Lecture 1: Herman Curves

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Rotation curves

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If $\mathbf{H}$ is not contained in the closure of a rotation domain, we call it a **Herman curve**.

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Trichotomy: When $\text{rot}(f|_{\mathbf{H}})$ is of bounded type, there are 3 cases:

a. $\mathbf{H}$ is an analytic curve contained in a rotation domain,

b. $\mathbf{H}$ is the boundary of a rotation domain containing a critical point of f,

h. $\mathbf{H}$ is a Herman curve containing inner and outer critical points of f.

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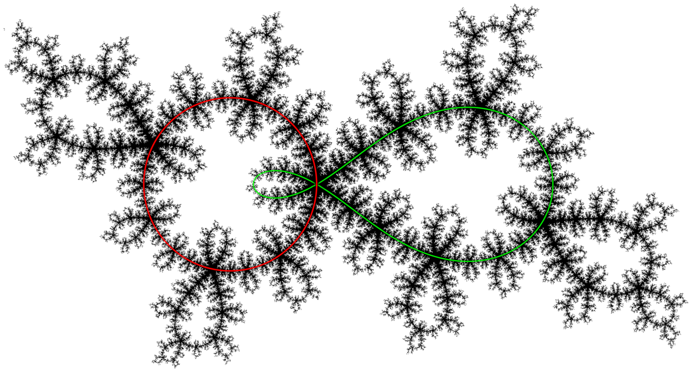
We'll focus on a Herman curve $\mathbf{H}$ with a single critical point c.

It comes with an inner criticality d_0 and an outer criticality d_∞.
The local degree at c is equal to $d_0 + d_\infty - 1$.

Old example: $d_0 = d_\infty = 2$

For any irrational θ, there is a unique $\zeta_\theta \in \mathbb{T}$ such that the unit circle is a Herman curve of rotation number θ for the map

$$f_\theta(z) = \zeta_\theta z^2 \frac{z-3}{1-3z}.$$



Arbitrary criticality (d_0, d_∞)

Fix a bounded type θ and a pair (d_0, d_∞).

Theorem

There exists a unique degree $d_0 + d_\infty - 1$ rational map F such that

1. *F has critical fixed points at 0 and ∞ with local degrees d_0 and d_∞,*
2. *F has a critical point 1 with local degree $d_0 + d_\infty - 1$,*
3. *F has a Herman quasicircle $\mathbf{H}$ of rotation number θ,*
4. *$\mathbf{H}$ passes through 1 and separates 0 and ∞.*

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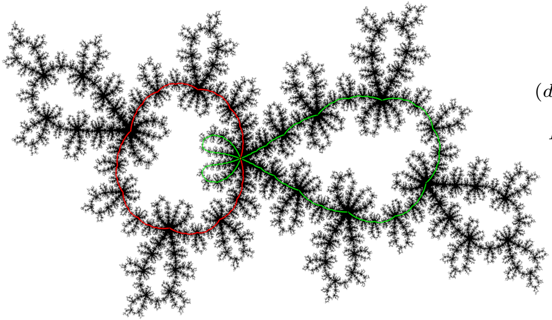


$\theta =$ golden mean

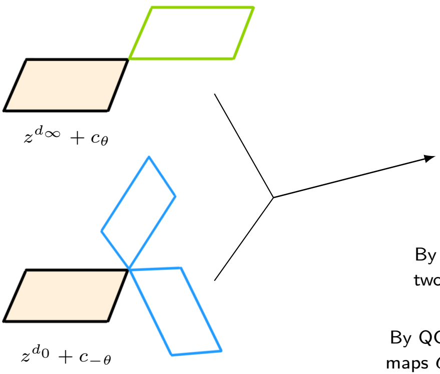
$(d_0, d_\infty) = (3, 2)$

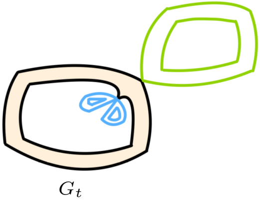
$$F_{c_*}(z) = c_* z^3 \, \frac{4 - z}{1 - 4z + 6z^2}$$

$$c_* \approx -1.144208 - 0.964454i$$

Proof of realization

Shishikura's surgery:





By QC surgery, we can combine two Siegel disks to get a Herman ring.

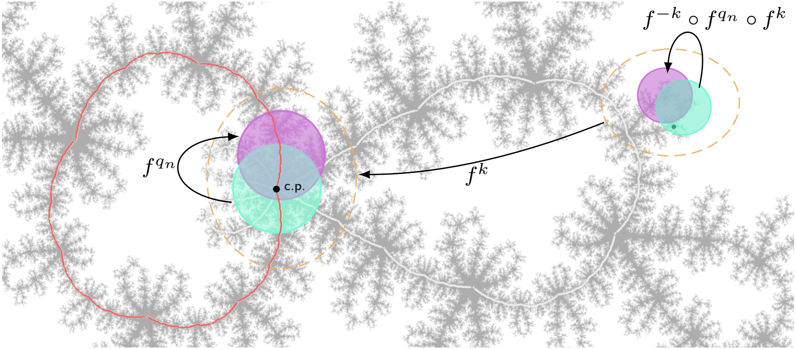
By QC deformation, we obtain a family of rational maps G_t with a Herman ring $\mathbb{H}_t$ of modulus t.

Theorem (A priori bounds)

$\partial\mathbb{H}_t$ are K-quasicircles, where K is independent of t.

Then, as $t \to 0$, $F = \lim_{t\to 0} G_t$ exists and has the desired Herman quasicircle $\mathbf{H} = \lim_{t\to 0} \overline{\mathbb{H}_t}$.

Proof of uniqueness (combinatorial rigidity)



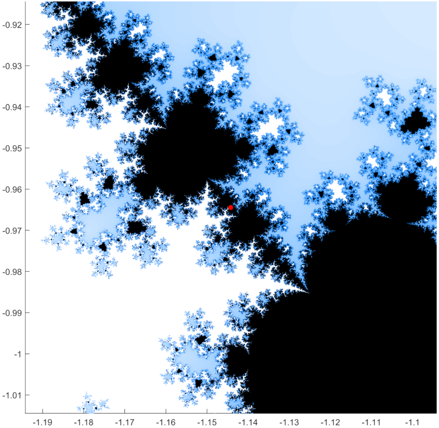
Theorem

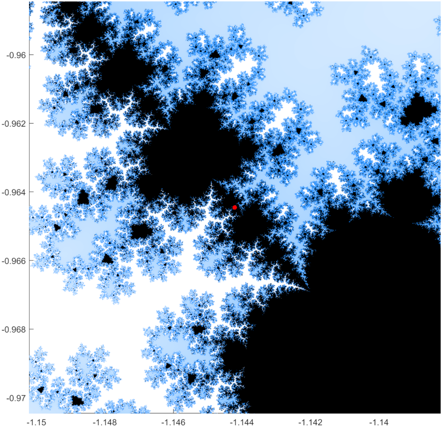
$J(F)$ *supports no invariant line field.*

Given two such maps with equal θ and (d_0, d_∞),

combinatorial equivalence $\xrightarrow[\text{argument}]{\text{pullback}}$ QC conjugacy, conformal on the Fatou set $\xrightarrow{\text{NILF}}$ conformal conjugacy

The parameter space picture





Conjecture: The bifurcation locus of $\left\{ F_c = cz^3 \, \dfrac{4 - z}{1 - 4z + 6z^2} \right\}_{c \in \mathbb{C}^*}$ is self-similar at $c_\star$.

Critical quasicircle maps

(uni-)critical quasicircle map $= \begin{cases} \text{analytic self homeomorphism } f \text{ of a quasicircle } \mathbf{H} \\ \text{with a unique critical point } c \text{ on } \mathbf{H} \end{cases}$

Petersen: rot$(f|_{\mathbf{H}})$ is of bounded type iff it is qs conjugate to irrational rotation.

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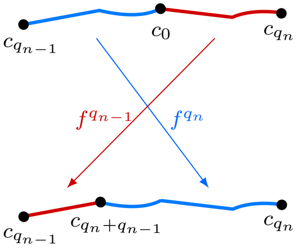
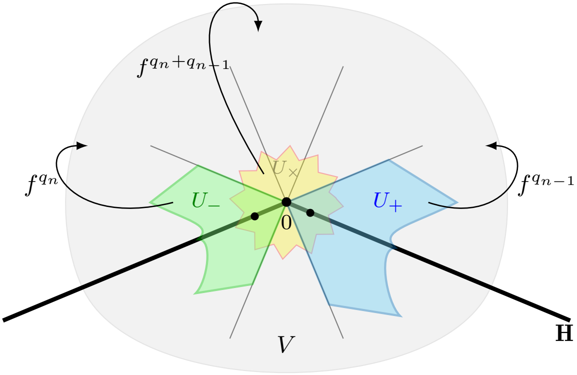
The pre-renormalization $p\mathcal{R}^n f$ is the commuting pair

$$\left(f^{q_n}|_{[c_{q_{n-1}}, c_0]}, f^{q_{n-1}}|_{[c_0, c_{q_n}]} \right).$$

The renormalization $\mathcal{R}^n f$ is obtained by affine rescaling $c_{q_{n-1}} \mapsto -1$ and $c_0 \mapsto 0$.



Butterflies



A $(3,2)$-critical 🦋 structure for $\mathcal{R}^n f$.

Theorem (Complex bounds)

For $n \gg 0$, the disks $U_\times, U_-, U_+, V$ can be chosen to be uniform quasidisks.

$C^{1+\alpha}$ Rigidity

Theorem

Given two critical quasicircle maps $f_1 : \mathbf{H}_1 \to \mathbf{H}_1$ *and* $f_2 : \mathbf{H}_2 \to \mathbf{H}_2$ *of the same criticalities* (d_0, d_∞) *and bounded type rotation number,*

- *there is a QC conjugacy* ϕ *between* f_1 *and* f_2 *on a neighborhood of* $\mathbf{H}_1$*;*
- ϕ *is uniformly* $C^{1+\alpha}$*-conformal on* $\mathbf{H}_1$*.*

Ingredients of the proof:

1. Construct QC conjugacy ϕ via complex bounds and pullback argument.
2. No inv. line fields $\Longrightarrow$ ϕ has zero dilatation on $K_1 := \overline{\text{iterated preimages of } \mathbf{H}_1}$.
3. Points on $\mathbf{H}$ are uniformly deep in K_1.

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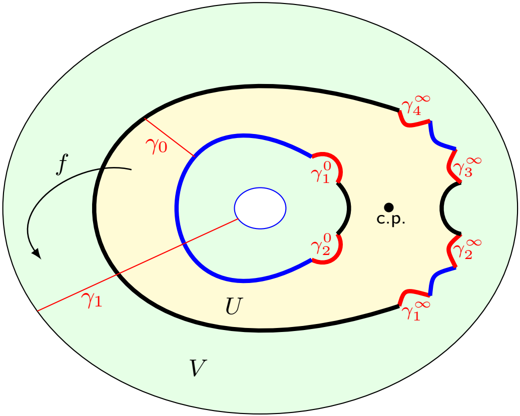
Corollary

If $\theta = [0; N, N, \ldots]$, $\exists!$ *normalized commuting pair* ζ_* *with rot*$(\zeta_*) = \theta$ *and* $\mathcal{R}\zeta_* = \zeta_*$*.* *Given a critical quasicircle map* $f : \mathbf{H} \to \mathbf{H}$ *with rot*$(f) = [0; *, \ldots, *, N, N, \ldots]$,

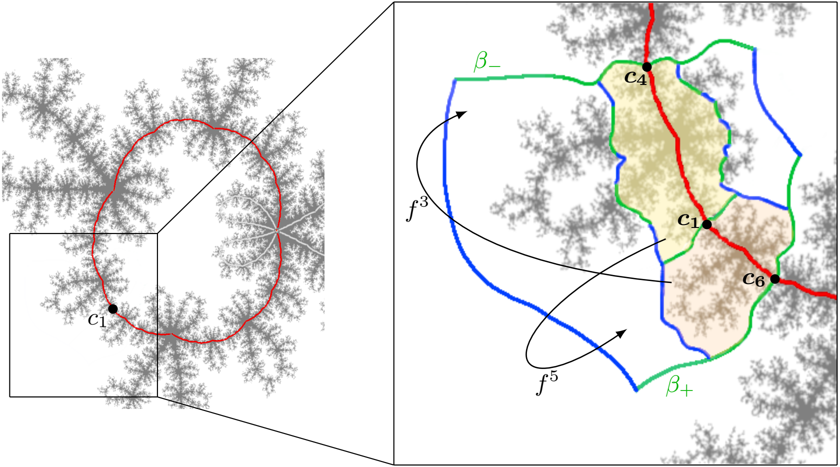
$$\mathcal{R}^n f \longrightarrow \zeta_* \quad \textit{exp. fast.}$$

Corona, an annular sibling of Pacman



A (2,3)-critical corona $f : (U, \gamma_0) \to (V, \gamma_1)$

We say that f is **rotational** if f contains a Herman quasicircle passing through c.p. essentially contained in U.



Gluing β_- and β_+ projects the pre-corona, i.e. the pair (f^5, f^3), into a rotational corona.

Fix criticalities (d_0, d_∞) and $\theta = [0; N, N, N, \ldots]$.

Theorem

There exists a corona renormalization operator $\mathcal{R} : \mathcal{U} \to \mathcal{B}$ with the following properties.

1. *$\mathcal{U}$ is an open subset of a Banach analytic manifold $\mathcal{B}$ consisting of (d_0, d_∞)-critical coronas.*
2. *$\mathcal{R}$ is a compact analytic operator with a unique fixed point f_* which is hyperbolic.*
3. *$\mathcal{W}^s =$ the space of rotational coronas with rotation number θ in $\mathcal{B}$.*
4. $\dim(\mathcal{W}^u) = 1$.

Hyperbolicity

Fix criticalities (d_0, d_∞) and $\theta = [0; N, N, N, \ldots]$.

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Corollary

Within the space of unicritical holomorphic maps on an annulus, the space of critical quasicircle maps of rotation number θ is an analytic submanifold of codimension ≤ 1.

How to prove hyperbolicity

Similar to the story of pacmen,

- $\mathcal{R}$ is analytic (holomorphic motions)
- $\mathcal{R}$ is compact (complex bounds)
- If $\mathcal{R}^n f$ is close to f_* for all $n \in \mathbb{N}$, then f is rotational and in $\mathcal{W}^s$. (renorm. tiling)

Similar to both pacmen and Feigenbaum,

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Remaining obstacle: $\dim(\mathcal{W}^u) \leq 1$?

Unlike pacman, we have no α fixed points.
Unlike Feigenbaum, we don't have "hybrid lamination" or "external maps".

Key: Identify $\mathcal{W}^u$ as a parameter space of transcendental maps of unknown dimension.

For $f \in \mathcal{W}^u$, the pre-corona (f_+, f_-) admits a maximal σ-proper extension

$$(\mathbf{f}_+ : W_+ \to \mathbb{C},\, \mathbf{f}_- : W_- \to \mathbb{C}).$$

If $f_n = \mathcal{R}^n f$ where $n < 0$, then

$(\mathbf{f}_+, \mathbf{f}_-)$ is the rescaling by $A_*^n(z) = \mu_*^n z$ of an iterate of $(\mathbf{f}_{n,+}, \mathbf{f}_{n,-})$.

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There exists a dense sub-semigroup $\mathbf{T}$ of $(\mathbb{R}_{\geq 0}, +)$ generated by $\{\mathbf{t}^n \mathbf{a}_+, \mathbf{t}^n \mathbf{a}_-\}_{n \in \mathbb{Z}}$, and we get a **cascade** of transcendental maps

$$\mathbf{F} = \left(\mathbf{F}^P : \mathrm{Dom}(\mathbf{F}^P) \to \mathbb{C} \right)_{P \in \mathbf{T}}$$

where $\mathbf{F}^{\mathbf{t}^n \mathbf{a}_\pm} = \mathbf{f}_{n,\pm}$.

When $f = f_*$,

$$\mathbf{F}_*^P = A_*^{-n} \circ \mathbf{F}_*^{\mathbf{t}^n P} \circ A_*^n \qquad \text{for all } P \in \mathbf{T}, n \in \mathbb{Z}.$$

Dynamical sets for cascades

For $f \in \mathcal{W}^u$, we define...

- Fatou set:

$$\mathfrak{F}(\mathbf{F}) = \text{points of normality of } \left(\mathbf{F}^P\right)_{P \in \mathbf{T}}$$

- Julia set:

$$\mathfrak{J}(\mathbf{F}) = \mathbb{C} \backslash \mathfrak{F}(\mathbf{F})$$

- postcritical set:

$$\mathfrak{P}(\mathbf{F}) = \text{ closure of the critical orbit } \left(\mathbf{F}^P(0)\right)_{P \in \mathbf{T}}$$

- finite-time escaping set:

$$\mathbf{I}_{<\infty}(\mathbf{F}) = \bigcup_{P \in \mathbf{T}} \mathbb{C} \backslash \mathrm{Dom}\left(\mathbf{F}^P\right)$$

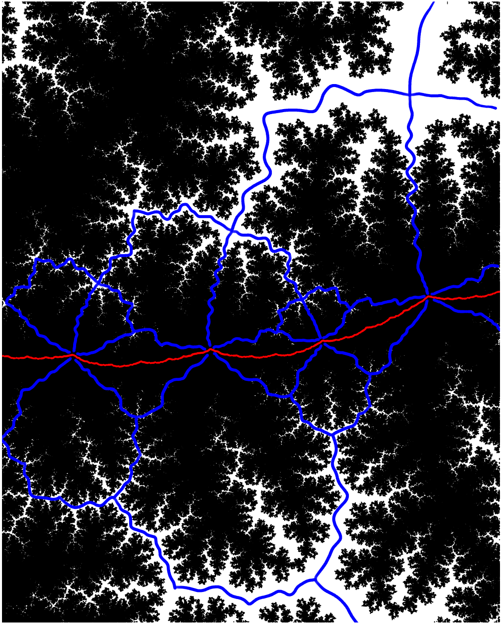
- infinite-time escaping set:

$$\mathbf{I}_{\infty}(\mathbf{F}) = \text{ points } x \text{ where } \mathbf{F}^P(x) \to \infty \text{ as } P \to \infty$$

- full escaping set:

$$\mathbf{I}(\mathbf{F}) = \mathbf{I}_{<\infty}(\mathbf{F}) \cup \mathbf{I}_{\infty}(\mathbf{F}).$$

Approximate dynamical picture for $\mathbf{F}_*$, the $\mathcal{R}$ fixed point



In blue:
Some rays in $\mathbf{I}_{<\infty}(\mathbf{F}_*)$
landing at critical points of $\mathbf{F}_*$

$\mathfrak{P}(\mathbf{F}_*)$

$\mathcal{W}^u$ is one-dimensional

Proposition: If $\mathfrak{J}(\mathbf{F})$ has no interior, then for almost every $z \in \mathfrak{J}(\mathbf{F})$,

$$\text{either } z \in \mathbf{I}(\mathbf{F}) \quad \text{or} \quad \operatorname{dist}\big(\mathbf{F}^P(z), \mathfrak{P}(\mathbf{F})\big) \to 0.$$

Theorem (Rigidity of escaping dynamics)

$\mathbf{I}(\mathbf{F})$ *supports no invariant line field & moves conformally away from the pre-critical pts. If* $\mathbf{F}$ *is hyperbolic, then* $\mathfrak{J}(\mathbf{F})$ *also supports no invariant line field.*

At last,

$$\begin{matrix}\text{unicriticality} \\ + \\ \text{combinatorial rigidity}\end{matrix} \Longrightarrow \begin{matrix}\exists \text{ hyperbolic component} \\ \Omega \subset \mathcal{W}^u \text{ near } f_*\end{matrix} \xRightarrow[\text{above}]{\text{theorem}} \dim(\Omega) \leq 1$$