Degeneration of Herman rings

Willie Rush Lim

Stony Brook University

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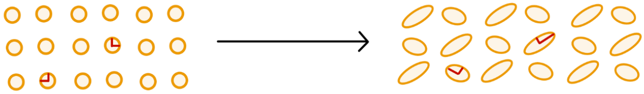

QC maps and Quasicircles

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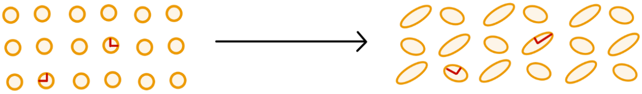
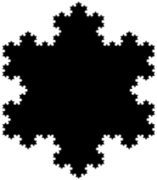
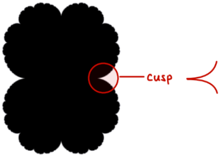


A ***K*-quasidisk** is the image of the unit disk $\mathbb{D} \subset \hat{\mathbb{C}}$ under a K-qc map on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Its boundary is called a ***K*-quasicircle**.





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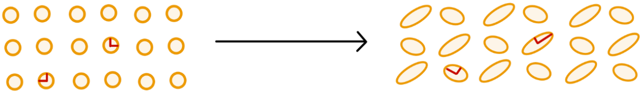

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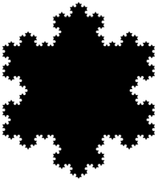

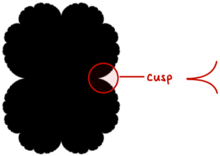

- Moduli spaces of Riemann surfaces can be described in terms of qc maps.
- The universal Teichmüller space can be described as the space of quasicircles.
- Quasicircles appear naturally in the study of Kleinian groups and rational maps.

Rotation domains

A maximal invariant domain U of a holomorphic map f is called a **rotation domain** if $f|_U$ is conjugate to irrational rotation $R_\theta(z) = e^{2\pi i\theta} z$.

There are only 2 types:

1. U is simply connected, i.e. a **Siegel disk**;
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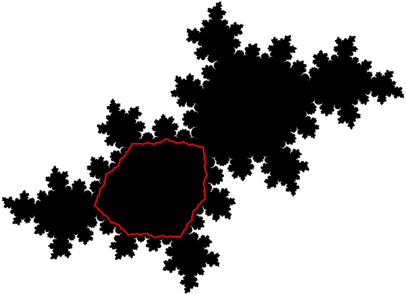
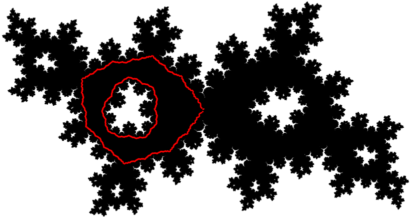


$f(z) = z^2 + c$ where $c \approx -0.3905 - 0.5868i$



$f(z) = e^{2\pi i t} z^2 \dfrac{z-4}{1-4z}$ where $t \approx 0.61517$

Conjecture:
The boundary components of rotation domains of rational maps are Jordan curves.

Deforming invariant annuli

Unlike Siegel disks, Herman rings come with a natural "Teichmüller space".

Two ways of deforming a Herman ring U:

1. Radial stretch, i.e. increase/decrease $\mathrm{mod}(U)$,
2. Twist ∂U

Cutting U along a radial line gives us a rectangle (where the horizontal sides are to be identified). The two moves above correspond to:

1. Vertical stretch,
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Naturally, the “moduli space“ of (f, U) is isomorphic to $\mathbb{R}_{>0} \times S^1$.

Question: What happens at the boundary of the “moduli space“?

- When $\mathrm{mod}(U) \to \infty$, this is easy;
- When $\mathrm{mod}(U) \to 0$, ...?

Bounded type assumption

Fix an irrational $\theta \in (0,1)$. Assume it is of **bounded type**, i.e. there is $B \in \mathbb{N}$ such that

$$\sup_{n\geq 1} a_n \leq B \qquad \text{where} \qquad \theta = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}.$$

E.g. golden mean $\frac{\sqrt{5}-1}{2} = \cfrac{1}{1 + \frac{1}{1+\frac{1}{1+\dots}}}$

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Theorem (G.F. Zhang '11)

If U is a rotation domain of a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with rotation number θ, every component of ∂U is a quasicircle containing a critical point.

Fix $d_0, d_\infty \geq 2$. Consider the family $\mathcal{H}$ of degree $d_0 + d_\infty - 1$ rational maps f where

- 0 and ∞ are critical fixed points with local degree d_0 and d_∞,
- f has a Herman ring $\mathbb{H}_f$ of rotation number θ,
- all other critical points are on $\partial\mathbb{H}_f$.

Theorem (A Priori Bounds)

For all $f \in \mathcal{H}$, the boundary of $\mathbb{H}_f$ consists of K-quasicircles, where K depends only on $\deg(f)$ *and B and not on* $\operatorname{mod}(\mathbb{H}_f)$.

How to prove *a priori bounds*?

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$I =$ an interval in H of (combinatorial) length $|I| < 0.1$.
$10I =$ the interval of length $10|I|$ having the same midpoint as I.
$W_{10}(I) =$ the extremal width of curves connecting I and $H \setminus 10I$.

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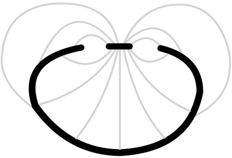
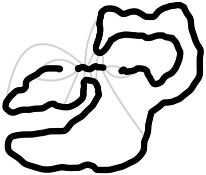


small $W_{10}(I)$



large $W_{10}(I)$

$W_{10}(I)$ encodes the local (near-)degeneration of H near the interval I.

Near-Degenerate Regime

To prove *a priori bounds*, it is sufficient to find constants ε and $\mathbf{K}$ depending only on B, d_0, d_∞ such that:

$$\text{every interval } I \subset H \text{ of length } |I| < \varepsilon \text{ satisfies } W_{10}(I) < \mathbf{K}.$$

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Our goal is reduced to showing:

Theorem (Amplification)

If

there is an interval $I \subset H$ *with length* $|I| \ll 1$ *and width* $W_{10}(I) = \mathbf{K} \gg 1$,

then

there is another interval $J \subset H$ *with length* $|J| \ll 1$ *and width* $W_{10}(J) \geq 2\mathbf{K}$.

(All bounds depend only on d_0, d_∞, B.*)*

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The proof relies on the near-degenerate machinery, including ideas from: Kahn-Lyubich '05, Kahn '06, and Dudko-Lyubich '22.

Rotation curves

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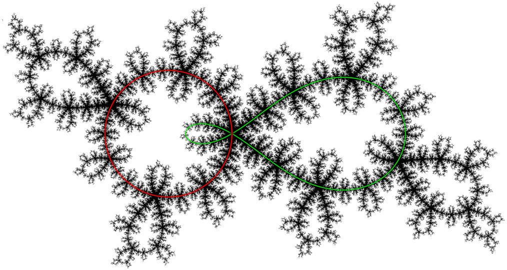
Trivial example: For any irrational θ, there is a unique $\zeta_\theta \in \mathbb{T}$ such that the unit circle is a Herman curve of rotation number θ for the rational map

$$f_\theta(z) = \zeta_\theta z^2 \frac{z-3}{1-3z}.$$



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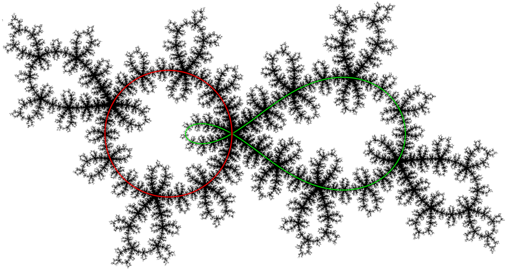

Question: Can non-trivial Herman curves exist?

Non-trivial Herman curves

Theorem (Realization)

Given any points

$$a_1, \ldots, a_{d_0 - 1}, b_1, \ldots, b_{d_\infty - 1} \in S^1,$$

there exists a rational map f in $\partial\mathcal{H}$ admitting a Herman curve $\mathbf{H}$ *such that* $rot(f|_{\mathbf{H}}) = \theta$ *and in linearizing coordinates, the inner critical points of* $f|_{\mathbf{H}}$ *are* $a_1, \ldots, a_{d_0-1}$*, and the outer critical points are* $b_1, \ldots, b_{d_\infty - 1}$*.*

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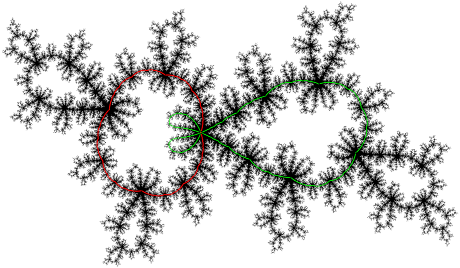
Unicritical example:



$\theta =$ golden mean

2 inner critical pts

1 outer critical pt

$$F_{c_*}(z) = c_* z^3 \, \frac{4 - z}{1 - 4z + 6z^2}$$

$$c_* \approx -1.144208 - 0.964454i$$

Proof of realization

X.G. Wang '12 :
$\exists$ a rational map f_1 in $\mathcal{H}$ admitting a Herman ring $\mathbb{H}_1$ with inner and outer critical points combinatorially positioned at $a_1, \ldots, a_{d_0-1}$ and $b_1, \ldots, b_{d_\infty-1}$.

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By QC deformation,
$\exists$ 1-par family $\{f_t\}_{0<t\leq 1} \subset \mathcal{H}$ where f_t has a Herman ring $\mathbb{H}_t$ with the same combinatorics and $\mathrm{mod}(\mathbb{H}_t) \to 0$ as $t \to 0$.

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By *a priori bounds*,

- $\partial\mathbb{H}_t$ are K-quasicircles for all t;
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Result: $f_0 = \lim\limits_{t\to 0} f_t$ exists and has a Herman curve with the same combinatorics as f_1.

Description of $\partial\mathcal{H}$

Theorem (Rigidity)

If two rational maps f, g in $\partial\mathcal{H}$ are combinatorially equivalent, then

$$f = L \circ g \circ L^{-1}$$

for some linear map $L(z) = \lambda z$.

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- the support E is a positive-measure totally invariant subset of $\hat{\mathbb{C}}$,
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Corollary

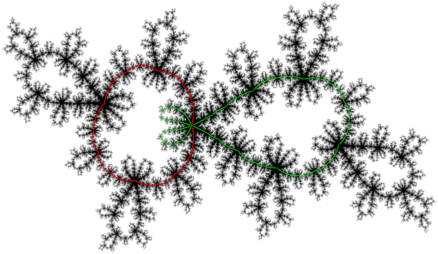
$\partial\mathcal{H}/_\sim$ is homeomorphic to

$$\mathrm{SP}^{d_0 - 1}(S^1) \times \mathrm{SP}^{d_\infty - 1}(S^1)/_{\text{rigid rotation}}$$

which is a compact connected topological orbifold of dimension $d_0 + d_\infty - 3$.

What's next?

Recall the unicritical example:



θ = golden mean

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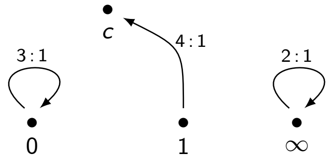
The 1-par family of degree 4 rational maps

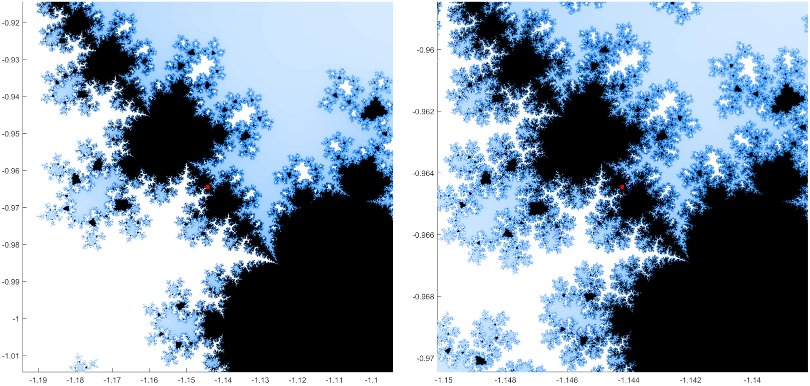
$$F_c(z) = cz^3 \; \frac{4 - z}{1 - 4z + 6z^2}$$

is characterized by the data on the right.



In general, for any bounded type θ, $\exists!$ parameter c_θ such that F_{c_θ} has a Herman curve with rotation number θ.

Parameter space picture



Bifurcation locus of $\{F_c\}$ magnified around the parameter $c_\star = c_\theta$ where θ=golden mean.

Conjecture: The bifurcation locus of $\{F_c\}$ is self-similar at $c_\star$.

Thank you!