# From Herman rings to Herman curves

Willie Rush Lim

Stony Brook University

Quasiworld Seminar
November 1st 2023

# Complex dynamics

Complex dynamics = study of holomorphic self maps $f$ of a complex manifold $M$

$$x \mapsto f(x) \mapsto f^2(x) \mapsto f^3(x) \mapsto \dots$$

Complex dynamics $=$ study of holomorphic self maps $f$ of a complex manifold $M$

$$x \mapsto f(x) \mapsto f^2(x) \mapsto f^3(x) \mapsto \dots$$

Dichotomy:

$$\text{Fatou set } F(f) = \text{ set of points } z \in M \text{ near which } \{f^n\}_{n \geq 0} \text{ is normal,}$$
$$\text{Julia set } J(f) = M \backslash F(f).$$

Complex dynamics = study of holomorphic self maps $f$ of a complex manifold $M$

$$x \mapsto f(x) \mapsto f^2(x) \mapsto f^3(x) \mapsto \dots$$

Dichotomy:

$$\text{Fatou set } F(f) = \text{ set of points } z \in M \text{ near which } \{f^n\}_{n\geq 0} \text{ is normal,}$$
$$\text{Julia set } J(f) = M \setminus F(f).$$

In this talk, we take $M = \hat{\mathbb{C}}$ and $f \in \mathrm{Rat}_d$ is a degree $d \geq 2$ rational map.

# Rotation domains

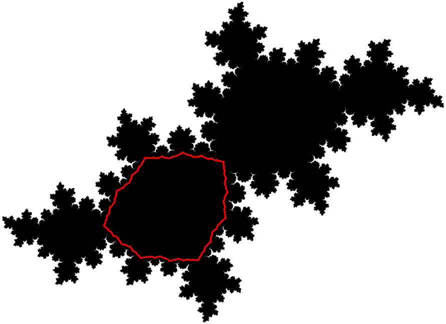
A maximal invariant domain $U \subset \hat{\mathbb{C}}$ of $f$ is called a **rotation domain** if $f|_U$ is conjugate to a rigid rotation. There are 2 types:

1. $U$ is simply connected, i.e. a **Siegel disk**;
2. $U$ is an annulus, i.e. a **Herman ring**.

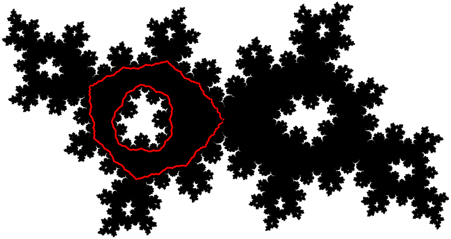
# Rotation domains

A maximal invariant domain $U \subset \hat{\mathbb{C}}$ of $f$ is called a **rotation domain** if $f|_U$ is conjugate to a rigid rotation. There are 2 types:

1. $U$ is simply connected, i.e. a **Siegel disk**;
2. $U$ is an annulus, i.e. a **Herman ring**.



$f(z) = z^2 + c$ where $c \approx -0.3905 - 0.5868i$



$f(z) = e^{2\pi i t} z^2 \dfrac{z-4}{1-4z}$ where $t \approx 0.61517$

# Bounded type assumption

Fix an irrational $\theta \in (0,1)$ and assume it is of **bounded type**, i.e. there is some $B \in \mathbb{N}$ such that $\sup_n a_n \leq B$ where

$$\theta = [0; a_1, a_2, a_3, \ldots] = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}.$$

E.g. golden mean $\frac{\sqrt{5}-1}{2} = [0; 1, 1, 1, \ldots]$

# Bounded type assumption

Fix an irrational $\theta \in (0,1)$ and assume it is of **bounded type**,
i.e. there is some $B \in \mathbb{N}$ such that $\sup_n a_n \leq B$ where

$$\theta = [0; a_1, a_2, a_3, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}.$$

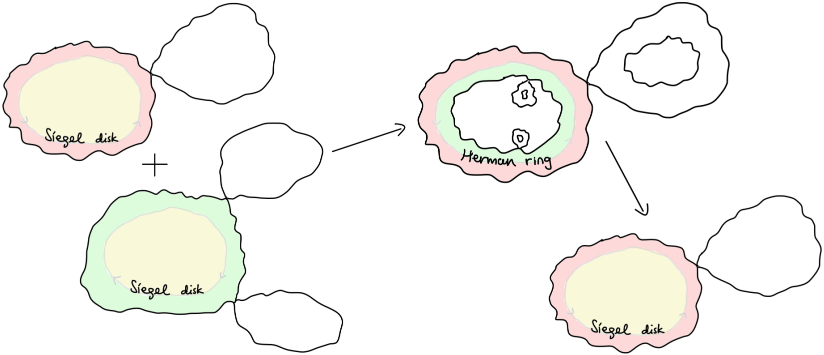
E.g. golden mean $\frac{\sqrt{5}-1}{2} = [0; 1, 1, 1, \ldots]$

**Theorem (Zhang '11)**

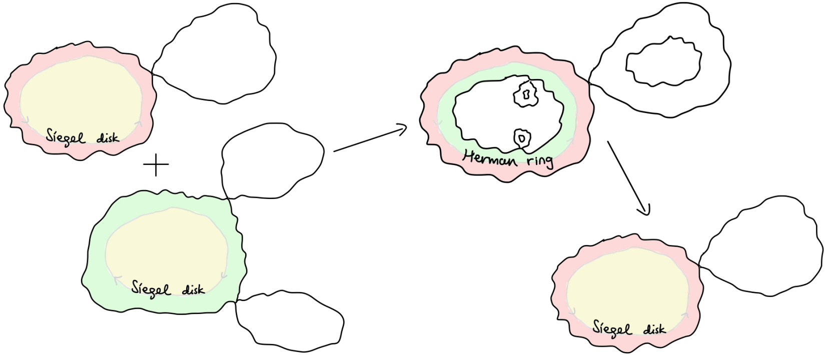
*Every invariant Siegel disk of a map $f \in \mathrm{Rat}_d$ with rotation number $\theta$ is a $K(d, B)$-quasidisk containing a critical point on the boundary.*

# Shishikura's surgery

Siegel disks can can be converted into Herman rings, and vice versa, via QC surgery.



Siegel disks can can be converted into Herman rings, and vice versa, via QC surgery.




**Corollary**

*The boundary components of an invariant Herman ring of $f \in \mathrm{Rat}_d$ with rotation number $\theta$ and modulus $\boldsymbol{\mu}$ are $K(d, B, \boldsymbol{\mu})$-quasicircle containing a critical point.*

# A nice class of Herman rings

Fix integers $d_0, d_\infty \geq 2$.

Let $\mathcal{H}$ = space of degree $d_0 + d_\infty - 1$ rational maps $f$ such that

1. $f$ has critical fixed points at 0 and $\infty$ of local degree $d_0$ and $d_\infty$,
2. $f$ has a Herman ring $\mathbb{H}$ of rotation number $\theta$,
3. $\mathbb{H}$ separates 0 and $\infty$,
4. all other critical points lie on $\partial\mathbb{H}$.

# A nice class of Herman rings

Fix integers $d_0, d_\infty \geq 2$.

Let $\mathcal{H}$ = space of degree $d_0 + d_\infty - 1$ rational maps $f$ such that

1. $f$ has critical fixed points at 0 and $\infty$ of local degree $d_0$ and $d_\infty$,
2. $f$ has a Herman ring $\mathbb{H}$ of rotation number $\theta$,
3. $\mathbb{H}$ separates 0 and $\infty$,
4. all other critical points lie on $\partial\mathbb{H}$.

**Theorem** (*A priori bounds*, L'23)

*The boundary components of the Herman ring of $f \in \mathcal{H}$ are $K(d_0, d_\infty, B)$-quasicircles. In particular, dilatation is independent of* $\mathrm{mod}(\mathbb{H})$.

# How to prove *a priori bounds*?

Let $H$ be a boundary component of $\mathbb{H}$.
Endow $H$ with the combinatorial metric, i.e. the unique normalized $f$-invariant metric.

# How to prove *a priori bounds*?

Let $H$ be a boundary component of $\mathbb{H}$.
Endow $H$ with the combinatorial metric, i.e. the unique normalized $f$-invariant metric.

$I$ = an interval in $H$ of (combinatorial) length $|I| < 0.1$.
$10I$ = the interval of length $10|I|$ having the same midpoint as $I$.

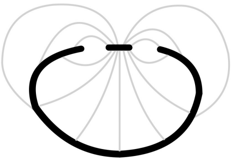
$W_{10}(I)$ = the extremal width of curves connecting $I$ and $H \backslash 10I$.

# How to prove *a priori bounds*?

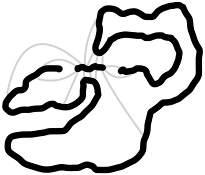
Let $H$ be a boundary component of $\mathbb{H}$.
Endow $H$ with the combinatorial metric, i.e. the unique normalized $f$-invariant metric.

$I =$ an interval in $H$ of (combinatorial) length $|I| < 0.1$.
$10I =$ the interval of length $10|I|$ having the same midpoint as $I$.

$W_{10}(I) =$ the extremal width of curves connecting $I$ and $H \setminus 10I$.



small $W_{10}(I)$



large $W_{10}(I)$

$W_{10}(I)$ encodes the local (near-)degeneration of $H$ near the interval $I$.

# Near-degenerate regime

It is sufficient to find constants $\varepsilon$ and $\mathbf{K} > 1$ depending only on $d_0, d_\infty, B$ such that:

$$\text{every interval } I \subset H \text{ of length } |I| < \varepsilon \text{ satisfies } W_{10}(I) < \mathbf{K}.$$

# Near-degenerate regime

It is sufficient to find constants $\varepsilon$ and $\mathbf{K} > 1$ depending only on $d_0, d_\infty, B$ such that:

every interval $I \subset H$ of length $|I| < \varepsilon$ satisfies $W_{10}(I) < \mathbf{K}$.

Our goal is reduced to showing:

**Theorem (Amplification)**

*If*

*there is an interval $I \subset H$ with length $|I| \ll 1$ and width $W_{10}(I) = \mathbf{K} \gg 1$,*

*then*

*there is another interval $J \subset H$ with length $|J| \ll 1$ and width $W_{10}(J) \geq \mathbf{2K}$.*

*(All bounds depend only on $d_0, d_\infty, B$.)*

# Near-degenerate regime

It is sufficient to find constants $\varepsilon$ and $\mathbf{K} > 1$ depending only on $d_0, d_\infty, B$ such that:

$$\text{every interval } I \subset H \text{ of length } |I| < \varepsilon \text{ satisfies } W_{10}(I) < \mathbf{K}.$$

Our goal is reduced to showing:

**Theorem (Amplification)**

*If*

*there is an interval $I \subset H$ with length $|I| \ll 1$ and width $W_{10}(I) = \mathbf{K} \gg 1$,*

*then*

*there is another interval $J \subset H$ with length $|J| \ll 1$ and width $W_{10}(J) \geq \mathbf{2K}$.*

*(All bounds depend only on $d_0, d_\infty, B$.)*

The proof relies on the analysis of near-degenerate surfaces via quasi-additivity law, covering lemma, canonical arc diagrams, including ideas from Kahn-Lyubich '05, Kahn '06, and D.Dudko-Lyubich '22.

# Rotation curves

An invariant curve $X \subset \hat{\mathbb{C}}$ of a holomorphic map $f$ is a **rotation curve** if $f|_X$ is conjugate to irrational rotation.

If $X$ is not contained in the closure of a rotation domain, we call it a **Herman curve**.

# Rotation curves

An invariant curve $X \subset \hat{\mathbb{C}}$ of a holomorphic map $f$ is a **rotation curve** if $f|_X$ is conjugate to irrational rotation.

If $X$ is not contained in the closure of a rotation domain, we call it a **Herman curve**.

**Proposition (Trichotomy)**

*When rot($f|_X$) is of bounded type, there are 3 possibilities:*

a. *$X$ is an analytic curve contained in a rotation domain,*

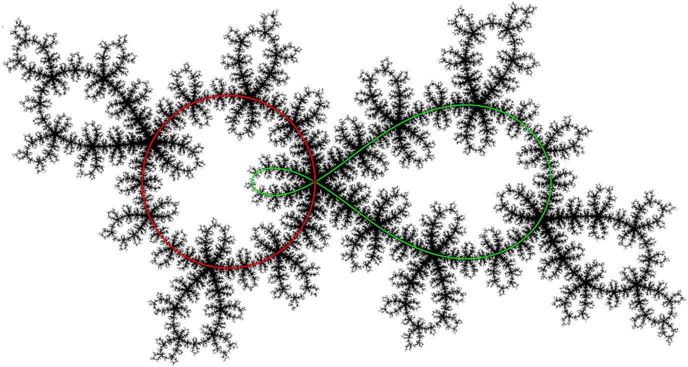
b. *$X$ is the boundary of a rotation domain containing a critical point of $f$,*

h. *$X$ is a Herman curve containing inner and outer critical points of $f$.*

# Example #0: trivial Herman curve

For any irrational $\theta$, there is a unique $\zeta_\theta \in \mathbb{T}$ such that the unit circle is a Herman curve of rotation number $\theta$ for the map

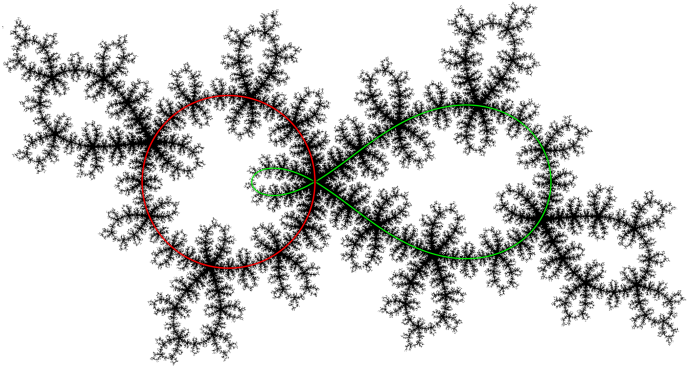
$$f_\theta(z) = \zeta_\theta z^2 \frac{z-3}{1-3z}.$$



# Example #0: trivial Herman curve

For any irrational $\theta$, there is a unique $\zeta_\theta \in \mathbb{T}$ such that the unit circle is a Herman curve of rotation number $\theta$ for the map

$$f_\theta(z) = \zeta_\theta z^2 \frac{z-3}{1-3z}.$$



Question by Eremenko: Can non-trivial Herman curves exist?

# Realizing arbitrary combinatorics

The **combinatorics** of a Herman curve $\mathbf{H}$ refers to the relative combinatorial position and the criticalities of critical points on $\mathbf{H}$.

# Realizing arbitrary combinatorics

The **combinatorics** of a Herman curve **H** refers to the relative combinatorial position and the criticalities of critical points on **H**.

**Theorem (Realization + Rigidity)**

*For bounded type $\theta$ and any chosen combinatorial data,*

- *there exists $f \in \partial\mathcal{H}$ admitting a Herman quasicircle that has a rotation number $\theta$ and the prescribed combinatorics;*
- *$f$ is unique up to conformal conjugacy.*

# Realization

Step 1: Apply *a priori bounds*.

For $\varepsilon > 0$,

$$\{f \in \mathcal{H} \ : \ \text{mod}(\mathbb{H}) < \varepsilon\}/_{\underset{\text{conf}}{\sim}}$$

is precompact inside of $\text{Rat}_{d_0+d_\infty-1}/_{\underset{\text{conf}}{\sim}}$.

# Realization

Step 1: Apply *a priori bounds*.

For $\varepsilon > 0$,

$$\{f \in \mathcal{H} \ : \ \text{mod}(\mathbb{H}) < \varepsilon\}/_{\underset{\text{conf}}{\sim}}$$

is precompact inside of $\text{Rat}_{d_0+d_\infty-1}/_{\underset{\text{conf}}{\sim}}$.

Step 2: Use a Thurston-type result for Herman rings (Wang '12).

There exists $f_1 \in \mathcal{H}$ whose Herman ring has combinatorics similar to the chosen one.

# Realization

Step 1: Apply *a priori bounds*.

For $\varepsilon > 0$,

$$\{f \in \mathcal{H} \,:\, \mathrm{mod}(\mathbb{H}) < \varepsilon\}/_{\underset{\mathrm{conf}}{\sim}}$$

is precompact inside of $\mathrm{Rat}_{d_0+d_\infty-1}/_{\underset{\mathrm{conf}}{\sim}}$.

Step 2: Use a Thurston-type result for Herman rings (Wang '12).

There exists $f_1 \in \mathcal{H}$ whose Herman ring has combinatorics similar to the chosen one.

Step 3: Apply QC deformation.

There is a normalized family of maps $\{f_t\}_{0<t\leq 1}$ in $\mathcal{H}$ of the same combinatorics, with modulus $\to 0$ as $t \to 0$.

# Realization

Step 1: Apply *a priori bounds*.

For $\varepsilon > 0$,

$$\{f \in \mathcal{H} \,:\, \text{mod}(\mathbb{H}) < \varepsilon\}/_{\underset{\text{conf}}{\sim}}$$

is precompact inside of $\text{Rat}_{d_0+d_\infty-1}/_{\underset{\text{conf}}{\sim}}$.

Step 2: Use a Thurston-type result for Herman rings (Wang '12).

There exists $f_1 \in \mathcal{H}$ whose Herman ring has combinatorics similar to the chosen one.

Step 3: Apply QC deformation.

There is a normalized family of maps $\{f_t\}_{0<t\leq 1}$ in $\mathcal{H}$ of the same combinatorics, with modulus $\to 0$ as $t \to 0$.

Result: $f_t \to f_0 \in \partial\mathcal{H}$
$f_0$ has a Herman quasicircle with the same combinatorics as $f_1$.

# Rigidity

An **invariant line field** of $f$ is a measurable Beltrami differential $\mu = \mu(z)\frac{d\bar{z}}{dz}$ on $\hat{\mathbb{C}}$ where

- $f^*\mu = \mu$ a.e.,
- $\text{supp}(\mu) =$ positive area subset of $J(f)$,
- $|\mu(z)| = 1$ on $\text{supp}(\mu)$.

# Rigidity

An **invariant line field** of $f$ is a measurable Beltrami differential $\mu = \mu(z)\frac{d\bar{z}}{dz}$ on $\hat{\mathbb{C}}$ where

- $f^*\mu = \mu$ a.e.,
- $\text{supp}(\mu) =$ positive area subset of $J(f)$,
- $|\mu(z)| = 1$ on $\text{supp}(\mu)$.

**Theorem (NILF, L'23)**

*Suppose $f$ is a rational map that is* **J-rotational**, *i.e. every critical point in $J(f)$ either*

- *has finite orbit, or*
- *is eventually mapped to a bounded type rotation quasicircle.*

*Then, $J(f)$ supports no invariant line field of $f$.*

# Rigidity

An **invariant line field** of $f$ is a measurable Beltrami differential $\mu = \mu(z)\frac{d\bar{z}}{dz}$ on $\hat{\mathbb{C}}$ where

- $f^*\mu = \mu$ a.e.,
- $\text{supp}(\mu) =$ positive area subset of $J(f)$,
- $|\mu(z)| = 1$ on $\text{supp}(\mu)$.

**Theorem (NILF, L'23)**

*Suppose $f$ is a rational map that is **J-rotational**, i.e. every critical point in $J(f)$ either*

- *has finite orbit, or*
- *is eventually mapped to a bounded type rotation quasicircle.*

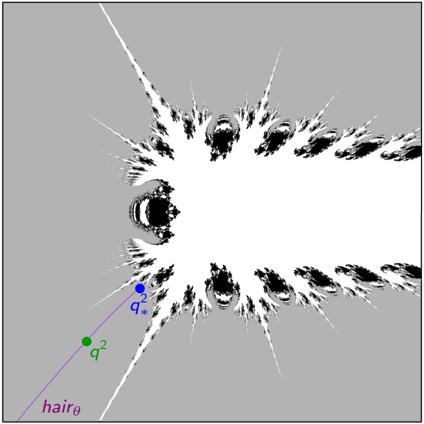
*Then, $J(f)$ supports no invariant line field of $f$.*

Given two maps in $\partial\mathcal{H}$,

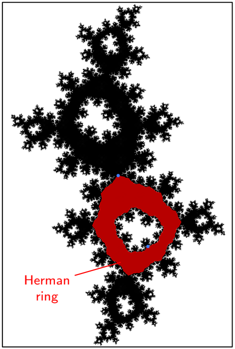
combinatorial equivalence $\xrightarrow[\text{argument}]{\text{pullback}}$ QC conjugacy, conformal on the Fatou set $\xrightarrow{\text{NILF}}$ conformal conjugacy

# Example #1: antipode-preserving rational maps (Bonifant-Buff-Milnor)

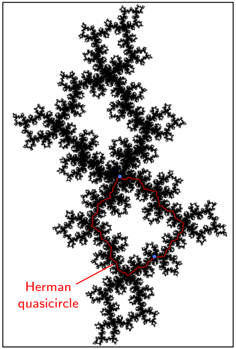
$q^2$ parameter plane for $f_q(z) = z^2 \dfrac{q - z}{1 + \bar{q}z}$




Dynamical plane of $f_q$



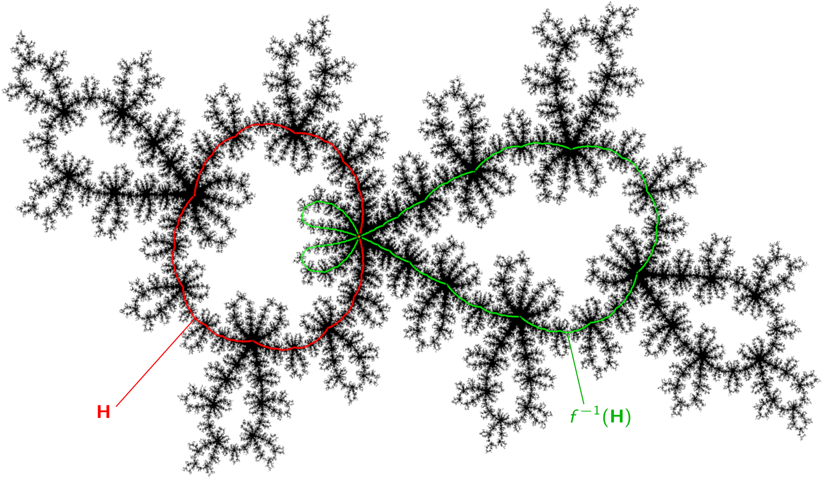

Dynamical plane of $f_{q_*}$



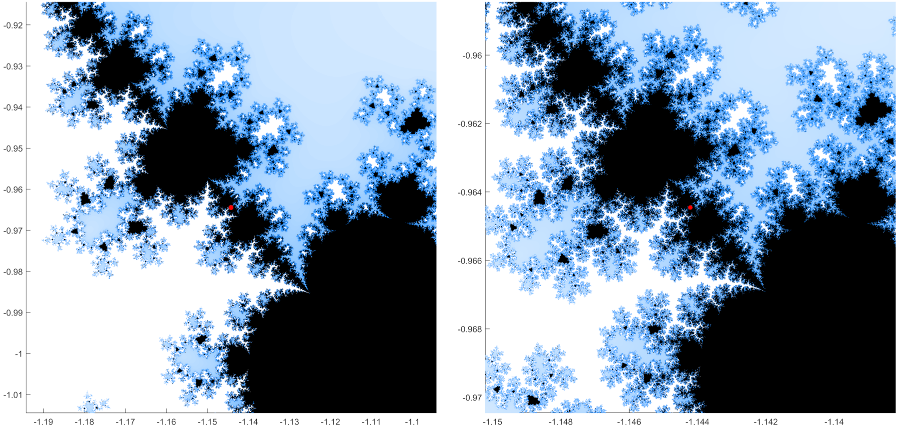

For every bounded type $\theta$, there is an analytic curve "$hair_\theta$" of parameters $q^2$ where $f_q$ has a Herman ring of rotation number $\theta$. $hair_\theta$ lands at a unique parameter $q_*^2$.

# Example #2: an imbalanced unicritical Herman curve

$$F_{c_*}(z) = c_* z^3 \; \frac{4 - z}{1 - 4z + 6z^2}, \qquad c_* \approx -1.144208 - 0.964454i$$



# Example #2: the parameter space picture




Conjecture: Bifurcation locus of $\{F_c\}_{c\in\mathbb{C}^*}$ is self-similar at the special parameter $c_\star$

# Beyond the realm of rational maps

$$\text{critical quasicircle map} = \begin{cases} \text{analytic self homeomorphism } f \text{ of a quasicircle } \mathbf{H} \\ \text{with a unique critical point on } \mathbf{H} \end{cases}$$

# Beyond the realm of rational maps

$$\text{critical quasicircle map} = \begin{cases} \text{analytic self homeomorphism } f \text{ of a quasicircle } \mathbf{H} \\ \text{with a unique critical point on } \mathbf{H} \end{cases}$$

**Theorem ($C^{1+\alpha}$ rigidity, L'23)**

*Given two critical quasicircle maps $f_1 : \mathbf{H}_1 \to \mathbf{H}_1$ and $f_2 : \mathbf{H}_2 \to \mathbf{H}_2$ of the same criticalities $(d_0, d_\infty)$ and bounded type rotation number,*

- *there is a QC conjugacy $\phi$ between $f_1$ and $f_2$ on a neighborhood of $\mathbf{H}_1$;*
- *$\phi$ is uniformly $C^{1+\alpha}$-conformal on $\mathbf{H}_1$.*

# Consequences of $C^{1+\alpha}$ rigidity

Given a critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$ with bounded type rotation number $\theta$ and inner and outer criticalities $d_0, d_\infty$,

1. $\mathbf{H}$ is $C^1$ smooth $\longleftrightarrow \dim(\mathbf{H}) = 1 \longleftrightarrow d_0 = d_\infty$;
2. $\dim(\mathbf{H})$ is universal;
3. if $\theta$ is an quadratic irrational, $\mathbf{H}$ is self-similar at the critical point with universal self-similar constant;
4. renormalizations $\mathcal{R}^n f$ converge exponentially fast to a unique $\mathcal{R}$-invariant horseshoe attractor.

# Open questions

1. Can we describe $\partial\mathcal{H}$ when $\theta$ is of unbounded type?
   $\Rightarrow$ For $d_0 = d_\infty = 2$ and high type $\theta$, there exist smooth Herman curves. [Yang Fei '22]
2. Is every limit of degenerating Herman rings always a Herman curve?
3. Is every bounded type Herman curve a limit of degenerating Herman rings?
4. For $f \in \partial\mathcal{H}$, is area $J(f) = 0$? Is dim $J(f) < 2$?

Thank you!