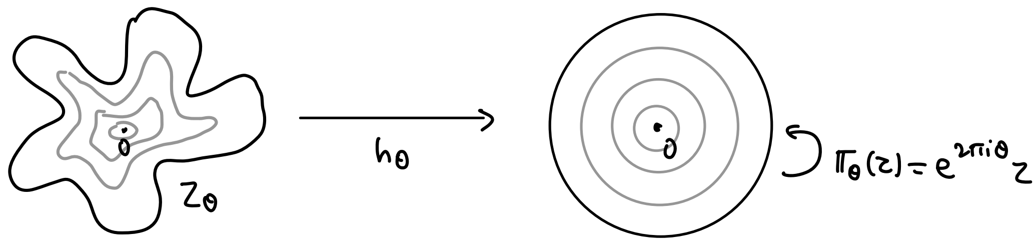
# Lecture 2

<u>Goal</u>: Describe the dynamics of $f_\theta(z) = e^{2\pi i\theta} z + z^2$ when $\theta$ is of bounded type.

Recall that since $\theta$ is Brjuno, $f_\theta$ admits a Siegel disk:

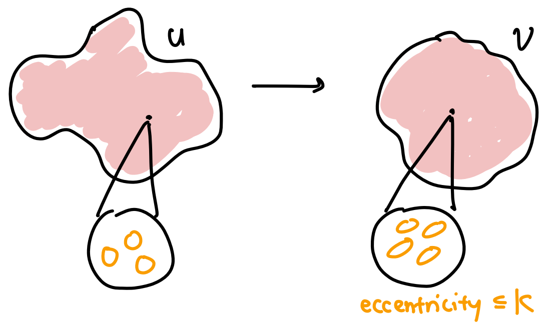



What's the regularity of $Z_\theta$? $h_\theta$? What happens outside of $Z_\theta$?

I'll give you a crash course on quasiconformal theory.

<u>Def</u> A map $h: U \to V$ between domains in $\mathbb{C}$ is <u>K-qc</u> for some $K \geq 1$ if

- $h$ is an orientation preserving homeo,
- $h$ is in $W^{1,2}(U)$,
- $|\bar{\partial} h| \leq \underbrace{\frac{K-1}{K+1}}_{<1} |\partial h|$.




<u>Facts:</u>

① 1-qc implies conformal.

② The space of K-qc maps $\mathbb{C} \to \mathbb{C}$ fixing 0, 1 is compact.

③ Every $\underbrace{\text{quasisymmetric}}_{\forall x \in \mathbb{R},\ \varepsilon > 0,\ |h(x+\varepsilon) - h(x)| \asymp |h(x) - h(x-\varepsilon)|}$ map $h: \mathbb{R} \to \mathbb{R}$ extends to a qc map $H: \mathbb{C} \to \mathbb{C}$.

<u>Def</u> A <u>quasicircle</u> is the image $h(\mathbb{T})$ of $\mathbb{T} \subset \mathbb{C}$ under a qc map $h: \mathbb{C} \to \mathbb{C}$.

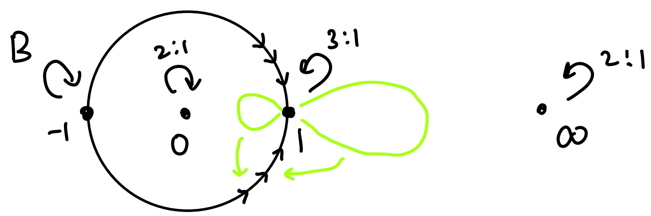
$\boxed{\text{E.g.}}$ ✓ $\mathbb{T}$, regular polygon, von Koch snowflake

⊗ Jordan curves with a cusp. 

From now on, we fix $\theta$ bdd type, i.e. $M = \sup_n \bar{a}_n < \infty$.

<u>Theorem</u> [Douady - Ghys - Herman - Swiatek '88]
The Siegel disk $Z_\theta$ of $f_\theta$ is a $K(M)$-quasidisk containing the critical point $c_\theta$ of $f_\theta$.

<u>proof</u>: Consider Blaschke product $B(z) = z^2 \frac{z-3}{1-3z}$.
$B$ has fixed critical pts at $0, 1, \infty$. $B$ is an orientation preserving homeo of $\mathbb{T}$.

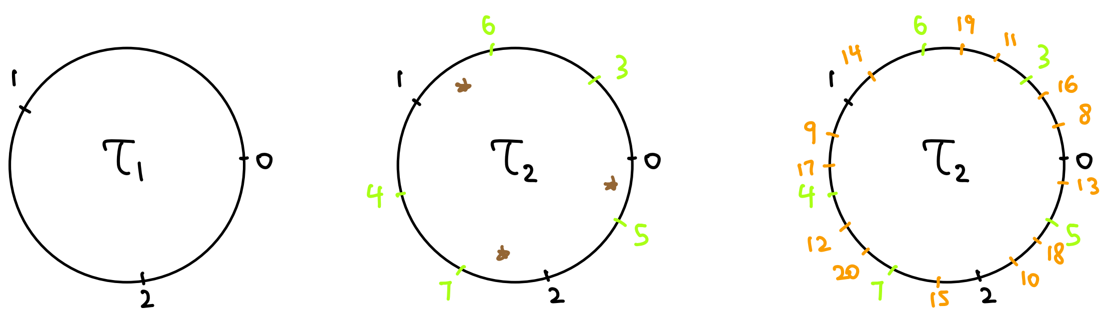



<u>General fact</u>: $\exists!\ \lambda_\theta \in \mathbb{T}$ such that $B_\theta(z) = \lambda_\theta z^2 \frac{z-3}{1-3z}$ has rotation number $\theta$ on $\mathbb{T}$.

<u>Real a priori bounds</u>
Let $\mathcal{T}_n$ = tiling of $\mathbb{T}$ with $\{B_\theta^k(1)\}_{0 \le k < q_{[n]}}$. Then,
(i) for any adjacent tiles $I, J \in \mathcal{T}_n$, $|I| \asymp |J|$.
(ii) if $I \in \mathcal{T}_n$, $J \in \mathcal{T}_{n+1}$, $J \subset I$, $I \setminus J$ is connected, then $|I| \asymp |J|$.

actually independent of $\theta$

The proof involves Denjoy cross ratio distortion estimates.
For $\theta = \theta_{gm}$ being the golden mean, the tiling looks like this:




$q_{[1]} = 3$ $\quad$ $q_{[2]} = 8$ $\quad$ $q_{[3]} = 21$

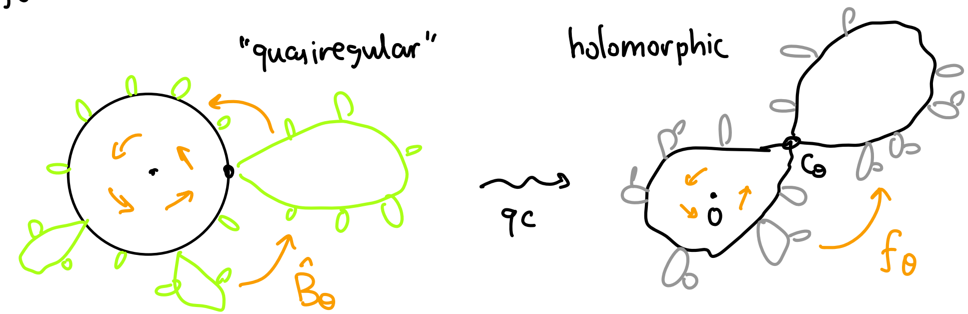
<u>Corollary</u> The conjugacy $h_\theta : \mathbb{T} \to \mathbb{T}$ from $B_\theta : \mathbb{T} \to \mathbb{T}$ to $r_\theta(z) = e^{2\pi i \theta} z$

is quasisymmetric.

$h_\theta$ extends to a qc map $\mathbb{D} \to \mathbb{D}$. Let's replace the inside with rotation, namely:

$$\hat{B}_\theta(z) = \begin{cases} h_\theta^{-1} \circ \Gamma_\theta \circ h_\theta & \text{on } \mathbb{D}, \\ B_\theta & \text{on } \hat{\mathbb{C}} \setminus \mathbb{D}, \end{cases}$$

a non-holomorphic deg. 2 branched covering of $\hat{\mathbb{C}}$. By using a powerful theorem (Ahlfors-Bers measurable Riemann mapping thm), it can be shown that $\hat{B}_\theta$ is indeed qc conjugate to a degree 2 polynomial which can be normalized to be $f_\theta(z) = e^{2\pi i \theta} z + z^2$. $\square$




There are many consequences, e.g. settles conjectures by physicists Manton & Nauenberg in 1983.

<u>Thm</u> [Graczyk, Jones '02] $1 < \dim \partial Z_\theta < 2$.

The important bit is the "$1<$". It uses $\beta$-numbers from geometric measure theory.

Recall: Julia set $J(f_\theta)$ = boundary of basin of attraction of $\infty$,
= closure of repelling periodic points,
= closure of iterated preimages of $\partial Z_\theta$.

<u>Thm</u> [Petersen '96] • $\mathrm{Leb}(J(f_\theta)) = 0$.
• $J(f_\theta)$ is locally connected.

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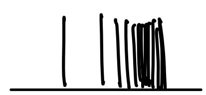


<u>Thm</u> [McMullen '98] $J(f_\theta)$ is porous and so $\dim_H J(f_\theta) < 2$.

rough proof of McM:

Step 1: Since $\partial Z_\theta$ is a quasicircle, then $\partial Z_\theta \subset \{\text{points of porosity of } J(f_\theta)\}$

Step 2: Every point in $J(f_\theta)$ can be "effectively" approximated by pre-critical points.

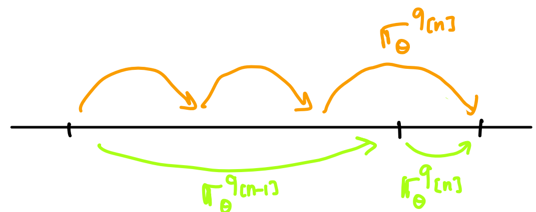
Now, I will restrict my attention to quadratic irrationals. To be concrete, I will pick the golden mean:

$$\theta = \theta_{gm} = \frac{3-\sqrt{5}}{2} \sim \langle (+,2), (+,2), (+,2), \ldots \rangle$$

Recall that the continuants $q_{[n]}$ satisfy

$$q_{[n+1]} = 3\, q_{[n]} - q_{[n-1]}.$$

The dynamics at all levels look like $\longrightarrow$

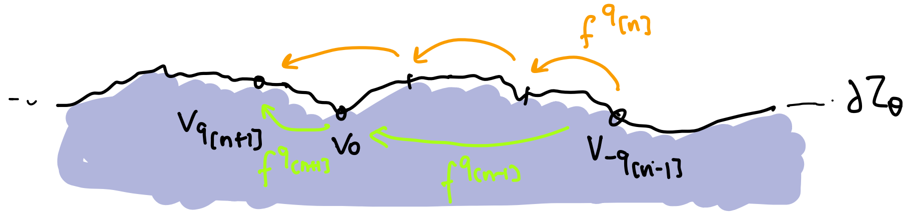



Let $f = f_\theta$,

$v_0 = -\dfrac{e^{4\pi i\theta}}{4}$, critical value of $f$.

$v_n = f_\theta^n(v_0) \in \partial Z_\theta$ for $n \in \mathbb{Z}$.

On $\partial Z_\theta$, combinatorially we have self-similarity. At every level $n$, we have:




<u>Thm</u> [McMullen '98]

$\partial Z_\theta$ and the semigroup $\{f_\theta^m\}_{m\geq 1}$ are asymptotically self-similar at $v_0$.

Let me make this more precise. Consider rescaling affine maps

$$u_n : \mathbb{C} \to \mathbb{C}, \quad u_n(v_0) = 0, \quad u_n(v_{-q_{[n]}}) = 1.$$

Then, as $n \to \infty$,

① $u_n(Z_\theta)$ converges to a quasi-half plane $\mathbf{Z} \subset \mathbb{C}$.

② $u_n f^{q_{[n+k]}} u_n^{-1}$ converges to an infinite degree branched covering map onto $\mathbb{C}$

$$F^{[k]} : D_k \to \mathbb{C}$$

uniformly on compact subsets of a dense simply connected $D_k \subsetneq \mathbb{C}$.

③ $F^{[k]}$, $k \in \mathbb{Z}$ generates an ordered abelian semigroup $\mathcal{F}$ with relations

$$\left(F^{[k]}\right)^3 = F^{[k-1]} \circ F^{[k+1]}.$$

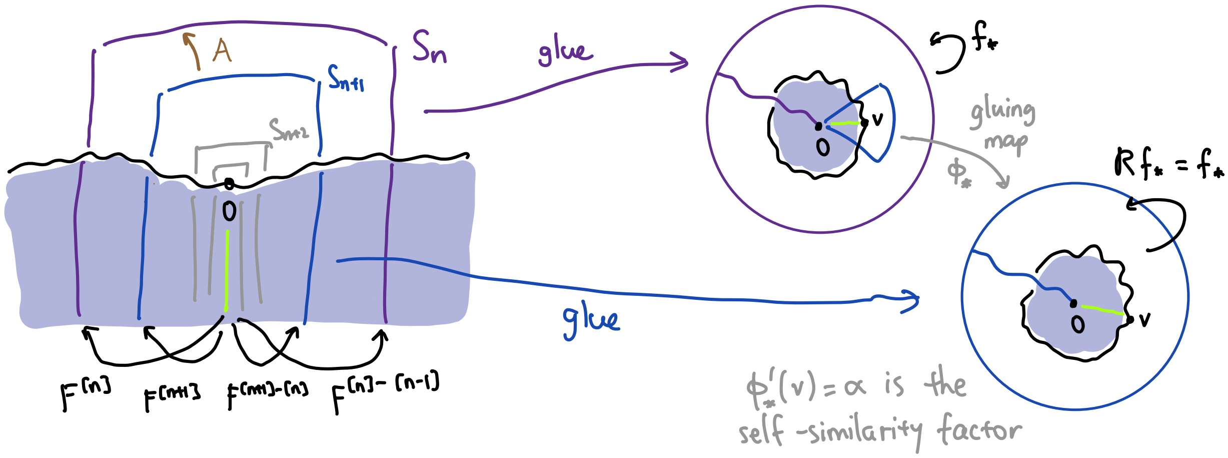
④ $\mathbf{Z}$ is invariant under $\mathcal{F}$. $\exists$ conformal map $h : \mathbf{Z} \to \mathbb{H}$ such that

$$h F^{[k]} h^{-1}(z) = z - \theta^k .$$

⑤ $\exists$ expanding linear map $A(z) = \alpha z$ such that

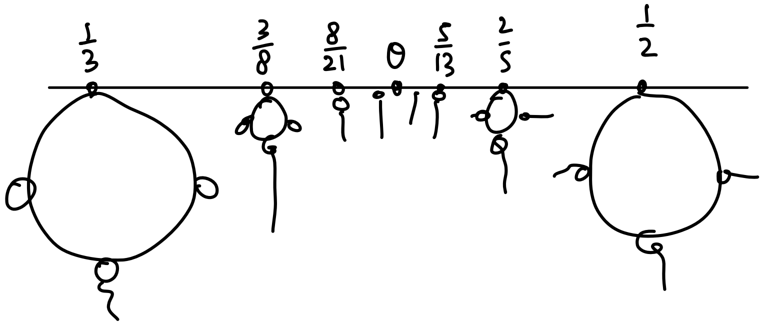
$$F^{[k+1]} = A^{-1} F^{[k]} A \quad \text{and} \quad L(\mathbf{Z}) = \mathbf{Z}.$$

This thm implies the existence of a renormalization fixed point $f_*$ :



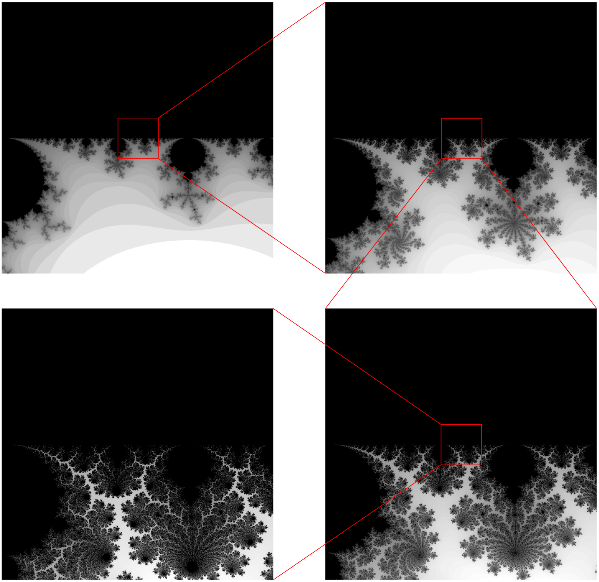

<u>Thm</u> [Dudko - Lyubich - Selinger '20]

The bifurcation locus of $f_\lambda(z) = e^{2\pi i \lambda} z + z^2$ is in some sense asymptotically self-similar at $\lambda = \theta$.

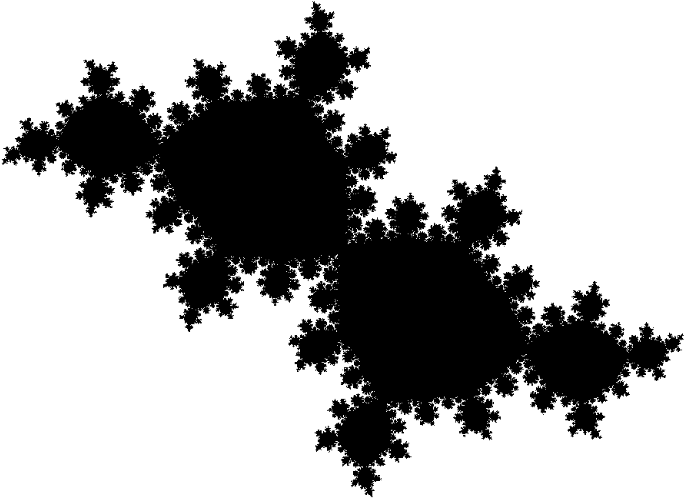



This is a deep result that comes from hyperbolicity of renormalization.

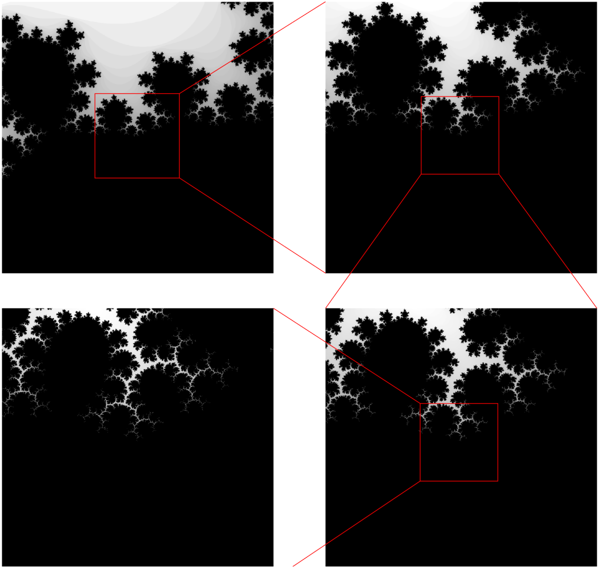
# Parameter self similarity for $e^{2\pi i\lambda}z + z^2$ at $\lambda = \frac{3-\sqrt{5}}{2}$



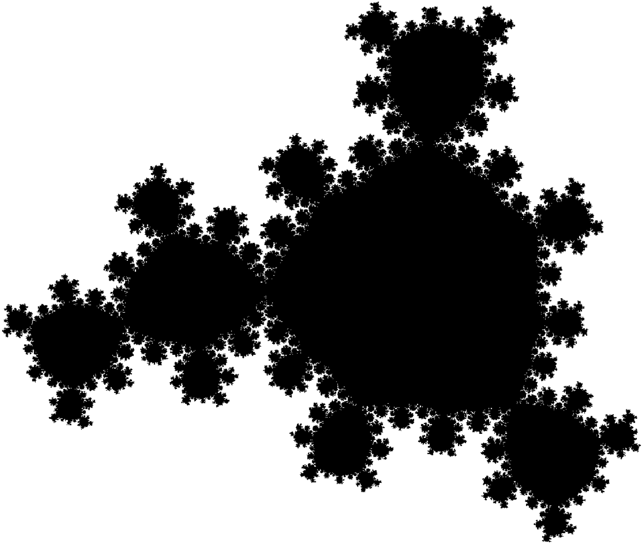
# Filled Julia set of $f_\theta(z) = e^{2\pi i\theta} z + z^2$ when $\theta = \frac{3-\sqrt{5}}{2}$



# Self-similarity at the critical value, $\alpha \approx 3.300..$



# Filled Julia set of $g_\theta(z) = e^{2\pi i\theta} z(1-z)^2$ when $\theta = \frac{3-\sqrt{5}}{2}$



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