Rigidity for Rotational Dynamics

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Invariant Line Field

Given a degree $d \geq 2$ rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, an **invariant line field** of f is a measurable Beltrami differential $\mu = \mu(z)\frac{d\bar{z}}{dz}$ on $\hat{\mathbb{C}}$ where

- $f^*\mu = \mu$ a.e.,
- $\text{supp}(\mu) =$ positive area subset of $J(f)$,
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Also, the conjecture implies...

Conjecture (Density of hyperbolicity)

Hyperbolic rational maps form a dense open subset of Rat_d.

Historical background 1

Consider a finitely generated Kleinian group $\Gamma < \mathrm{PSL}_2\mathbb{C}$.

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Somewhat related results:

Theorem (Sullivan '84, Tukia '84, Bishop-Jones '97)

$\dim(\Lambda(\Gamma)) < 2$ *if and only if Γ is geometrically finite.*

Theorem (Agol '04, Calegari-Gabai '06)

Either $\Lambda(\Gamma) = \hat{\mathbb{C}}$ or $\Lambda(\Gamma)$ has zero area.

Historical background 2

Similar results have been established, e.g.:

Theorem (McMullen '00)

If every critical point in $J(f)$ is pre-periodic (geometrically finite), then

$$\textit{either} \quad J(f) = \hat{\mathbb{C}} \quad \textit{or} \quad \dim(J(f)) < 2.$$

Theorem (Przytycki, Urbański '01)

If every critical point in $J(f)$ is non-recurrent, then

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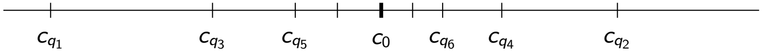
Qn: What happens when critical points are recurrent?
$\Rightarrow$ a common source of recurrence: **rotational dynamics**

Consider the rigid rotation

$$R_\theta : S^1 \to S^1, \quad z \mapsto e^{2\pi i\theta} z.$$

The closest returns of the orbit $\{c_i := R_\theta^i(c)\}_{i\geq 0}$ back to any point $c \in S^1$ are:



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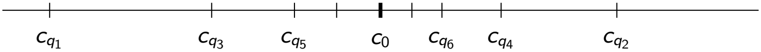


where p_n/q_n are the n^{th} best rational approximations of θ.

We say that an irrational number $\theta \in (0,1)$ is of **bounded type** if there is some $B \in \mathbb{N}$ such that $\sup_n a_n \leq B$ where

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}.$$

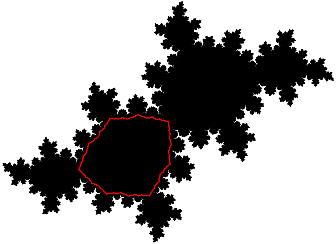
Then,

$$\text{bounded type} \quad \Longleftrightarrow \quad \log|c_{q_n} - c_0| \asymp -n.$$

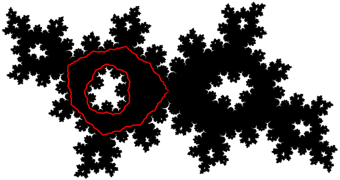
Rotation domains

Theorem (GF Zhang '11)

If D is a rotation domain of a rational map with bounded type rotation number, then every component of ∂D is a quasicircle containing a critical point.



$f(z) = z^2 + c$ where $c \approx -0.3905 - 0.5868i$



$f(z) = e^{2\pi it} z^2 \dfrac{z-4}{1-4z}$ where $t \approx 0.61517$

Rotation curves

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(1) X is a boundary component of a rotation domain, or

(2) X is not (1) (**Herman curve**) and contains both "inner" and "outer" critical points.

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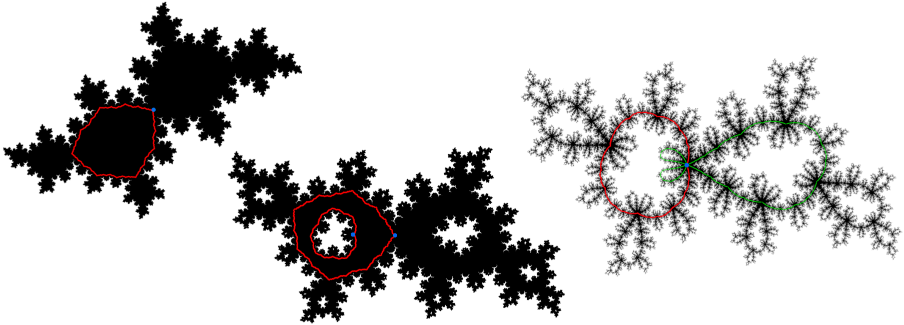

Note: all of the examples above are actually quasicircles too!

Rigidity of J-rotational rational maps

A rational map f is **J-rotational** if it admits bdd type rotation quasicircles $X_1, X_2, \ldots, X_k$ such that

$$P(f) \cap J(f) = \bigcup_{i=1}^{k} X_i \cup \{\text{finite set}\}.$$

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Theorem (L. '23)

Consider a J-rotational rational map f.

1. $J(f)$ *supports no invariant line field.*
2. *If f has no Herman curves,* $area(J(f)) = 0$.
3. *If f has no Herman curves and* $\{\text{finite set}\} = \emptyset$, *then* $\dim(J(f)) < 2$.

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2. *If f has no Herman curves, area$(J(f)) = 0$.*
3. *If f has no Herman curves and $\{\text{finite set}\} = \emptyset$, then dim$(J(f)) < 2$.*

Question: If $P(f) \cap J(f) =$ a single Herman curve, can $J(f)$ have positive area?
The complexity is similar to Feigenbaum Julia sets.

$$\text{critical quasicircle map} = \begin{cases} \text{holomorphic self homeomorphism } f \text{ of a quasicircle } X \\ \text{with a unique critical point on } X \end{cases}$$

Beyond the realm of rational maps

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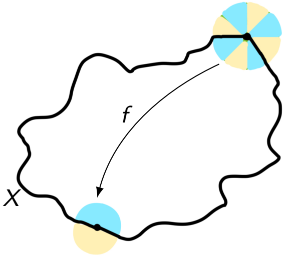
There are three obvious invariants:

- $\theta =$ rotation number,
- $d_0 =$ inner criticality of the critical point,
- $d_\infty =$ outer criticality of the critical point.



Example of $(d_0, d_\infty) = (2, 3)$

The total local degree of the critical point is $d_0 + d_\infty - 1$.

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Consider two critical quasicircle maps

$$f_1 : X_1 \to X_1 \quad \text{and} \quad f_2 : X_2 \to X_2$$

of the same criticalities (d_0, d_∞) and bounded type rotation number θ.

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One can adapt techniques for critical circle maps (de Faria-de Melo '99) as well as quasicritical circle maps (Avila-Lyubich '22) to prove:

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Moreover, due to our NILF Theorem and a deep point argument, we have:

Theorem (L. '23)

The conjugacy ϕ is $C^{1+\alpha}$ on X_1.

Consequences of $C^{1+\alpha}$ rigidity

Given a critical quasicircle map $f : X \to X$ with bdd type rotation number θ and criticalities (d_0, d_∞),

1. $\dim(X)$ is universal (depending only on θ, d_0, d_∞);
2. $d_0 = d_\infty \longleftrightarrow X$ is C^1 smooth $\longleftrightarrow \dim(X) = 1$;
3. if θ is a quadratic irrational, X is self-similar at the critical point with universal scaling factor;
4. *renormalizations* $\mathcal{R}^n f$ converge exponentially fast to a unique $\mathcal{R}$-invariant horseshoe attractor.

Thank you!