From Herman rings to Herman curves

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Rotation domains

Let $f \in \text{Rat}_d$ be a degree $d \geq 2$ rational map. A maximal invariant domain $U \subset \hat{\mathbb{C}}$ is a rotation domain if $f|_U$ is conjugate to a rigid rotation. There are 2 types:

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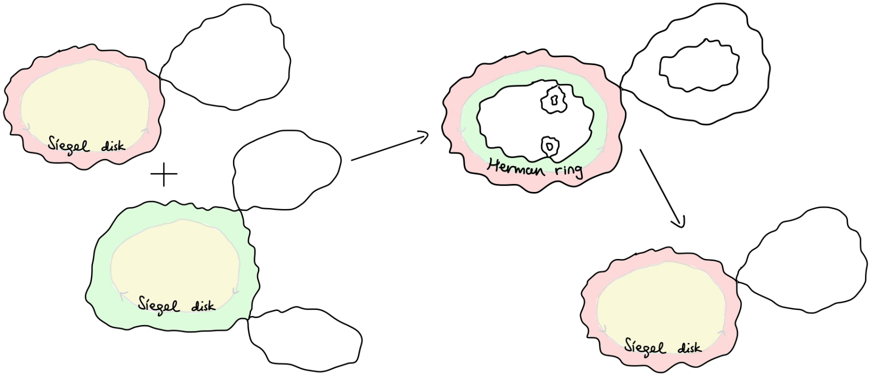
2. U is an annulus, i.e. a Herman ring.

The two can be converted into one another via quasiconformal surgery. (Shishikura '87)



Bounded type rotation domains

Assume from now on that $\theta \in (0,1)$ is an irrational number of bounded type, i.e. there is some $B \in \mathbb{N}$ such that $\sup_n a_n \leq B$ where $\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$.

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Applying Shishikura's surgery, we have:

Corollary

Every boundary component of an invariant Herman ring of $f \in Rat_d$ with rotation number θ and modulus μ is a $K(d,B,\mu)$-quasicircle containing a critical point.

Characterization of $\mathcal{H}_{d_0,d_\infty,\theta}$

Proposition

$\mathcal{H}_{d_0,d_\infty,\theta}$ consists of all rational maps that can be obtained from Shishikura surgery out of a pair of maps P_0, P_∞ such that for $\star \in \{0, \infty\}$,

- *$P_\star$ is a degree $d_\star$ polynomial;*
- *$P_\star$ has a Siegel disk $Z_\star$;*
- *$rot(Z_0) = \theta$ and $rot(Z_\infty) = -\theta$;*
- *all free critical points of $P_\star$ lie in $\partial Z_\star$.*

Rotation curves

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Proposition (Trichotomy)

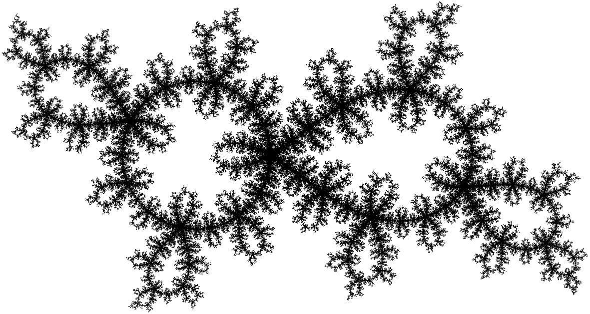
When the rotation number of $f|_X$ is of bounded type, there are 3 possibilities:

a. *X is an analytic curve contained in a rotation domain,*
b. *X is the boundary of a rotation domain containing a critical point of f,*
h. *X is a Herman curve containing a critical point of f.*

A trivial Herman curve

For any irrational θ, there is a unique $\zeta_\theta \in \mathbb{T}$ such that the unit circle is a Herman curve of rotation number θ for the map

$$f_\theta(z) = \zeta_\theta z^2 \frac{z-3}{1-3z}.$$



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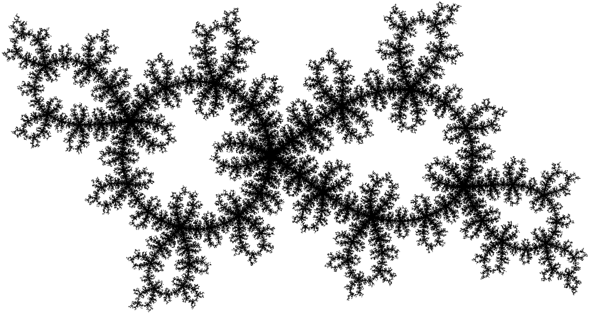

Question: Can non-trivial Herman curves exist?

Realization

Step 1: Use *a priori bounds*.

For any $\varepsilon > 0$, the family

$$\{f \in \mathcal{H}_{d_0,d_\infty,\theta} \ : \ \mathrm{mod}(\mathbb{H}) < \varepsilon\}/_{\underset{\text{conf}}{\sim}}$$

is precompact inside $\mathrm{Rat}_{d_0+d_\infty-1}/_{\underset{\text{conf}}{\sim}}$.

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Step 2: Use a Thurston-type result for Herman rings (Wang '12).

There is some $f_1 \in \mathcal{H}_{d_0,d_\infty,\theta}$ having a Herman ring with combinatorics similar to the chosen one.

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Result: Limit f_0 of f_t exists and $f_0 \in \mathcal{H}^{\partial}_{d_0,d_\infty,\theta}$ has the same combinatorics as f_1.

Rigidity of $\mathcal{X}_{d_0,d_\infty,\theta}$

Step 1: Apply pullback argument.

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Step 2: Small Julia sets everywhere.

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$$f^k : \mathbb{D}(y, r) \to \mathbb{C}$$

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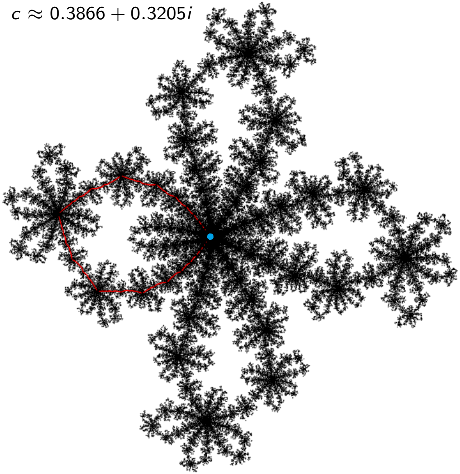
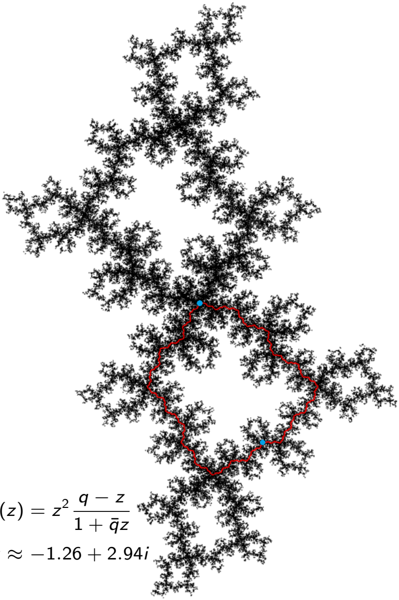
Result: Conformal conjugacy!

Non-trivial examples of golden mean Herman curves

$$f(z) = cz^2 \, \frac{z^3 - 5z^2 + 10z - 10}{5z - 1}$$

$c \approx 0.3866 + 0.3205i$





$$f(z) = z^2 \frac{q - z}{1 + \bar{q}z}$$

$q \approx -1.26 + 2.94i$

Let H be a boundary component of the Herman ring of $f \in \mathcal{H}_{d_0,d_\infty,\theta}$.
Endow H with the combinatorial metric.

How to prove *a priori bounds*?

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$I =$ an interval in H of (combinatorial) length $|I| < 0.1$.
$10I =$ the interval of length $10|I|$ having the same midpoint as I.
$W_{10}(I) =$ the extremal width of curves connecting I and $H \setminus 10I$.

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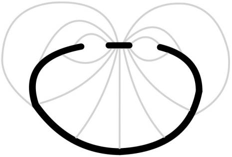

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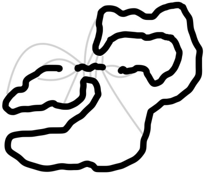

large $W_{10}(I)$

$W_{10}(I)$ encodes the local (near-)degeneration of H near the interval I.

Near-Degenerate Regime

To prove *a priori bounds*, it is sufficient to find some ε and $\mathbf{K} > 1$ depending only on d_0, d_∞, B such that:

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Our goal is reduced to showing:

Theorem (Amplification)

If

there is an interval $I \subset H$ with length $|I| \ll 1$ and width $W_{10}(I) = K \gg 1$,

then

there is another interval $J \subset H$ with length $|J| \ll 1$ and width $W_{10}(J) \geq 2K$.

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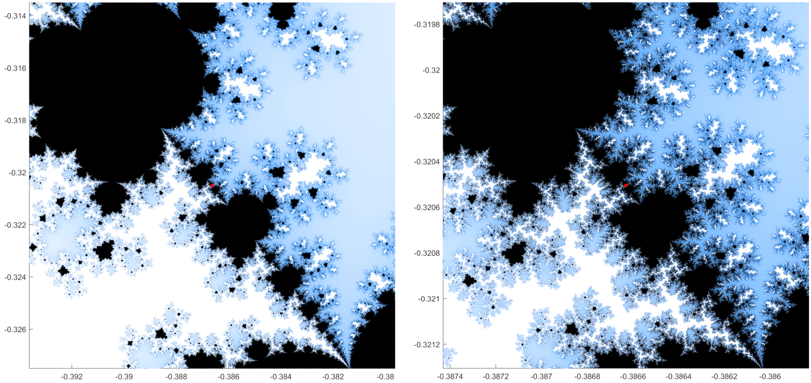
The proof relies on the near-degenerate machinery, including ideas from:
Kahn-Lyubich '05, Kahn '06, and Dudko-Lyubich '22.

Let F_c be the rational map with critical points $0, \infty, 1$ of local degrees d_0 ,d_∞, $d_0 + d_\infty - 1$, satisfying $f(0) = 0, f(\infty) = \infty, f(1) = c$.

Conjecture: Bifurcation locus of $\{F_c\}$ is self-similar at the unique parameter $c_\star$ where $F_{c_\star}$ has a golden mean Herman curve.



Bifurcation locus for $d_0 = 2, d_\infty = 4$ magnified around $c_\star$ at different scales.

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3. Is every Herman curve a limit of degenerating Herman rings?
4. For $f \in \mathcal{X}_{d_0,d_\infty,\theta}$, is it true that $\mathrm{Leb}\, J(f) = 0$? $\dim_{\mathrm{H}} J(f) < 2$?

Thank you!