Critical quasicircle maps

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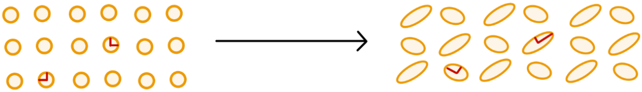
QC maps

A K-**quasiconformal** (QC) map $f\colon X \to X$ is an orientation-preserving homeomorphism of a Riemann surface X sending a (measurable) field of circles to a field of ellipses of eccentricity uniformly bounded above by $K \geq 1$.



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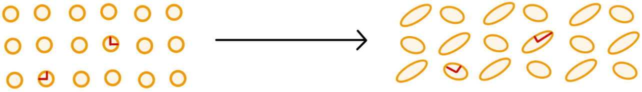
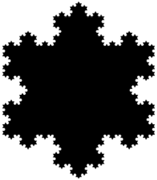
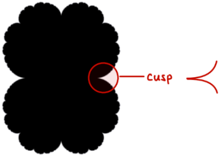


A K-**quasidisk** is the image of the unit disk $\mathbb{D}$ under a K-QC map on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Its boundary is called a K-**quasicircle**.





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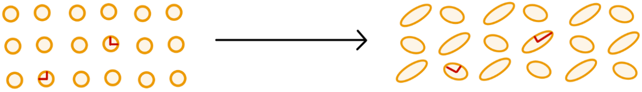

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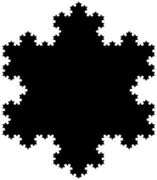

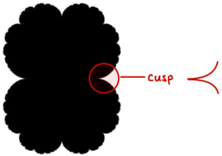

- Moduli spaces of Riemann surfaces can be described in terms of QC maps.
- The universal Teichmüller space can be described as the space of quasicircles.
- Quasicircles appear naturally in the study of Kleinian groups and rational maps.

Diophantine assumption

Fix an irrational $\theta \in (0,1)$ and write

$$\theta = [a_1, a_2, a_3, \dots] := \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}.$$

θ is called

- **bounded type** if $\sup a_n < \infty$.
- **periodic type** if $a_{n+p} = a_n$ for all n.

E.g. golden mean $= [1, 1, 1, \dots] = \frac{\sqrt{5}-1}{2}$

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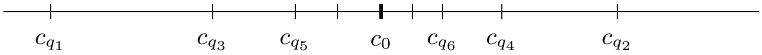
Consider the rigid rotation

$$R_\theta : S^1 \to S^1, \quad z \mapsto e^{2\pi i \theta} z.$$

Let $p_n/q_n = [a_1, \ldots, a_n]$ be the n^{th} best rational approximation of θ.
The closest returns of the orbit $\{c_i := R_\theta^i(c)\}_{i \geq 0}$ back to any point $c \in S^1$ is:



(uni-)critical quasicircle map $= \begin{cases} \text{analytic self homeomorphism } f \text{ of a quasicircle } X \\ \text{with a unique critical point } c \text{ on } X \end{cases}$

[1] Carsten Lunde Petersen. On holomorphic critical quasicircle maps. ETDS, 24(5):1739–1751, 2004.

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It follows from a result by Petersen[1] that if $f : X \to X$ has irrational rotation number θ,

1. X has no wandering intervals,
2. $f|_X$ is conjugate to rigid rotation $R_\theta : S^1 \to S^1$;
3. θ is of bounded type iff the conjugacy $X \to S^1$ extends to a QC map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

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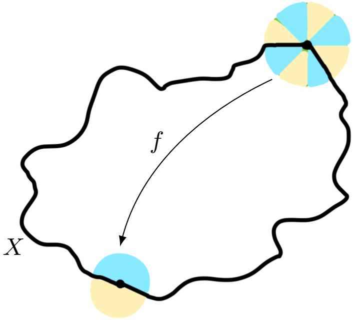
Inner & outer criticalities

Let $d_0 =$ inner criticality of the critical point
and $d_\infty =$ outer criticality.
The total local degree of the critical point is $d_0 + d_\infty - 1$.



Example of
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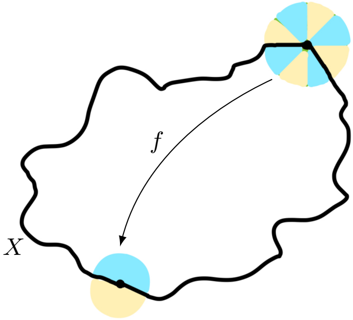

Example of
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Critical circle maps (when $X = S^1$) automatically have $d_0 = d_\infty$.
E.g. an example of $(d_0, d_\infty) = (2, 2)$ is the Arnold family:

$$A_t(x) = x + t - \frac{1}{2\pi}\sin(2\pi x), \quad x \in \mathbb{R}/\mathbb{Z}.$$

Realization of arbitrary criticalities

Fix a bounded type θ and a pair of integers $d_0 \geq 2$ and $d_\infty \geq 2$.

Theorem I: Realization

There exist a rational map $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and an invariant quasicircle X such that $F : X \to X$ is a (d_0, d_∞)-critical quasicircle map with rot. no. θ.

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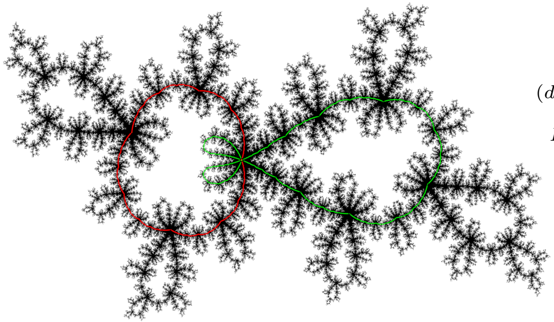

$\theta =$ golden mean

$$(d_0, d_\infty) = (3, 2)$$

$$F_{c_*}(z) = c_* z^3 \, \frac{4 - z}{1 - 4z + 6z^2}$$

$$c_* \approx -1.14421 - 0.96445i$$

Idea behind the proof

There exists a 1-par family of degree $d_0 + d_\infty - 1$ rational maps $\{F_m\}_{m>0}$ where

1. F_m has critical fixed points at 0 and ∞ with local degrees d_0 and d_∞,
2. F_m has a **Herman ring** $\mathbb{H}_m$ (rotation annulus) with rot. no. θ and modulus m;
3. $\mathbb{H}_m$ separates 0 and ∞;
4. the inner (resp. outer) boundary of $\mathbb{H}_m$ contains a critical point of local degree d_0 (resp. d_∞).

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As $m \to 0$, $F = \lim\limits_{m\to 0} F_m$ exists and has the desired invariant quasicircle $X = \lim\limits_{t\to 0} \overline{\mathbb{H}_m}$.

Consider two (d_0, d_∞)-critical quasicircle maps

$$f : X \to X \quad \text{and} \quad g : Y \to Y$$

with rot. no. θ. There's a unique conjugacy $\phi : X \to Y$ preserving the critical pts.

[2] Edson de Faria, Welington de Melo. Rigidity of critical circle mappings II. JAMS, 13:343–-370, 1999.

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Theorem II: Rigidity

ϕ extends to a QC conjugacy on a nbh of X. Also, $\phi|_X$ is $C^{1+\alpha}$-conformal.

In the special case $X = Y = S^1$, this was proven by de Faria and de Melo[2].

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Rigidity has many consequences, e.g.

1. H-dim$(X) =$ H-dim(Y);
2. H-dim$(X) = 1$ iff X is C^1-smooth iff $d_0 = d_\infty$;
3. if θ is of periodic type, X is self-similar at the crit. pt. with universal scaling const.

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Renormalization

To prove rigidity, we need the concept of renormalization!

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Fix $f : X \to X$ and let $\{c_i := f^i(c)\}_{i\geq 0}$ be the orbit of the critical point c of f.

The **$\mathbf{n^{th}}$ pre-renormalization** $p\mathcal{R}^n f$ is the pair

$$\left(f^{q_n}|_{[c_{q_{n-1}},c_0]}, f^{q_{n-1}}|_{[c_0,c_{q_n}]}\right)$$

which is the first return map of f back to the interval $[c_{q_{n-1}}, c_{q_n}] \subset X$.

The **$\mathbf{n^{th}}$ renormalization** $\mathcal{R}^n f$ is
the normalized pair obtained by affine rescaling
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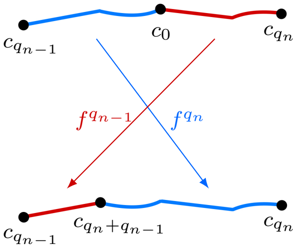

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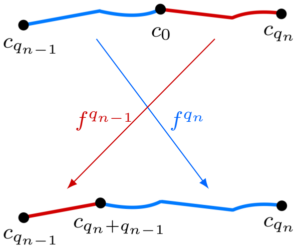

$\mathcal{R}$ acts on rotation number as the Gauss map:

$$\text{rot}(f) = \theta = [a_1, a_2, \ldots] \quad \Longrightarrow \quad \text{rot}(\mathcal{R}^n f) = G^n\theta = [a_{n+1}, a_{n+2}, \ldots].$$

Proof of $C^{1+\alpha}$ Rigidity

To construct a QC conjugacy ϕ on a nbh of X,

1. Obtain "complex bounds", i.e. uniform geometric control of domain of analyticity of $f^{q_n}, f^{q_{n-1}}$ for $n \gg 1$.
2. Construct QC conjugacy between $p\mathcal{R}^n f$ and $p\mathcal{R}^n g$ using complex bounds.
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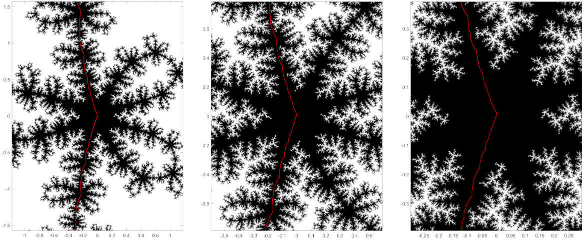

As we zoom in near the critical pt, J converges to the whole plane exp. fast.

Renormalization fixed point

Fix $\theta_* = [N, N, N, N, \ldots]$ (fixed type) and $\theta' = [\text{whatever}, N, N, N, N, \ldots]$ (pre-fixed).

Corollary

There is a unique normalized pair ζ_* with rot. no. θ_* satisfying

$$\mathcal{R}\zeta_* = \zeta_*.$$

Given any critical quasicircle map $f : X \to X$ with rot. no. θ',

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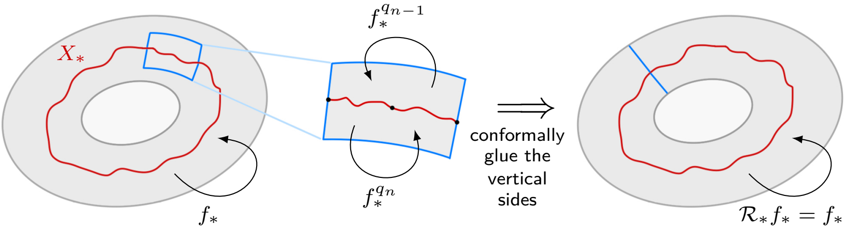
$$\mathcal{R}^n f \longrightarrow \zeta_* \quad \textit{exp. fast.}$$

One can also glue the two ends of ζ_* to obtain a critical quasicircle map $f_* : X_* \to X_*$ fixed by a renormalization operator $\mathcal{R}_*$:



Banach neighborhood

Given a critical quasicircle map $f : X \to X$, fix a small $\varepsilon > 0$ and a skinny annular nbh A of X, and define the Banach ball:

$$\mathcal{B}_\varepsilon(f) := \left\{ g \in \mathsf{Hol}(A, \mathbb{C}) \;\middle|\; g \text{ has a unique critical point and } \sup_{z \in A} |g(z) - f(z)| < \varepsilon \right\}.$$

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We will extend our renormalization operator $\mathcal{R}_*$ on a Banach nbh $\mathcal{B}_\varepsilon(f_*)$ in a natural way.

Hyperbolicity of renormalization

Theorem III: Hyperbolicity

$\mathcal{R}_*$ can be naturally extended to a compact analytic operator on $\mathcal{B}_\varepsilon(f_*)$ such that:

1. $\mathcal{R}_*$ has a unique fixed point f_*, which is hyperbolic.
2. $\mathcal{W}^s_{\mathrm{loc}}(f_*) = \{g \in \mathcal{B}_\varepsilon(f_*) \,|\, g$ is a critical quasicircle map with rot. no. $\theta_*\}$.
3. $\dim \mathcal{W}^u_{\mathrm{loc}}(f_*) = 1$.

In the circle case ($d_0 = d_\infty$), the real version of this was proven by Yampolsky[3].

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Corollary

Consider a critical quasicircle map $f : X \to X$ with preperiodic rot. no. θ'. Then,

$$S_\varepsilon(f) := \left\{ g \in \mathcal{B}_\varepsilon(f) \;\middle|\; \begin{array}{c} g \text{ has an invariant quasicircle } X_g \text{ on which} \\ g \text{ is a critical quasicircle map with rot. no. } \theta' \end{array} \right\}.$$

is an analytic submanifold of $\mathcal{B}_\varepsilon(f)$ of codimension ≤ 1. Moreover, X_g moves holomorphically in $g \in S_\varepsilon(f)$.

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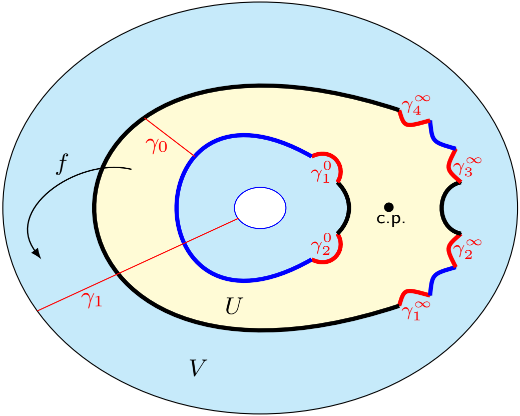
Key ingredient: Corona structure

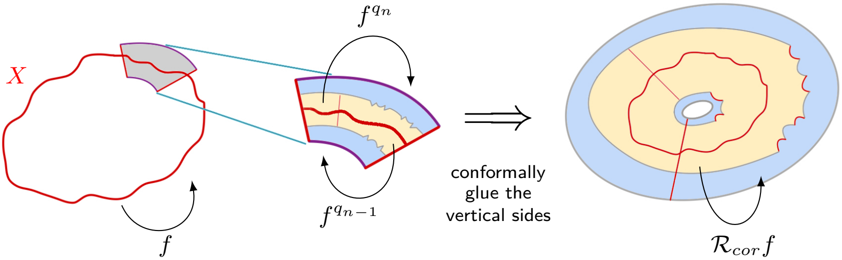
A **corona** is a holomorphic map $f : U \to V$ between nested annuli with radial arcs $\gamma_0 \subset U$ and $\gamma_1 \subset V$ such that $f : U \backslash \gamma_0 \to V \backslash \gamma_1$ is a covering map branched at a unique crit. pt.



A corona $f : (U, \gamma_0) \to (V, \gamma_1)$ with criticalities $(d_0, d_\infty) = (2, 3)$

Corona renormalization operator

Every critical quasicircle map $f : X \to X$ can be renormalized to a corona:



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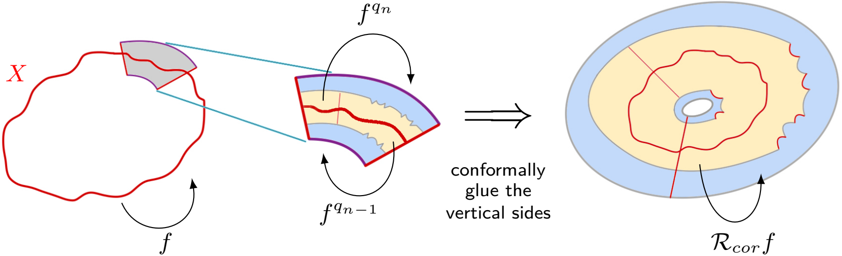

$\mathcal{R}_{cor}$ naturally extends to an analytic operator on $\mathcal{B}_\varepsilon(f)$.

Since $f_* : X_* \to X_*$ can be renormalized to itself, f_* admits a corona structure.
We extend $\mathcal{R}_* : f_* \mapsto f_*$ to an analytic renormalization operator on $\mathcal{B}_\varepsilon(f_*)$.

Most difficult part of the proof

With the corona framework, we can prove most of the theorem somewhat easily.

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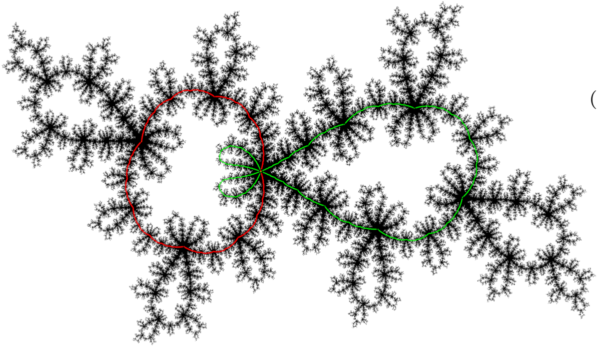
Idea:

1. Infinite anti-renormalization tower induces global transcendental dynamics.
2. Identify $\mathcal{W}^u_{\mathrm{loc}}$ with a parameter space of transcendental dynamical systems.
3. Study the rigidity properties of the escaping set of such transcendental maps.

Recall this example...



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$(d_0, d_\infty) = (3, 2)$

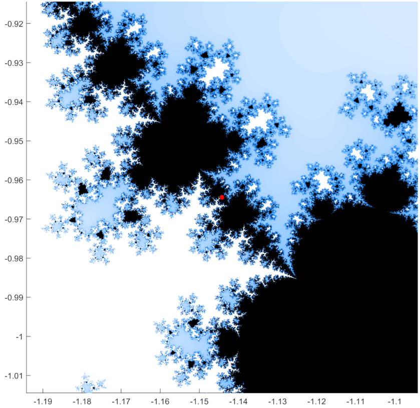
$$F_{c_*}(z) = c_* z^3 \, \frac{4-z}{1-4z+6z^2}$$

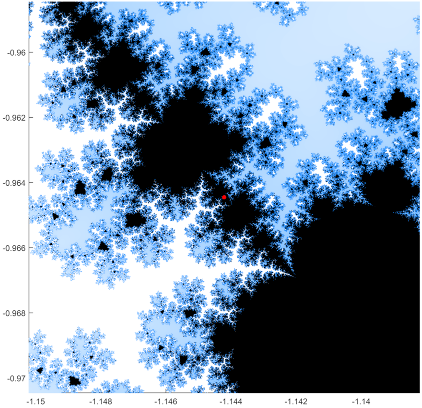
$c_* \approx -1.14421 - 0.96445i$

The map F_{c_*} naturally lives in the 1-parameter family

$$\left\{ F_c = cz^3 \, \frac{4-z}{1-4z+6z^2} \right\}_{c \in \mathbb{C}^*} .$$

The parameter space picture





Conjecture: The bifurcation locus of $\{F_c\}_{c\in\mathbb{C}^*}$ is self-similar at $c_\star$.

Rmk: Self-similarity of the Mandelbrot set at the Feigenbaum param. was proven by Lyubich[4].

[4] Mikhail Lyubich. Feigenbaum-Coullet-Tresser universality and Milnor's hairiness conjecture. Annals, 149(2):319–-420, 1999.

Thank you!