Lecture 2: Herman rings

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December 11, 2023

Bounded type rotation domains

Fix a bounded type irrational $\theta \in (0,1)$. $\exists\, B \in \mathbb{N}$ such that $\sup_n a_n \le B$ where

$$\theta = [0; a_1, a_2, a_3, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}.$$

Theorem (Zhang '11)

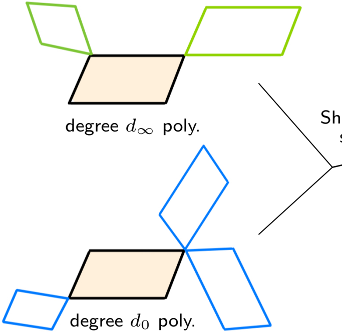
Every invariant Siegel disk of a map $f \in \mathrm{Rat}_d$ with rotation number θ is a $K(d,B)$-quasidisk containing a critical point on the boundary.

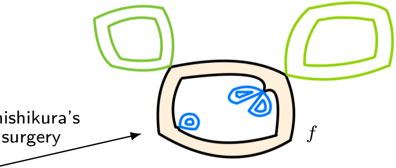
Corollary

The boundary components of an invariant Herman ring of $f \in \mathrm{Rat}_d$ with rotation number θ and modulus $\boldsymbol{\mu}$ are $K(d,B,\boldsymbol{\mu})$-quasicircle containing a critical point.

Herman rings of the simplest configuration

Let $\mathcal{H}$ = space of degree $d_0 + d_\infty - 1$ rational maps f that can be obtained by:





f has...

- superattractors at 0 and ∞ of local degree d_0 and d_∞,
- Herman ring $\mathbb{H}$ of rotation number θ,
- all critical points contained in $\partial\mathbb{H}$.

Theorem (*A priori bounds*)

The boundary components of the Herman ring of $f \in \mathcal{H}$ are $K(d_0, d_\infty, B)$-quasicircles. In particular, dilatation is independent of $\text{mod}(\mathbb{H})$.

Proof via near-degenerate regime

Let H be a boundary component of $\mathbb{H}$.
Endow H with the **combinatorial metric**, i.e. the unique normalized f-invariant metric.

Goal: Find constants ε and $\mathbf{K} > 1$ (depending only on d_0, d_∞, B) such that:

every interval $I \subset H$ of length $|I| < \varepsilon$ satisfies $W_{10}(I) < \mathbf{K}$.

Theorem (Amplification)

If

there is an interval $I \subset H$ with length $|I| < \varepsilon$ and width $W_{10}(I) = \mathbf{K} \gg 1$,

then

there is another interval $J \subset H$ with length $|J| < \varepsilon$ and width $W_{10}(J) \geq \mathbf{2K}$.

(All bounds depend only on d_0, d_∞, B.)

Herman scale vs. Siegel scale

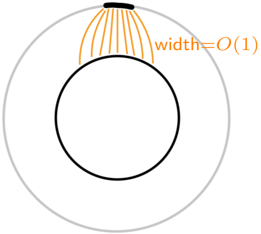
Assume mod($\mathbb{H}$) is very small. Given an interval $I \subset H$...

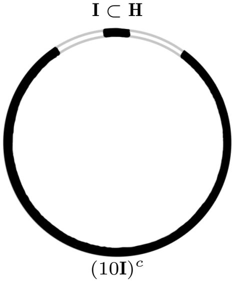
- Siegel scale: $|I| <$ mod($\mathbb{H}$),
- Herman scale: mod($\mathbb{H}$) $< |I| < \varepsilon$.

At the Siegel scale,
width of curves connecting I and $\partial\mathbb{H}\backslash H$ $= O(1)$.

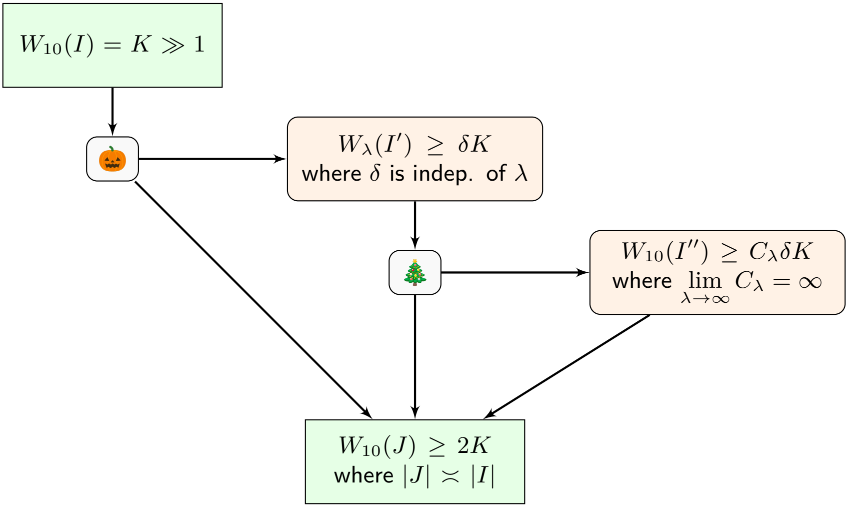
The Herman scale is the **main** case.
Replace:

- H with $\mathbf{H} := \overline{\mathbb{H}}$,
- intervals $I \subset H$ with "pieces" $\mathbf{I} \subset \mathbf{H}$,
- $W_{10}(I)$ with $W_{10}(\mathbf{I})$.





Strategy


$W_{10}(I) = K \gg 1$
$W_\lambda(I') \geq \delta K$
where δ is indep. of λ
$W_{10}(I'') \geq C_\lambda \delta K$
where $\lim_{\lambda\to\infty} C_\lambda = \infty$
$W_{10}(J) \geq 2K$
where $|J| \asymp |I|$

Proof of 🎄

Assume $\exists$ level n combinatorial piece I width $W_\lambda(I) = K$.

Step 1: Spreading around:

either $W_{10}(f^j(I)) \geq 2K$ for some j. or $W_\lambda(f^j(I)) \succ K$ for all $j = 1, 2, \ldots, q_{n+1}$.

Step 2: Apply quasiadditivity law:

Set $N \approx 0.001\lambda$ and consider $2N + 1$ islands

$$J_{-N},\ J_{-N+1},\ \ldots,\ J_0,\ \ldots,\ J_{N-1},\ J_N$$

from the tiling $\{f^j(I)\}_j$ that are 10-separated from one another.
Set $U = \hat{\mathbb{C}} \backslash (\rho J_0)^c$, a disk containing λJ_i for all i.

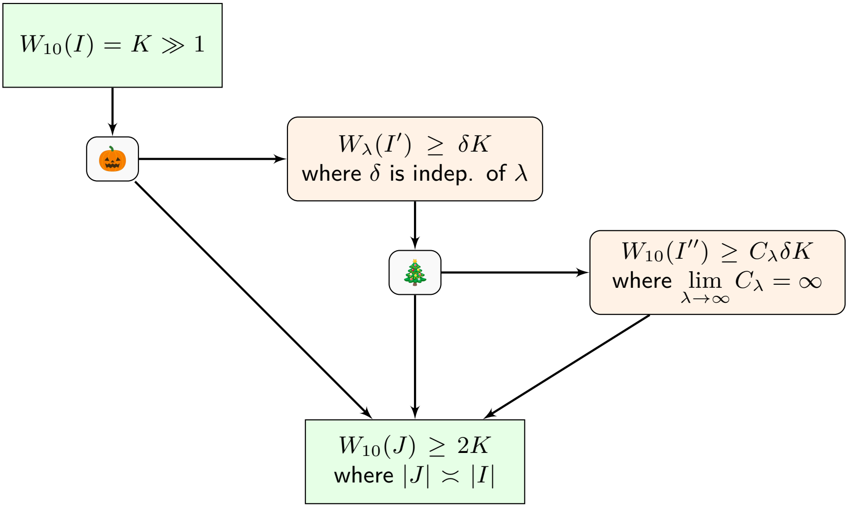
Either

$$\exists\ i \text{ such that } W_{10}(J_i) \succ \sqrt{\lambda} K,$$

or

$$W_{10}(P) \succ \sqrt{\lambda} K \text{ where } \cup_i J_i \subset P.$$

Strategy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$W_{10}(J) \geq 2K$
where $|J| \asymp |I|$

Proof of 🎃

Assume $\exists$ level n combinatorial piece I with $W_{10}(I) \geq K$.

Step 1: Spreading around:

$$W_{10}(f^j(I)) \succ K \quad \text{for all } j = 1, 2, \ldots q_{n+1}.$$

Step 2: Localization: either

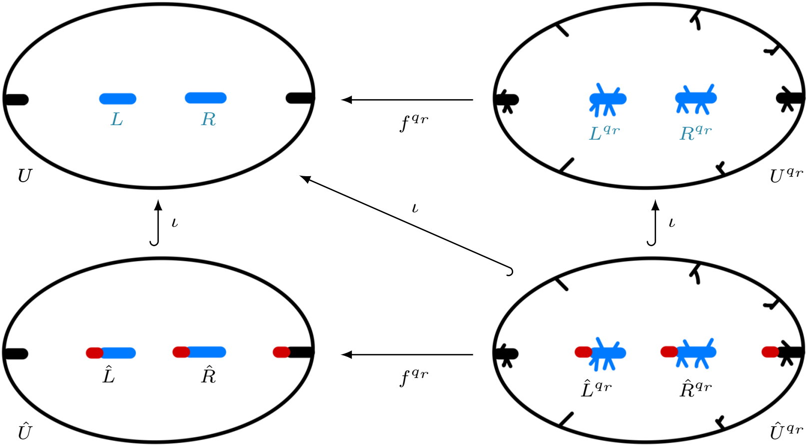
$$\exists \text{ piece } J \text{ where } W_{10}(J) \geq 2K, \text{ or}$$

$\exists$ pieces $L = f^i(I)$ and $R \subset (10L)^c$ such that

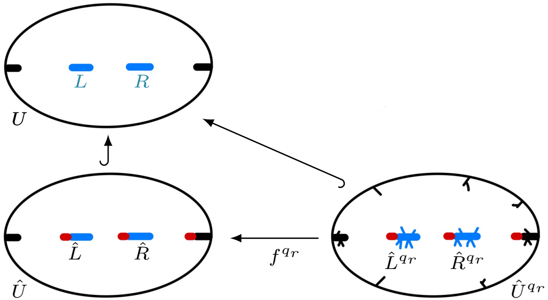
$$\text{dist}(L, R) \asymp |R| \asymp |L| \quad \text{and} \quad \text{width}\{\gamma \in \mathcal{F}_{10}(L) : \gamma \text{ lands on } R\} \asymp K.$$

Two islands in a lake

Consider the disk $U = \hat{\mathbb{C}} \backslash (\lambda L)^c$. Then, $W_U(L, R) \succ K$.



Pick high $r > n$. We define $\hat{U}$, $\hat{L}$, $\hat{R}$ by removing $(\lambda f^{q_r}(L))^c$, $f^{q_r}(L)$, and $f^{q_r}(R)$.

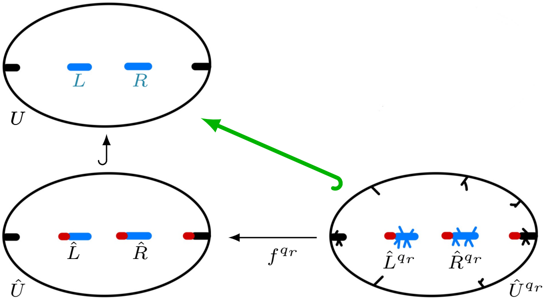


<u>Step 3</u>: Either the symmetric difference in red gives $W_{10}(J) \geq 2K$ or $W_\lambda(J) \succ K$, or

$$W^h_{can}(U, L \cup R) \asymp W^h_{can}(\hat{U}, \hat{L} \cup \hat{R}).$$

<u>Step 4</u>: For large $r > n$, $\exists\ \delta(r) \to 0$ such that either

$$\exists\ J \text{ where } W_{10}(J) \geq 2K, \quad \text{or} \quad W^h_{can}(\hat{U}^{q_r}, \hat{L}^{q_r} \cup \hat{R}^{q_r}) \leq \delta \cdot W^h_{can}(\hat{U}, \hat{L} \cup \hat{R}).$$



From the green inclusion,

$$W^h_{can}(\hat{U}^{q_r}, \hat{L}^{q_r} \cup \hat{R}^{q_r}) + W^v_{can}(\hat{U}^{q_r}, \hat{L}^{q_r} \cup \hat{R}^{q_r}) \geq W^h_{can}(U, L \cup R) - O(1).$$

Step 5: $W^v_{can}(\hat{U}^{q_r}, \hat{L}^{q_r} \cup \hat{R}^{q_r}) \succ K$.

Step 6: By the covering lemma, $\exists\ J \in \{\hat{L}, \hat{R}\}$ such that either

$$W_{10}(J) \geq 2K \quad \text{or} \quad W_\lambda(J) \succ K.$$

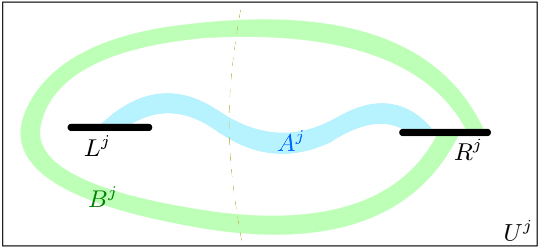
Step 4: Widthlifting 🏋 → width loss?

Notation: $W^j := W^h_{can}(U^j, L^j \cup R^j)$.

<u>KEY Proposition:</u> $\exists\ \delta < 1$ such that for large $r > n$,

$$W^0 \geq K \Longrightarrow \begin{array}{c} \exists\ J \text{ where } W_{10}(J) \geq 2K, \\ \text{or} \\ W^{q_r} \leq \delta \cdot W^0. \end{array}$$

<u>Proof:</u> Split W^j into $A^j + B^j$.



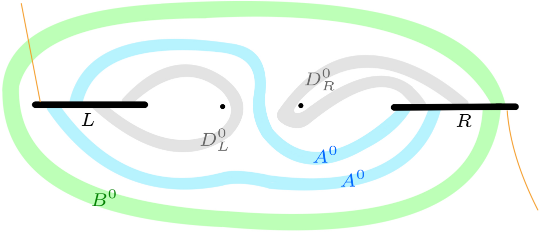
Preliminary observation: $A^j + 2B^j$ is monotone.

Let CV =critical values of f^{q_r} in U.

$$f^{q_r} : U^{q_r} \backslash f^{-q_r}(L \cup R \cup \mathsf{CV}) \to U \backslash (L \cup R \cup \mathsf{CV})$$

is an unbranched covering map of degree $d = d(\lambda)$.

Thick-thin decomposition $\mathcal{T}$ of $U \backslash (L \cup R \cup \mathsf{CV})$:



The thick-thin decomposition of $U^{q_r} \backslash f^{-q_r}(L \cup R \cup \mathsf{CV})$ is $(f^{q_r})^* \mathcal{T}$.
We say that a rectangle in $(f^{q_r})^* \mathcal{T}$ is "persistent" if it connects L^{q_r} and R^{q_r}.

<u>Claim 1:</u> Either

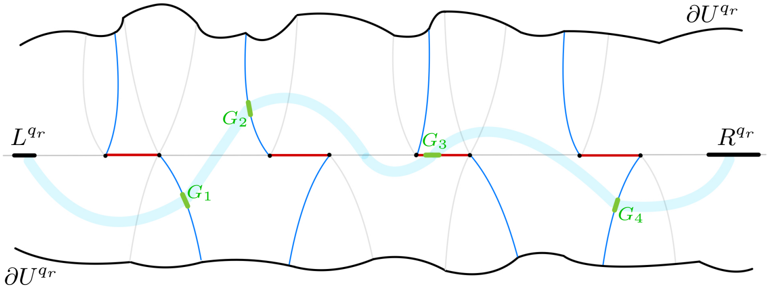
$$D_L^0 + D_R^0 \prec K \quad \text{or} \quad \exists\ J \text{ with } W_{10}(J) \geq 2K.$$

Claim 2: Either

$$A^{q_r} + 2B^{q_r} < \nu(A^0 + 2B^0)$$

for some $\nu < 1$, or the width of persistent rectangles in $(f^{q_r})^*\mathcal{T}$ is $W_{\text{per}} \asymp K$.

Persistent rectangles are represented by a single proper homotopy class rel $\text{CP}(f^{q_r})$.

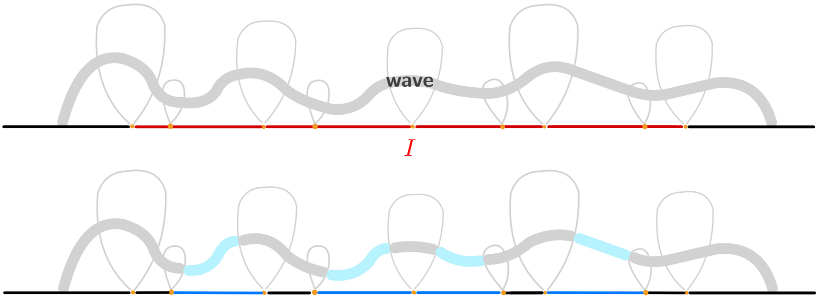


A'_{per} passes through many gates $G_i \subset f^{-q_r}(\mathbf{H})$.

Claim 3: If $W_{\text{per}} \asymp K$, then $\exists$ piece $J = f^{q_r}(G_i)$ such that $W_{10}(J) \geq 2K$.

Shallow level

<u>Caution!</u> Our application of quasi-additivity law may bring us to shallower level. At the shallow level ($|I| \asymp 1$), we need a different approach to ensure that the new degeneration is witnessed at a deeper level.

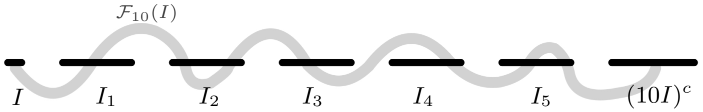


<u>Bubble-wave argument:</u>

Given a wave of width K protecting a shallow level n comb. piece I, $\exists$ level $n+m$ comb. piece J with $W_{10}(J) \geq C^m K$.

Suppose $W_{10}(I) = K$ where I is a comb. piece of shallow level n.

Introduce many level $n+m$ pieces I_i between I and $(10I)^c$ with 10-separation.



If curves in $\mathcal{F}_{10}(I)$ skip some I_i, they induce a wave. Ignoring this case, chop up $\mathcal{F}_{10}(I)$ via I_i's and apply Grotzsch inequality and conclude that $\exists\ i$ with

$$W_{10}(I_i) \geq 2K.$$