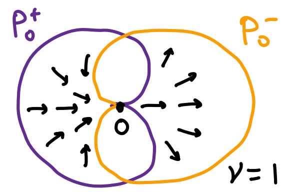
Lecture 3

Last time, we've explored θ = bounded type / badly approximable numbers.
To study what happens with all irrationals, we first need to study about the rational case $\theta = \frac{p}{q}$.

Assume $f(z) = e^{2\pi i \frac{p}{q}} z + O(z^2)$ is not of finite order. We have :

$$\hat{f}(z) = f^q(z) = z + c z^{\nu+1} + O(z^{\nu+2})$$

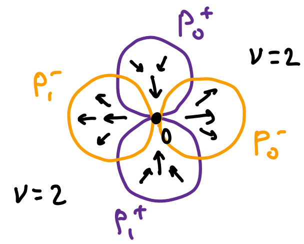
where $c \neq 0$ and $\nu \geq 1$.



Leau-Fatou Flower Theorem (1919)

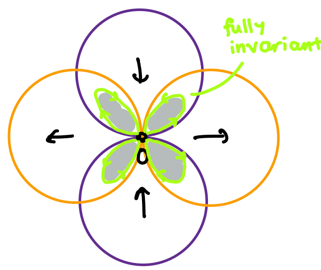
$\exists$ "petals" $P_0^+, P_1^+, \ldots P_{\nu-1}^+$ and $P_0^-, \ldots, P_{\nu-1}^-$ where

- each is a Jordan disk with 0 on the boundary,
- its union is a disk neighborhood of 0,
- they're arranged in this way $\Longrightarrow$
- On $P_j^{\pm}$, $\hat{f}^{\pm n}(z) \to 0$ as $n \to \infty$.



For f, there will be $2\nu q$ petals, each is rotated under f with rot number $\frac{p}{q}$.

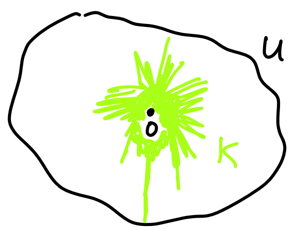
Observe that within the intersection, one can find fully invariant sub-flowers.



Theorem [Perez-Marco '97]

For any $\theta \in \mathbb{R}$, any germ $f(z) = e^{2\pi i \theta} z + O(z^2)$, and any closed disk U on which f & f^{-1} are univalent, there exists a set K (hedgehog) such that

① K is cpt, connected, full ($\mathbb{C} \setminus K$ is connected),
② $0 \in K \subset \bar{U}$ and $K \cap \partial U \neq \phi$,
③ $f : K \to K$ is a homeomorphism.



If f is not of finite order, the component of int K containing 0 is the Siegel disk of $f|_U$.

rough proof: If θ is rational, K = the fully invariant subflowers described above.
In general, take a Hausdorff limit $\theta_n \to \theta$, $n \to \infty$ where θ_n are rationals. □

This is special for analytic maps! It doesn't hold for smooth maps in general.

Open Problem: Does there exist hedgehogs for holomorphic germs of $(\mathbb{C}^n, 0)$ admitting a neutral eigenvalue?

Let's go back to neutral quadratic polynomials.

Theorem [Dudko - Lyubich '22]
For every irrational θ,
① $f_\theta(z) = e^{2\pi i\theta} z + z^2$ admits a unique Mother Hedgehog H_θ, i.e. a maximal hedgehog for f_θ.
② ∂H_θ is the postcritical set $P(f_\theta)$, the closure of critical orbit.
int H_θ is the Siegel disk Z_θ which is non-empty iff θ is Brjuno.

I'll give an idea behind this. When θ is of bounded type, we know that H_θ = closure of Siegel disk Z_θ since Z_θ is a Jordan curve containing the critical point. What happens in the unbounded case?

Pick θ in $\Theta = (-\frac{1}{2}, \frac{1}{2}) \setminus \mathbb{Q}$. Consider its symbolic representation

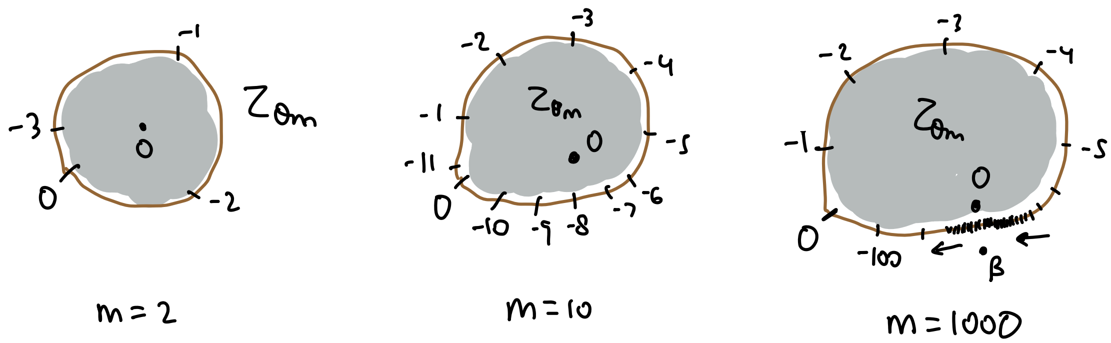
$$\langle (\varepsilon_n, \bar{a}_n) \rangle_{n \geq 1}, \quad \varepsilon_n \in \{-1, +1\}, \ \bar{a}_n \geq 2.$$

Fixed points of f_θ are

$$f_\theta(z) = z \longrightarrow \alpha = 0, \ \beta = 1 - e^{2\pi i\theta}.$$

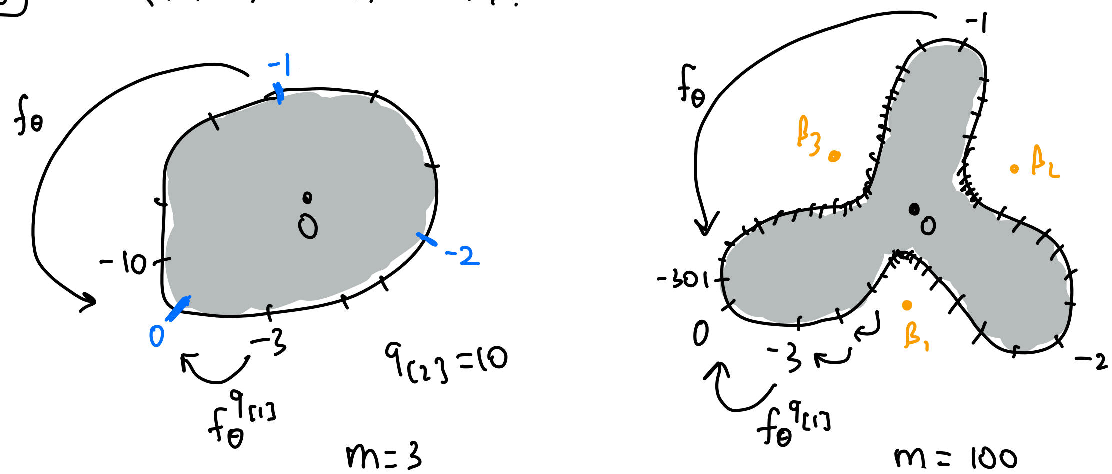
As $\theta \to 0$ (i.e. $\bar{a}_1 \to \infty$), then $\beta \simeq 2\pi i\theta \to 0$ and the two fixed points collide!

E.g. $\theta_m \sim \langle (+, m), \overline{(+, 2)} \rangle$:



What if $\bar{a}_2 \to \infty$ instead?

E.g. $\theta_m \sim \langle (+,3), (-,m), \overline{(+,2)} \rangle$.



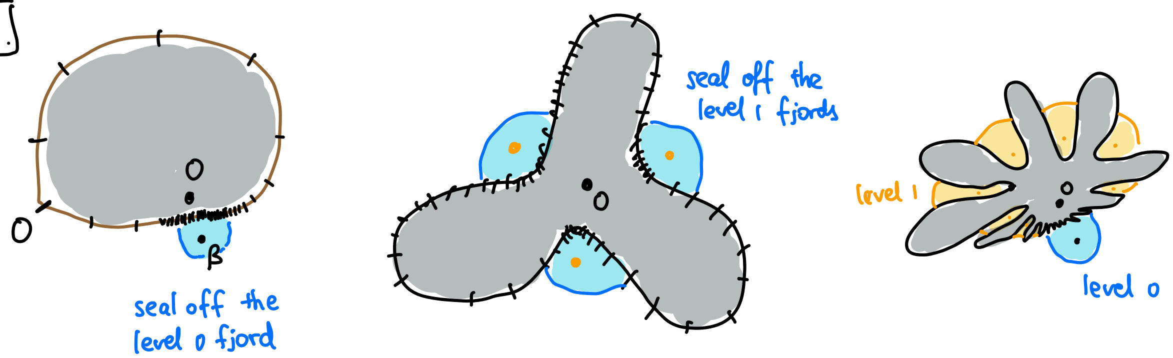
As $m \to \infty$, there is a cycle of period-3 points $\beta_1, \beta_2, \beta_3$ converging to 0. Geometrically, we see that ∂Z_θ has 3 inward fjords.

The solution to [DL'22] is a regularization procedure:

- Pick a threshold $\mathbb{M}$ for which $\bar{a}_{n+1} \geq \mathbb{M}$ is considered "NP" (near parabolic)
- When n is NP, seal a fjord with a hyperbolic geodesic of $\mathbb{C} \setminus Z_\theta$.

The result is called a <u>pseudo-Siegel disk</u> $\hat{Z}_\theta$. Unlike Z_θ, $\hat{Z}_\theta$ is only almost invariant. [DL'22] proved that $\hat{Z}_\theta$ is always a K-quasidisk. This is a "pre-compactness" result for bounded type θ. For any irrational θ, take bounded approximations $\theta_n \to \theta$. The limit $\hat{Z}_\theta = \lim_{n\to\infty} \hat{Z}_{\theta_n}$ is again a K-quasidisk!

E.g.



$\langle (+,10^6), \overline{(+,2)} \rangle$ $\langle (+,3), (-,10^3), \overline{(+,2)} \rangle$ $\langle (+,10^5), (+,10^7), \overline{(+,2)} \rangle$

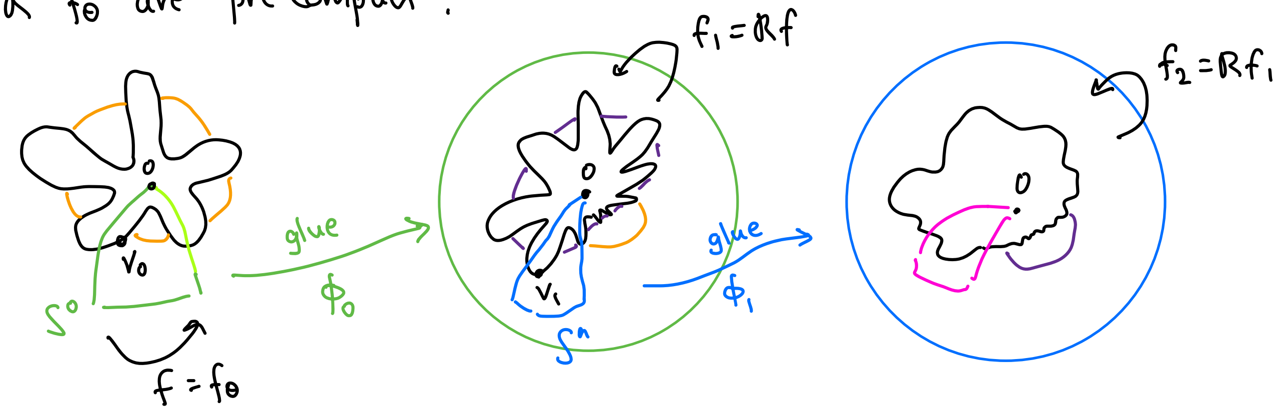
Key properties:

* $f_\theta : \hat{Z}_\theta \to \mathbb{C}$ is injective, $\hat{Z}_\theta$ is almost invariant.
* $P(f_\theta)$ is contained in $\hat{Z}_\theta$
* Mother Hedgehog H_θ = non-escaping set of $f_\theta : \hat{Z}_\theta \to \mathbb{C}$.

These pseudo-Siegel disks also allow for uniform control of renormalizations.

Thm [Dudko-Lyubich '26]

For all irrational θ, sector renormalizations of f_θ can be defined such that $\mathcal{R}^n f_\theta$ are "pre-compact".



The precompactness means the following. Sectors S_n can be constructed such that:

* the gluing map Φ_n projects 1st return map of f_n back to S_n to a new map $f_{n+1} = \mathcal{R}f_n$,

* the pseudo-Siegel disk $\hat{Z}_n$ of f_n projects to a "pseudo-Siegel disk" $\hat{Z}_n$ of f_{n+1}, etc...
* $\mathbb{D}(0,R) \subset \hat{Z}_n \subset \mathbb{D}(0,R')$ for some uniform $0<R<R'<1$.
* the non-escaping set of $f_n : \hat{Z}_n \to \mathbb{C}$ is the Mother Hedgehog H_n.

One, not so obvious, consequence is:

<u>Thm</u> [Lim'26] The postcritical set P_θ of f_θ has 0 area. ☺

As a corollary, a.e. point in $J(f_\theta)$ is non-recurrent.
The proof involves Yoccoz-style estimates on the gluing maps ϕ_n.
With more work, it should be possible to prove this conjecture:

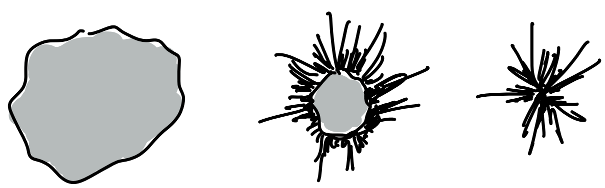
<u>Trichotomy Conjecture</u> [Cheraghi]

(a) If θ is a Herman irrational, P_θ is a Jordan curve.

(b) If θ is Brjuno but not Herman, P_θ is a 1-sided hairy Jordan curve.

(c) If θ is a non-Brjuno irrational, P_θ is a Cantor bouquet.

A Herman irrational θ is one where every analytic circle diffeomorphism is analytically conjugate to rigid rotation.



(a) (b) (c)

Another consequence of sectorial precompactness is convergence to an attractor:

$$\mathcal{R}^n f_\theta \xrightarrow{n\to\infty} \mathcal{A}.$$

The attractor $\mathcal{A}$ has elements that can be written as a bi-infinite renormalization tower $\underline{f} = \{ \langle f_n \rangle_{n\in\mathbb{Z}},\ \mathcal{R} f_n = f_{n+1} \}$.

⚠ Some f_n's may have rational rotation numbers!!

To describe the combinatorics of elements of $\mathcal{A}$, we consider

$$\underleftarrow{\bar{\Theta}} = \left(\{-,+\} \times \{2,3,4,\ldots,\infty\} \right)^{\mathbb{Z}}.$$

$\underline{f} = \langle f_n \rangle_{n\in\mathbb{Z}}$ has combinatorics $\underline{\theta} = \langle \varepsilon_n, \bar{a}_n \rangle_{n\in\mathbb{Z}} \in \underleftarrow{\bar{\Theta}}$ if

- ε_{n+1} = orientation of f_n,
- $\bar{a}_{n+1}$ = 1st return time of f_n back to S_n (∞ if $f_n'(0) = 1$).

<u>Thm</u> [Dudko - Lim - Lyubich '26]

The attractor $\mathcal{A}$ is combinatorially rigid.

(i) If $\underline{f}$ and $\underline{g}$ have the same combinatorics, then for all $n \in \mathbb{Z}$, f_n is conformally conjugate to g_n.

(ii) Modulo conf. conjugacy, $\mathcal{R} : \mathcal{A}/\!\sim \,\circlearrowleft$ is conjugate to the shift map on $\underleftarrow{\bar{\Theta}}$.

This theorem is a complete extension of McMullen's renormalization fixed points + dynamical universality.

<u>Problem</u>: Uniform hyperbolicity of renormalization $\mathcal{R} : \mathcal{A}/\!\sim \,\circlearrowleft$?