^2)$?

Assume: $f(z)$ is injective on $\mathbb{D}$

A topological disk $U \subset \mathbb{C}$ is called a <u>Siegel disk</u> of f if it's the largest disk nbh of 0 in $\mathbb{D}$ on which $f: U \to U$ is conjugate to $\mathbb{r}_\theta : \mathbb{D} \to \mathbb{D}$.

Step 1 On a disk of radius $\sim|\theta|$, f is close to R_θ.

A renormalization sector S_0 of size $c|\theta|$ can be constructed.



Step 2 Identify $z \sim f(z)$ to get a conformal gluing map $\Phi_0 : S_0 \to \mathbb{D}^*$.

Φ_0 is close to the power map $\left(\frac{z}{c|\theta|}\right)^{\frac{1}{|\theta|}}$.

Step 3 Φ_0 projects the 1st return map of f back to S to a univalent map

$f_1 : (\mathbb{D}, 0) \to (\mathbb{C}, 0)$ with $f_1'(0) = e^{2\pi i \theta_1}$ where $\theta_1 = g(\theta)$.

Repeat to get $(f_n)_{n\geq 0}$, a sequence of neutral germs with

$$f_n'(0) = e^{2\pi i \theta_n}, \quad \theta_n = g^n(\theta).$$

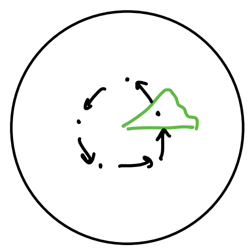
Step 4 f has a Siegel disk of conformal radius $\geq \text{const} \cdot e^{-Y(\theta)}$.

To see why, here's a rough idea:

$$e^{-Y(\theta)} \xrightarrow[\Phi_0]{} \left(\frac{e^{-Y(\theta)}}{c|\theta|}\right)^{\frac{1}{|\theta|}} = c^{-\frac{1}{|\theta|}} e^{-\frac{1}{|\theta|}\left(Y(\theta) - \log\frac{1}{|\theta|}\right)} \gtrsim e^{-Y(\theta_1)}.$$

$\therefore$ Points z in $D(0, \text{const}\, e^{-Y(\theta)})$ will visit S all the time,

and so it lies in a region of stability.

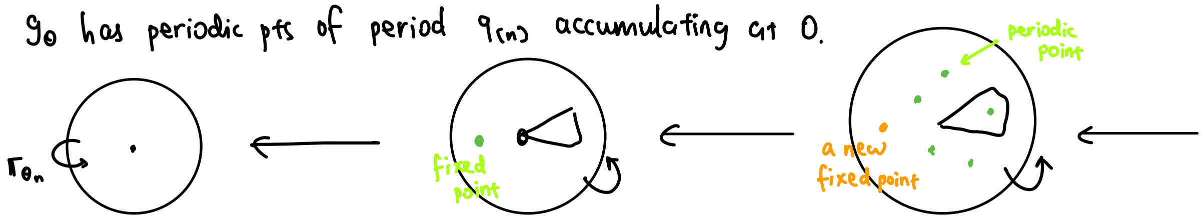


For the non Brjuno θ,

Step 1 Use C^∞ interpolation & antirenormalizations of $R_\theta(z) = e^{2\pi i\theta} z$

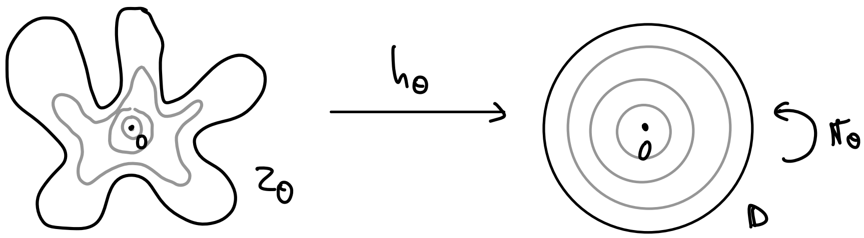
to build non-linearizable holomorphic map $g_\theta : (\mathbb{D}, 0) \to (\mathbb{C}, 0)$, $g_\theta'(0) = e^{2\pi i \theta}$.

g_θ has periodic pts of period $q_{(n)}$ accumulating at 0.



Step 2 Use quasiconformal tool (Douady-Hubbard) to convert g_θ to $f_\theta(z) = e^{2\pi i\theta} z + z^2$.

Let Z_θ = Siegel disk of $f_\theta(z) = e^{2\pi i\theta} z + z^2$ at 0.



Conjecture [Marmi-Moussa-Yoccoz]

Let $R(\theta) = |h_\theta'(0)|^{-1}$ = conformal radius of Z_θ.

Then, $Y(\theta) + \log R(\theta)$ is $\frac{1}{2}$-Hölder.

Conjecture If θ is Brjuno, Z_θ is a Jordan disk.

Some progress:

- Petersen-Zakeri '04: True for a.e. θ ($\log \bar{a}_n = O(\sqrt{n})$).
- Shishikura-F.Yang '24: True for high type ($\inf_n \bar{a}_n \geq N$).

It is known that $\partial Z_\theta \subset P_\theta = \overline{\{f_\theta^n(c_\theta)\}_{n\geq 0}}$, $c_\theta = -\frac{1}{2}e^{2\pi i\theta}$ is the critical pt.

Thm [wrl '26] For all θ, $P(f_\theta)$ = closure of critical orbit has measure 0. ☺