Critical quasicircle maps

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April 4, 2024

QC maps and Quasicircles

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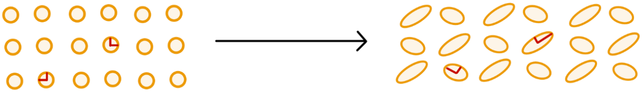

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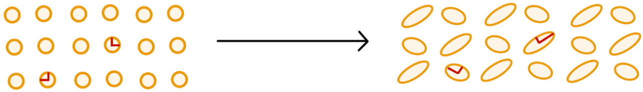
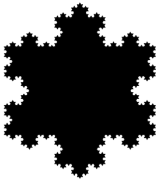
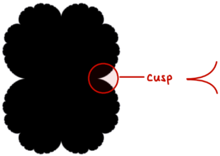


A K-**quasidisk** is the image of the unit disk $\mathbb{D}$ under a K-QC map on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Its boundary is called a K-**quasicircle**.





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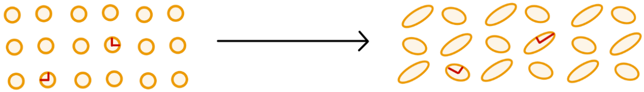

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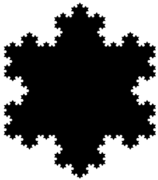

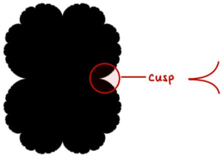

- Moduli spaces of Riemann surfaces can be described in terms of QC maps.
- The universal Teichmüller space can be described as the space of quasicircles.
- Quasicircles appear naturally in the study of Kleinian groups and rational maps.

Diophantine assumption

Fix an irrational $\theta \in (0,1)$ and write

$$\theta = [a_1, a_2, a_3, \ldots] := \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}.$$

θ is called

- **bounded type** if $\sup a_n \leq B$ for some $B \in \mathbb{N}$.
- **periodic type** if $a_{n+p} = a_n$ for all n.

E.g. golden mean $= [1, 1, 1, \ldots]$

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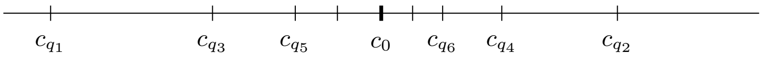
Consider the rigid rotation

$$R_\theta : S^1 \to S^1, \quad z \mapsto e^{2\pi i \theta} z.$$

Let $p_n/q_n = [a_1, \ldots, a_n]$ be the n^{th} best rational approximation of θ.
The closest returns of the orbit $\{c_i := R_\theta^i(c)\}_{i \geq 0}$ back to any point $c \in S^1$ is:



Critical quasicircle maps

(uni-)critical quasicircle map $= \begin{cases} \text{analytic self homeomorphism } f \text{ of a quasicircle } \mathbf{H} \\ \text{with a unique critical point } c \text{ on } \mathbf{H} \end{cases}$

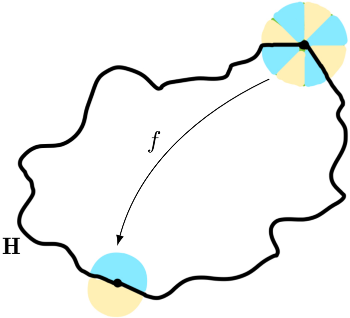
Theorem (Petersen '04)

When rot$(f|_{\mathbf{H}}) = \theta$ is irrational,

1. $f|_{\mathbf{H}}$ *has no wandering intervals and is conjugate to rigid rotation* $R_\theta : S^1 \to S^1$*;*
2. θ *is of bounded type iff the conjugacy* $\mathbf{H} \to S^1$ *extends to a QC map* $\mathbb{P}^1 \to \mathbb{P}^1$*.*

Inner & outer criticalities

Let d_0 = inner criticality of the critical point
and d_∞ = outer criticality.
The total local degree of the critical point is $d_0 + d_\infty - 1$.



Example of
$(d_0, d_\infty) = (2, 3)$

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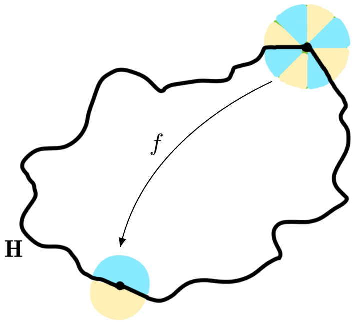

Example of
$(d_0, d_\infty) = (2, 3)$

When $d_0 = d_\infty$, examples can be found amongst critical circle maps.
E.g. when $(d_0, d_\infty) = (2, 2)$, we have the Arnold family:

$$f(x) = x + t - \frac{1}{2\pi} \sin(2\pi x), \quad x \in \mathbb{R}/\mathbb{Z}.$$

Realization of arbitrary criticalities

Fix a bounded type θ and a pair of integers $d_0 \geq 2$ and $d_\infty \geq 2$.

Theorem

There exists a unique degree $d_0 + d_\infty - 1$ rational map F such that

1. *F has critical fixed points at 0 and ∞ with local degrees d_0 and d_∞,*
2. *F has a critical point 1 with local degree $d_0 + d_\infty - 1$,*
3. *F has an invariant curve $\mathbf{H}$ passing through 1 and separates 0 and ∞;*
4. *$F : \mathbf{H} \to \mathbf{H}$ is a critical quasicircle map with rotation number θ.*

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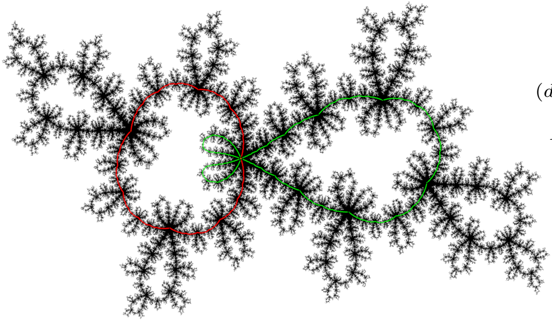

$\theta =$ golden mean

$(d_0, d_\infty) = (3, 2)$

$$F_{c_*}(z) = c_* z^3 \, \frac{4 - z}{1 - 4z + 6z^2}$$

$$c_* \approx -1.14421 - 0.96445i$$

Idea behind the proof

Consider a 1-par family of degree $d_0 + d_\infty - 1$ rational maps $\{F_m\}_{m>0}$ where

1. F_m has critical fixed points at 0 and ∞ with local degrees d_0 and d_∞,
2. F_m has a **Herman ring** $\mathbb{H}_m$ (rotation annulus) with rotation no. θ and modulus m;
3. $\mathbb{H}_m$ separates 0 and ∞;
4. the inner (resp. outer) boundary of $\mathbb{H}_m$ contains a critical point of local degree d_0 (resp. d_∞).

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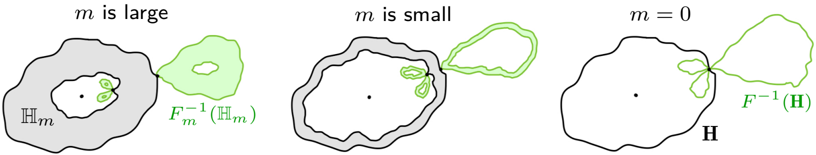
Theorem (A priori bounds)

$\partial\mathbb{H}_m$ are K-quasicircles, where K is independent of the conformal modulus m.



As $m \to 0$, $F = \lim_{m\to 0} F_m$ exists and has the desired invariant quasicircle $\mathbf{H} = \lim_{t\to 0} \overline{\mathbb{H}_m}$.

$C^{1+\alpha}$ Rigidity

Theorem

Given two critical quasicircle maps $f_1 : \mathbf{H}_1 \to \mathbf{H}_1$ *and* $f_2 : \mathbf{H}_2 \to \mathbf{H}_2$ *of the same criticalities* (d_0, d_∞) *and bounded type rotation number* θ*, there is a uniformly* $C^{1+\alpha}$ *conjugacy* $\phi : \mathbf{H}_1 \to \mathbf{H}_2$ *between* f_1 *and* f_2*.*

Corollary

Given a critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$*,*

1. $\dim(\mathbf{H})$ *is universal (depending only on* (d_0, d_∞) *and* θ*);*
2. $\mathbf{H}$ *is* C^1 *smooth iff* $\dim(\mathbf{H}) = 1$ *iff* $d_0 = d_\infty$*;*
3. *if* θ *is of periodic type,* $\mathbf{H}$ *is self-similar at the critical point with universal scaling.*

Renormalization

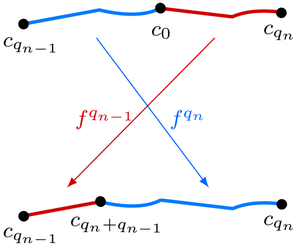
Fix $f : \mathbf{H} \to \mathbf{H}$ and let $\{c_i := f^i(c)\}_{i \geq 0}$ be the orbit of the critical point c of f.

The pre-renormalization $p\mathcal{R}^n f$ is the pair

$$\left(f^{q_n}|_{[c_{q_{n-1}}, c_0]}, f^{q_{n-1}}|_{[c_0, c_{q_n}]}\right)$$

which is the first return map of f back to the interval $[c_{q_{n-1}}, c_{q_n}] \subset \mathbf{H}$.

The renormalization $\mathcal{R}^n f$ is the normalized pair obtained by affine rescaling $c_{q_{n-1}} \mapsto -1$ and $c_0 \mapsto 0$.



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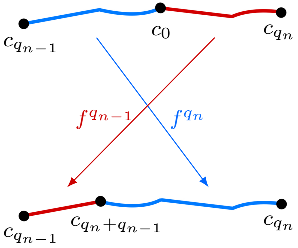

$\mathcal{R}$ acts on rotation number as the Gauss map:

$$\text{rot}(f) = \theta = [a_1, a_2, \dots] \quad \Longrightarrow \quad \text{rot}(\mathcal{R}^n f) = G^n \theta = [a_{n+1}, a_{n+2}, \dots].$$

Renormalization fixed point

Fix $\theta_* = [N, N, N, \ldots]$.

Corollary

There is a unique normalized pair ζ_* with

$$\text{rot}(\zeta_*) = \theta_* \quad \text{and} \quad \mathcal{R}\zeta_* = \zeta_*.$$

Given any critical quasicircle map $f : \mathbf{H} \to \mathbf{H}$ with rot$(f) = [???, N, N, N, \ldots]$,

$$\mathcal{R}^n f \longrightarrow \zeta_* \quad \text{exp. fast.}$$

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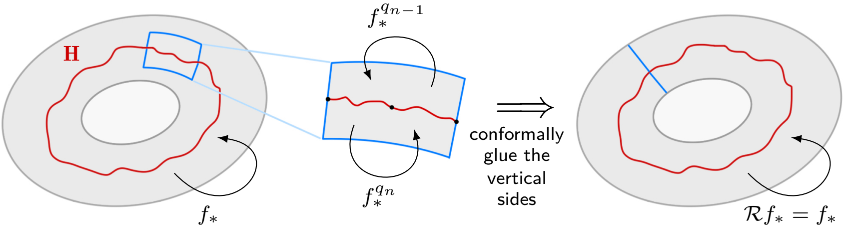
One can also glue the two ends of ζ_* to obtain a critical quasicircle map $f_* : \mathbf{H}_* \to \mathbf{H}_*$.



Hyperbolicity

Theorem

Consider a Banach neighborhood $\mathcal{B}$ of unicritical analytic maps on a neighborhood of $\mathbf{H}_*$ close to f_* in sup norm.

1. *$\mathcal{R}$ is a compact analytic operator on $\mathcal{B}$ with a unique fixed point f_* which is hyperbolic.*
2. $\mathcal{W}^s_{loc}(f_*) = \{g \in \mathcal{B} : g$ is a critical quasicircle map with rotation number $\theta_*\}$.
3. $\dim(\mathcal{W}^u_{loc}) = 1$.

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Corollary

Within the space of unicritical holomorphic maps on an annulus, the set of critical quasicircle maps with rotation number θ_* is an analytic submanifold of codimension ≤ 1.

Thank you!