

Degeneration of Herman rings

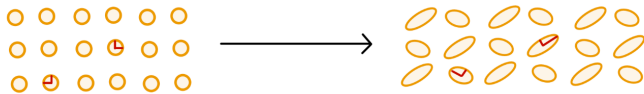
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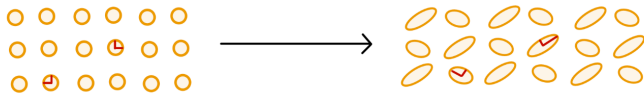
QC maps and Quasicircles

A V -quasiconformal map $H: \mathbb{C} \rightarrow \mathbb{C}$ is an orientation-preserving homeomorphism of a Riemann surface \mathbb{C} sending a (measurable) field of circles to a field of ellipses of eccentricity bounded by $V + 1$.

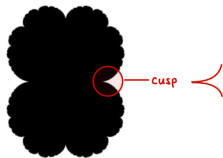
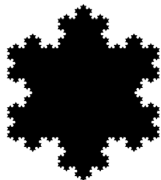


QC maps and Quasicircles

A V -qc map $H: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is an orientation-preserving homeomorphism of a Riemann surface $\hat{\mathbb{C}}$ sending a (measurable) field of circles to a field of ellipses of eccentricity bounded by $V > 1$.

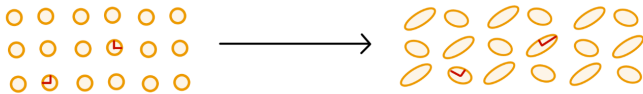


A V -qc is the image of the unit disk D under a V -qc map on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Its boundary is called a V -qc.

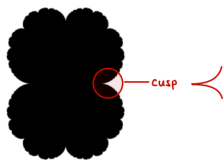
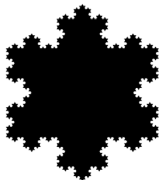


QC maps and Quasicircles

A V -qc map $H: \mathbb{D} \rightarrow \mathbb{C}$ is an orientation-preserving homeomorphism of a Riemann surface \mathbb{D} sending a (measurable) field of circles to a field of ellipses of eccentricity bounded by $V > 1$.



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- Moduli spaces of Riemann surfaces can be described in terms of qc maps.
- The universal Teichmüller space can be described as the space of quasicircles.
- Quasicircles appear naturally in the study of Kleinian groups and rational maps.

Rotation domains

A maximal invariant domain Ω of a holomorphic map H is called a \mathbb{C}^2 -rotation domain if $H|_{\Omega}$ is conjugate to irrational rotation $\rho(z) = e^{2\pi i \alpha} z$.

There are only 2 types:

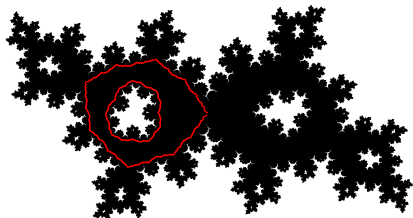
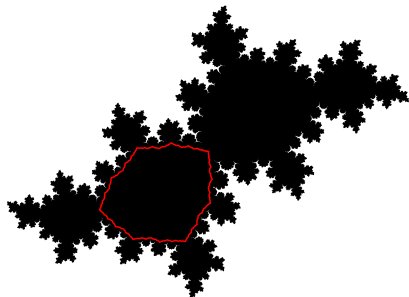
- 1 Ω is simply connected, i.e. a \mathbb{C} -rotation domain.
- 2 Ω is an annulus, i.e. a \mathbb{C}^* -rotation domain.

Rotation domains

A maximal invariant domain Ω of a holomorphic map H is called a rotation domain if $H|_{\Omega}$ is conjugate to irrational rotation $p(\zeta) = \zeta^S$.

There are only 2 types:

- Ω is simply connected, i.e. a disk.
- Ω is an annulus, i.e. a ring.



$$H(\zeta) = \zeta^2 + \zeta \text{ where } \zeta \in \{0.3905, 0.5868\}$$

$$H(\zeta) = \zeta^2 - \frac{\zeta}{1 - 4\zeta} \text{ where } \zeta \in \{0.61517, \dots\}$$

Conjecture:

The boundary components of rotation domains of rational maps are Jordan curves.

Unlike Siegel disks, Herman rings come with a natural “Teichmüller space”.

Two ways of deforming a Herman ring \mathcal{H} :

- 1 Radial stretch, i.e. increase/decrease $\text{mod}(\mathcal{H})$,
- 2 Twist \mathcal{H}

Cutting \mathcal{H} along a radial line gives us a rectangle (where the horizontal sides are to be identified). The two moves above correspond to:

- 1 Vertical stretch,
- 2 Horizontal shear.

Deforming invariant annuli

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Naturally, the “moduli space” of (\mathcal{H}) is isomorphic to $\mathbb{R}_{>0} \times \mathbb{R}^1$.

Question: What happens at the boundary of the “moduli space”?

- When $\text{mod}(\mathcal{H}) \rightarrow 1$, this is easy;
- When $\text{mod}(\mathcal{H}) \rightarrow 0$, ...?

Bounded type assumption

Fix an irrational $\alpha \in (0;1)$. Assume it is of bounded type, i.e. there is $C \in \mathbb{N}$ such that

$$\sup_{n \geq 1} \frac{1}{q_n} \leq C \quad \text{where} \quad q_n = \frac{1}{-1 + \frac{1}{-2 + \frac{1}{-3 + \dots}}}$$

E.g. golden mean $\frac{\sqrt{5}-1}{2} = \frac{1}{1 + \frac{1}{1 + \dots}}$

Bounded type assumption

Fix an irrational $\alpha \in (0;1)$. Assume it is of $4b\text{-}^{\wedge}\text{@}\text{C}\text{z}\%_{\text{C}}$, i.e. there is $3 \in \mathbb{N}$ such that

$$\sup_{n \geq 1} \frac{1}{q_n} \leq 3 \quad \text{where} \quad = \frac{1}{-1 + \frac{1}{-2 + \frac{1}{-3 + \dots}}}$$

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Theorem (G.F. Zhang '11)

$RH) \ S - \phi z - z^S \wedge @b \setminus - S^b H - q z^S \wedge - Y \setminus - e H: \hat{C} ! \hat{C} \dots SP \phi z - z^S \wedge \wedge \setminus 4Cq > C f C q \%_o$
 $\langle b \setminus e b \wedge C \wedge z b H @ \rangle S - I \sim s S q \setminus C \langle b \wedge z - S^S L - \langle q S - Y e b S^z i$

Herman rings of the simplest configuration

- Fix $\alpha; \alpha_1$ 2. Consider the family H of degree $\alpha + \alpha_1 - 1$ rational maps H where
- 0 and 1 are critical fixed points with local degree α and α_1 ,
 - H has a Herman ring H_H of rotation number θ ,
 - all other critical points are on ∂H_H .

Theorem (A Priori Bounds)

$\deg(H) \leq \frac{1}{\alpha} + \frac{1}{\alpha_1} + 3 - \frac{1}{\alpha_1} \text{ mod } (H_H) i$

How to prove - eq 4b-^@?

Let O be a boundary component of the Herman ring of H^2/H .
Equip O with the unique normalized H -invariant metric.

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R = an interval in O of (combinatorial) length $|R| < 0.1$.

$10R$ = the interval of length $10|R|$ having the same midpoint as R

$w_{10}(R)$ = the extremal width of curves connecting R and $O \setminus 10R$

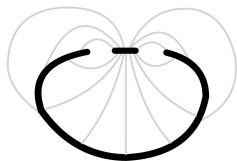
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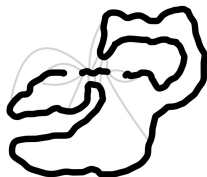
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$10R$ = the interval of length $10|R|$ having the same midpoint as R

$\mu_{10}(R)$ = the extremal width of curves connecting R and $O \setminus 10R$



small $\mu_{10}(R)$



large $\mu_{10}(R)$

$\mu_{10}(R)$ encodes the local (near-)degeneration of O near the interval R

Near-Degenerate Regime

To prove - eq 4.1, it is sufficient to find constants ϵ and V depending only on β, γ, δ such that:

every interval $R \subset O$ of length $|R| < \epsilon$ satisfies $\mu_{10}(R) < V$:

Near-Degenerate Regime

To prove $\epsilon \ll \delta \ll \epsilon^2$, it is sufficient to find constants η and V depending only on $3, \epsilon, \delta$ such that:

every interval $R \subset [0, 1]$ of length $|R| < \eta$ satisfies $\mu_{10}(R) < V$:

Our goal is reduced to showing:

Theorem (Amplification)

$$\begin{aligned} & \text{RH} \\ & \mu_{10}(R) = V \quad |R| < \eta \\ & \mu_{10}(T) < 2V \quad |T| < \eta \\ & f, \epsilon, \delta \end{aligned}$$

Near-Degenerate Regime

To prove $\epsilon_{\text{opt}} \leq \epsilon_{\text{opt}}^{\text{near-degenerate}}$, it is sufficient to find constants δ and V depending only on $\epsilon; \epsilon_1; \epsilon_2$ such that:

every interval $R \subset [0, 1]$ of length $|R| < \delta$ satisfies $\mu_{10}(R) < V$:

Our goal is reduced to showing:

Theorem (Amplification)

$$\begin{aligned} & \text{RH} \\ & \mu_{10}(R) = V \cdot |R| \\ & \mu_{10}(T) \leq 2V \cdot |T| \end{aligned}$$

$\epsilon, \epsilon_1, \epsilon_2$ are constants depending only on $\epsilon; \epsilon_1; \epsilon_2$.

The proof relies on the near-degenerate machinery, including ideas from: Kahn-Lyubich '05, Kahn '06, and Dudko-Lyubich '22.

Rotation curves

An invariant curve Γ of a holomorphic map H is a \mathbb{C}^* if $H|_{\Gamma}$ is conjugate to irrational rotation.

If Γ is not contained in the closure of a rotation domain, we call it a \mathbb{C}^* .

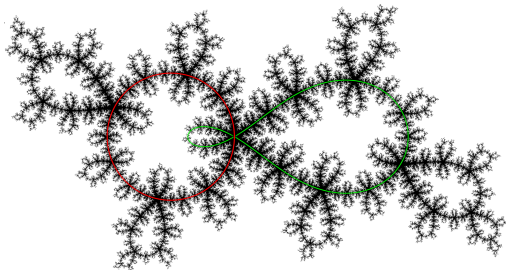
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Trivial example: For any irrational θ , there is a unique 2π such that the unit circle is a Herman curve of rotation number θ for the rational map

$$H(z) = z^2 \frac{z - 3}{1 - 3z}.$$



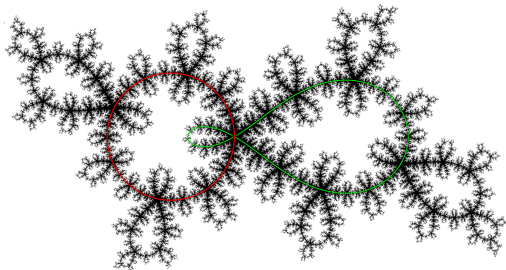
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Trivial example: For any irrational θ , there is a unique $\gamma \in \mathbb{C}^*$ such that the unit circle is a Herman curve of rotation number θ for the rational map

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Question: Can non-trivial Herman curves exist?

Theorem (Realization)

$K^{\mathbb{C}^n} \cong \mathbb{C}^n$

$$-1; \dots; -q_1; 4_1; \dots; 4_{q_1-1} \geq r^1;$$

$zPCqCtSzs - qzS^{\wedge} - Y \setminus - eHS^{\wedge} @H - @Szs^{\wedge}L - OCq \setminus -^{\wedge} < \sim qfCO s \sim PzP-z \phi z(Hj_0) =$
 $-^{\wedge} @S^{\wedge} S^{\wedge}C \phi S^{\wedge}L < bbq@S^{\wedge} - zCs > zPC S^{\wedge}Cq < qS - YebS^{\wedge}zs bHHj_0 - qC -1; \dots; -q_1 > -^{\wedge} @zPC$
 $b-zCq < qS - YebS^{\wedge}zs - qC 4_1; \dots; 4_{q_1-1} i$

Theorem (Realization)

$K \mathcal{F} C^{\wedge} - \wedge \% e b S^{\wedge} z s$

$$-1; \dots; -\varrho_1; 1; 4_1; \dots; 4_{\varrho_1} \quad 1 \quad 2 \quad r^1;$$

$z P C q C t S z s - q z S^{\wedge} - Y \setminus - e H S^{\wedge} @ H - @ \mathcal{S} z S^{\wedge} L - O C q \setminus - \wedge \leftarrow q f C O s \sim P z P - z \varphi z (H_j^0) =$
 $- \wedge @ S^{\wedge} \mathcal{S}^{\wedge} C - \varphi S^{\wedge} L \leftarrow b b q @ S^{\wedge} - z C s \rangle z P C S^{\wedge} \wedge C q \leftarrow \mathcal{S} S - Y e b S^{\wedge} z s b H H_j^0 - q C - 1; \dots; -\varrho_1; 1 \rangle - \wedge @ z P C$
 $b - z C q \leftarrow \mathcal{S} S - Y e b S^{\wedge} z s - q C 4_1; \dots; 4_{\varrho_1} \quad 1 \quad i$

Unicritical example:

= golden mean

2 inner critical pts

1 outer critical pt

$$G_{\zeta}(\zeta) = \zeta^3 \frac{4 - \zeta}{1 - 4\zeta + 6\zeta^2}$$

$$\zeta \quad 1.144208 \quad 0.964454S$$

X.G. Wang '12 :

\mathcal{H} a rational map H in H admitting a Herman ring H_1 with inner and outer critical points combinatorially positioned at $-1; \dots; -\infty$ and $4; \dots; 4$.

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\mathcal{H} a rational map H admitting a Herman ring H_1 with inner and outer critical points combinatorially positioned at $-1; \dots; -e_1$ and $4_1; \dots; 4_{e_1}$.

By QC deformation,

\mathcal{H} 1-par family f_z $g_{0 < z < 1}$ H where H_z has a Herman ring H_z with the same combinatorics and $\text{mod}(H_z) \neq 0$ as $z \neq 0$.

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\mathcal{H} 1-par family $f_{H_z} g_{0 < z < 1}$ H where H_z has a Herman ring H_z with the same combinatorics and $\text{mod}(H_z) \neq 0$ as $z \neq 0$.

By - eq 4b ~ 4c,

- H_z are V -quasicircles for all z ;
- $f_{H_z} g_{0 < z < 1}$ is pre-compact in $\text{Rat}_{\theta_1 + \theta_1}$.

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Result: $H = \lim_{z \rightarrow 0} H_z$ exists and has a Herman curve with the same combinatorics as H .

Theorem (Rigidity)

$H \cong \mathbb{R}^n$ as a topological space. The map $H \rightarrow \mathbb{R}^n$ is a homeomorphism.

$$H = X \times L \times X^{-1}$$

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Let (X, \mathcal{L}, μ) be a probability space with a measurable action of a group G . Let H be a subgroup of G . Then H admits no invariant line field.

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An (X, \mathcal{L}, μ) is a measurable collection of 1-D subspaces $\{X_t\}_{t \in \hat{C}}$ where

- the support B is a positive-measure totally invariant subset of \hat{C} ,
- for a.e. $t \in B$, $@H(X_t) = X_{H(t)}$.

In the proof of rigidity, we show that every $H \subset H$ admits no invariant line field.

Description of \mathcal{H}

Theorem (Rigidity)

Let (X, μ) be a probability space and T a measure-preserving transformation. Let \mathcal{H} be a measurable collection of 1-D subspaces $\{X_t\}_{t \in B}$ where

$$H = X \oplus L \oplus X^{-1}$$

for μ -a.e. $t \in B$, $\mathcal{H}(X_t) = X_{H(t)}$.

Let $\mathcal{H} = \{X_t\}_{t \in B}$ be a measurable collection of 1-D subspaces $\{X_t\}_{t \in B}$ where

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In the proof of rigidity, we show that every $H \in \mathcal{H}$ admits no invariant line field.

Corollary

Let (X, μ) be a probability space and T a measure-preserving transformation. Let \mathcal{H} be a measurable collection of 1-D subspaces $\{X_t\}_{t \in B}$ where

$$SP^{\otimes 1}(r^{-1}) \oplus SP^{\otimes 1}(r^{-1}) = \text{rigid rotation}$$

Let (X, μ) be a probability space and T a measure-preserving transformation. Let \mathcal{H} be a measurable collection of 1-D subspaces $\{X_t\}_{t \in B}$ where

What's next?

Recall the unicritical example:

= golden mean

2 inner critical pts

1 outer critical pt

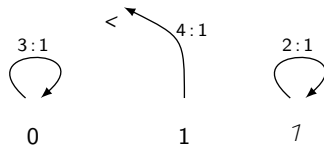
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ζ 1:144208 0:964454S

The 1-par family of degree 4 rational maps

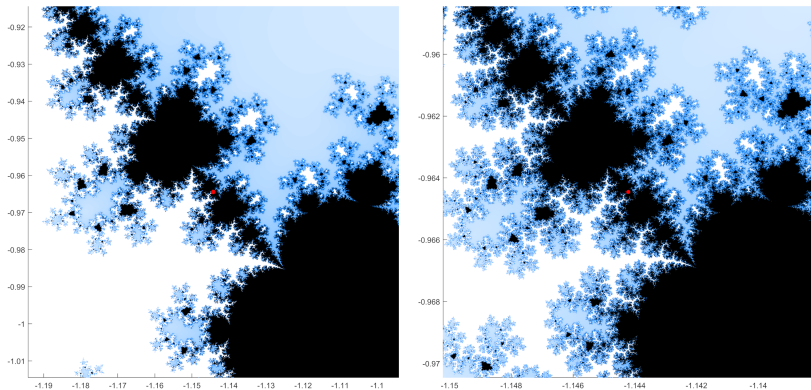
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is characterized by the data on the right.



In general, for any bounded type ζ , $\exists!$ parameter ζ such that G_{ζ} has a Herman curve with rotation number ζ .

Parameter space picture



Bifurcation locus of $fG_{\zeta}g$ magnified around the parameter $\zeta = \frac{1}{\phi}$ where $\phi = \text{golden mean}$.

Conjecture: The bifurcation locus of $fG_{\zeta}g$ is self-similar at $\zeta = \frac{1}{\phi}$.

Thank you!