

Critical quasicircle maps

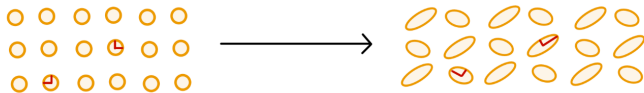
Willie Rush Lim

Brown University

Oct 7, 2024

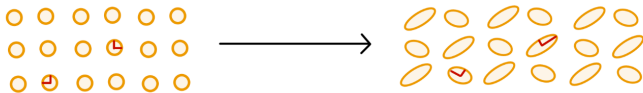
QC maps

A K -quasiconformal (QC) map $f: X \rightarrow X$ is an orientation-preserving homeomorphism of a Riemann surface X sending a (measurable) field of circles to a field of ellipses of eccentricity uniformly bounded above by $K \geq 1$.

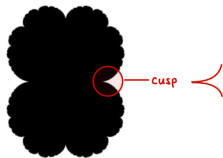


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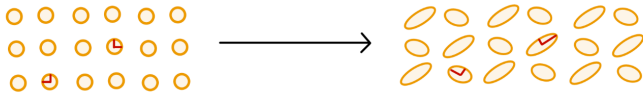


A K -QC map is the image of the unit disk D under a K -QC map on $\hat{C} = C \cup \{\infty\}$. Its boundary is called a K -QC boundary.

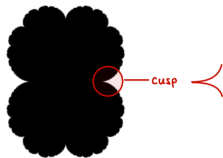


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A K -QC map is the image of the unit disk D under a K -QC map on $\hat{C} = C \cup \{\infty\}$. Its boundary is called a K -QC circle.



- Moduli spaces of Riemann surfaces can be described in terms of QC maps.
- The universal Teichmüller space can be described as the space of quasicircles.
- Quasicircles appear naturally in the study of Kleinian groups and rational maps.

Diophantine assumption

Fix an irrational $\alpha \in (0;1)$ and write

$$= [a_1; a_2; a_3; \dots] := \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

is called

- **bounded** if $\sup_n a_n < \infty$.
- **periodic** if $a_{n+p} = a_n$ for all n .

E.g. golden mean = $[1; 1; 1; \dots] = \frac{\sqrt{5}-1}{2}$

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- **eventually periodic** if $a_{n+p} = a_n$ for all n .

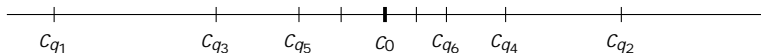
E.g. golden mean = $[1; 1; 1; \dots] = \frac{\sqrt{5}-1}{2}$

Consider the rigid rotation

$$R : S^1 \rightarrow S^1; z \mapsto e^{2\pi i \alpha} z$$

Let $p_n/q_n = [a_1; \dots; a_n]$ be the n^{th} best rational approximation of α .

The closest returns of the orbit $\{c_i := R^i(c) \mid i \geq 0\}$ back to any point $c \in S^1$ is:



$f^{-1}(c) = \{c\}$ (analytic self homeomorphism f of a quasicircle X with a unique critical point c on X)

¹Carsten Lunde Petersen. On holomorphic critical quasicircle maps. ETDS, 24(5):1739–1751, 2004.

Let $f: X \rightarrow X$ be an analytic self homeomorphism of a quasicircle X with a unique critical point c on X .

It follows from a result by Petersen¹ that if $f: X \rightarrow X$ has irrational rotation number α ,

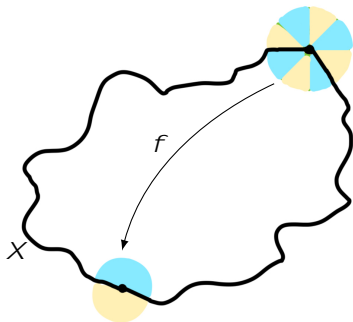
- ① X has no wandering intervals,
- ② $f|_X$ is conjugate to rigid rotation $R_\alpha: S^1 \rightarrow S^1$;
- ③ f is of bounded type iff the conjugacy $X \rightarrow S^1$ extends to a QC map $\hat{X} \rightarrow \hat{X}$.

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Inner & outer criticalities

Let d_0 = inner criticality of the critical point
and d_1 = outer criticality.

The total local degree of the critical point is $d_0 + d_1$ 1.

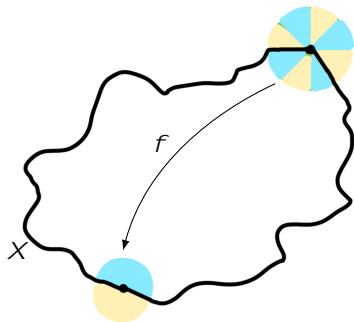


Example of
 $(d_0; d_1) = (2; 3)$

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and d_1 = outer criticality.

The total local degree of the critical point is $d_0 + d_1 = 1$.



Example of
 $(d_0; d_1) = (2; 3)$

Critical circle maps (when $X = S^1$) automatically have $d_0 = d_1$.
E.g. an example of $(d_0; d_1) = (2; 2)$ is the Arnold family:

$$A_t(x) = x + t \frac{1}{2} \sin(2x); \quad x \in \mathbb{R}/\mathbb{Z}$$

Realization of arbitrary criticalities

Fix a bounded type α and a pair of integers $d_0 \geq 2$ and $d_1 \geq 2$.

$\mathbb{P}^1 \rightarrow \mathbb{R}$ Realization

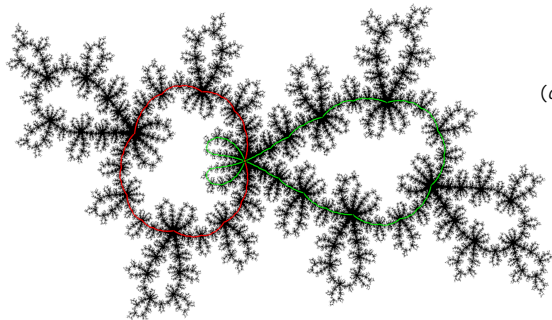
There exist a rational map $F : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ and an invariant quasicircle X such that $F : X \rightarrow X$ is a $(d_0; d_1)$ -critical quasicircle map with rot. no. α .

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$\alpha =$ golden mean

$$(d_0; d_1) = (3; 2)$$

$$F_c(z) = c z^3 \frac{4 - z}{1 - 4z + 6z^2}$$

$$c = 1.14421 - 0.96445i$$

Idea behind the proof

There exists a 1-par family of degree $d_0 + d_1 - 1$ rational maps f_m where

- 1 f_m has critical fixed points at 0 and 1 with local degrees d_0 and d_1 ,
- 2 f_m has a \mathbb{C}^* -invariant annulus H_m (rotation annulus) with rot. no. $\frac{1}{m}$ and modulus m ;
- 3 H_m separates 0 and 1;
- 4 the inner (resp. outer) boundary of H_m contains a critical point of local degree d_0 (resp. d_1).

Idea behind the proof

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Theorem (A priori bounds)

$@H_m - \text{C} \setminus K \setminus \text{O} \setminus \dots \setminus \text{S} \setminus \text{S} \setminus \text{K} \setminus \text{S} \setminus \text{S} \setminus \text{C} \setminus \text{C} \setminus \text{z} \setminus \text{b} \setminus \text{H} \setminus \text{m} \setminus \text{i}$

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- 2 F_m has a $\text{OC} \cap \text{L} H_m$ (rotation annulus) with rot. no. and modulus m ;
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Theorem (A priori bounds)

$\partial H_m \rightarrow \text{CK} \cap \text{L} \dots \text{PC} \cap \text{K} \text{ S S} @ \text{C} \wedge \text{C} \wedge \text{z} \text{ b H m i}$

As $m \neq 0$, $F = \bigcap_{m \neq 0} F_m$ exists and has the desired invariant quasicircle $X = \bigcap_{m \neq 0} \overline{H_m}$.

Consider two $(d_0; d_1)$ -critical quasicycle maps

$$f: X \rightarrow X \quad \text{and} \quad g: Y \rightarrow Y$$

with rot. no. α . There's a unique conjugacy $h: X \rightarrow Y$ preserving the critical pts.

²Edson de Faria, Welington de Melo. Rigidity of critical circle mappings II. JAMS, 13:343–370, 1999.

Rigidity

Consider two $(d_0; d_1)$ -critical quasidisk maps

$$f: X \rightarrow X \quad \text{and} \quad g: Y \rightarrow Y$$

with rot. no. θ . There's a unique conjugacy $h: X \rightarrow Y$ preserving the critical pts.

\mathbb{R} Rigidity

extends to a QC conjugacy on a nbh of X . Also, j_X is C^{1+} -conformal.

In the special case $X = Y = S^1$, this was proven by de Faria and de Melo².

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Rigidity has many consequences, e.g.

- 1 $H\text{-dim}(X) = H\text{-dim}(Y)$;
- 2 $H\text{-dim}(X) = 1$ iff X is C^1 -smooth iff $d_0 = d_1$;
- 3 if θ is of periodic type, X is self-similar at the crit. pt. with universal scaling const.

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Renormalization

To prove rigidity, we need the concept of renormalization!

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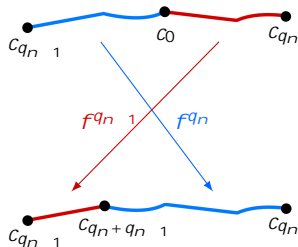
Fix $f: X \rightarrow X$ and let $\{c_i\} := \{f^i(c)\}_{i \geq 0}$ be the orbit of the critical point c of f .

The pair $(f^{q_n}|_{[c_{q_{n-1}}; c_0]}, f^{q_n-1}|_{[c_0; c_{q_n}]})$ is the pair

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which is the first return map of f back to the interval $[c_{q_{n-1}}; c_{q_n}] \subset X$.

The pair $(f^{q_n}|_{[c_{q_{n-1}}; c_0]}, f^{q_n-1}|_{[c_0; c_{q_n}]})$ is the normalized pair obtained by affine rescaling $c_{q_{n-1}} \mapsto 1$ and $c_0 \mapsto 0$.



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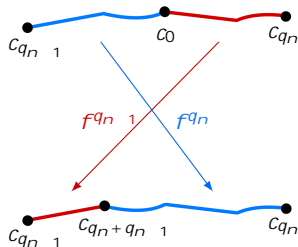
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R acts on rotation number as the Gauss map:

$$\text{rot}(f) = [a_1; a_2; \dots] \in \mathbb{E} \quad \text{rot}(R^n f) = G^n = [a_{n+1}; a_{n+2}; \dots]:$$

Proof of C^{1+} Rigidity

To construct a QC conjugacy on a nbh of X ,

- 1 Obtain “complex bounds”, i.e. uniform geometric control of domain of analyticity of $f^{q_n}; f^{q_{n-1}}$ for $n \geq 1$.
- 2 Construct QC conjugacy between $pR^n f$ and $pR^n g$ using complex bounds.
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- 2 Prove that points on X are uniformly deep in J :

As we zoom in near the critical pt, J converges to the whole plane exp. fast.

Renormalization fixed point

Fix $\tau = [N; N; N; N; \dots]$ (fixed type) and $\tau^0 = [\text{whatever}; N; N; N; N; \dots]$ (pre-fixed).

Corollary

$\tau \leq \tau^0$

$$\tau = \tau^0$$

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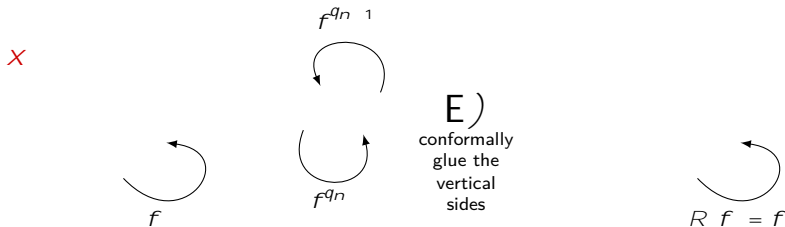
(Illegible text)

$$R = :$$

(Illegible text)

$$R^n f ! \quad \text{Ctei Hsz:}$$

One can also glue the two ends of θ to obtain a critical quasicircle map $f : X \rightarrow X$ fixed by a renormalization operator R :



Given a critical quasicircle map $f : X \rightarrow X$, fix a small $\epsilon > 0$ and a skinny annular nbh A of X , and define the Banach ball:

$$B_\epsilon(f) := \{ g \in \text{Hol}(A; \mathbb{C}) \mid g \text{ has a unique critical point and } \sup_{z \in A} |f(z) - g(z)| < \epsilon \}$$

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We will extend our renormalization operator R on a Banach nbh $B_\epsilon(f)$ in a natural way.

Hyperbolicity

R can be naturally extended to a compact analytic operator on $B^{\infty}(f)$ such that:

- 1 R has a unique fixed point f , which is hyperbolic.
- 2 $W_{\text{loc}}^s(f) = \{g \in B^{\infty}(f) \mid g \text{ is a critical quasicircle map with rot. no. } g\}$.
- 3 $\dim W_{\text{loc}}^u(f) = 1$.

In the circle case ($d_0 = d_1$), the real version of this was proven by Yampolsky³.

³Michael Yampolsky, Hyperbolicity of renormalization of critical circle maps. Publ. Math. IHES, 96:1–41, 2002.

Hyperbolicity of renormalization

Hyperbolicity

R can be naturally extended to a compact analytic operator on $B''(f)$ such that:

- 1 R has a unique fixed point f , which is hyperbolic.
- 2 $W_{loc}^s(f) = \{g \in B''(f) \mid g \text{ is a critical quasicycle map with rot. no. } \theta\}$
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Corollary

$S''(f) := \{g \in B''(f) \mid g \text{ has an invariant quasicycle } X_g \text{ on which } g \text{ is a critical quasicycle map with rot. no. } \theta\}$

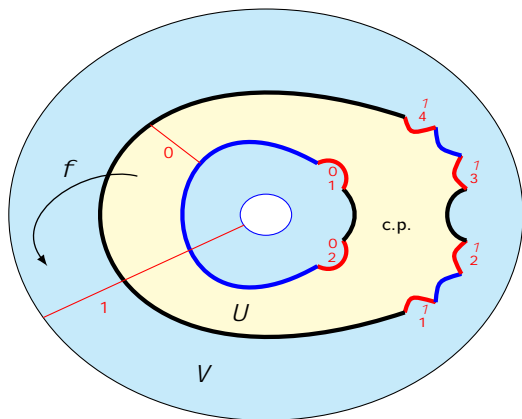
$S''(f) := \{g \in B''(f) \mid g \text{ has an invariant quasicycle } X_g \text{ on which } g \text{ is a critical quasicycle map with rot. no. } \theta\}$

$X_g \setminus \text{critical points} \setminus \text{critical values} \setminus \text{critical points} \setminus \text{critical values} \setminus \dots$

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Key ingredient: Corona structure

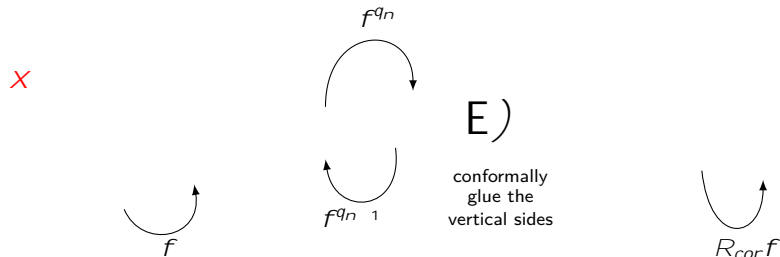
A $\langle \text{bc}p \rangle$ is a holomorphic map $f : U \rightarrow V$ between nested annuli with radial arcs $\gamma_0 \subset U$ and $\gamma_1 \subset V$ such that $f : U \rightarrow V$ is a covering map branched at a unique crit. pt.



A corona $f : (U; \gamma_0) \rightarrow (V; \gamma_1)$ with criticalities $(d_0; d_1) = (2; 3)$

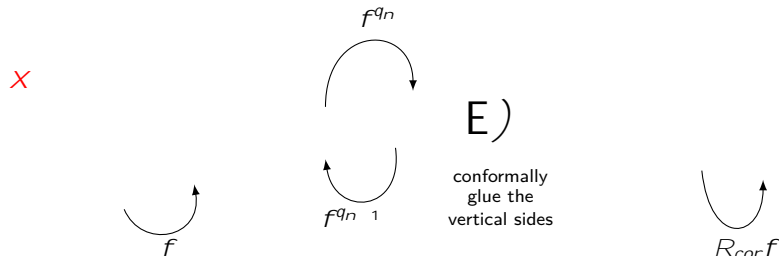
Corona renormalization operator

Every critical quasicircle map $f : X \rightarrow X$ can be renormalized to a corona:



Corona renormalization operator

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R_{cor} naturally extends to an analytic operator on $B''(f)$.

Since $f : X \rightarrow X$ can be renormalized to itself, f admits a corona structure. We extend $R : f \mapsto f$ to an analytic renormalization operator on $B''(f)$.

Most difficult part of the proof

With the corona framework, we can prove most of the theorem somewhat easily.

Most difficult part: there is only one unstable direction?

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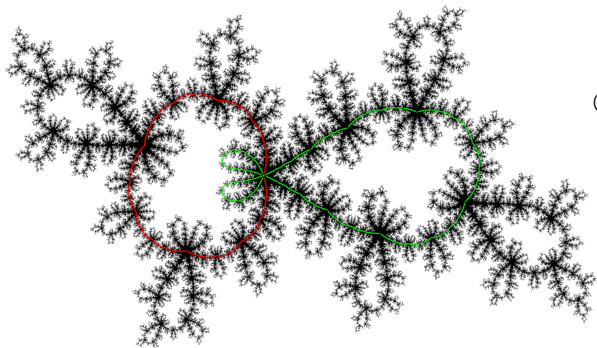
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Most difficult part: there is only one unstable direction?

Idea:

- 1 Infinite anti-renormalization tower induces global transcendental dynamics.
- 2 Identify W_{loc}^U with a parameter space of transcendental dynamical systems.
- 3 Study the rigidity properties of the escaping set of such transcendental maps.

Recall this example...



= golden mean

$$(d_0; d_1) = (3; 2)$$

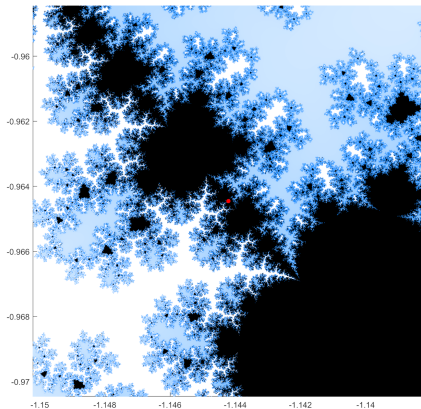
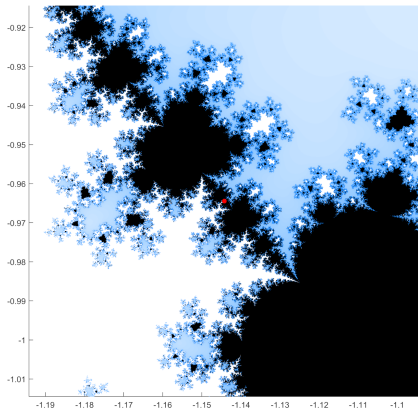
$$F_c(z) = c z^3 \frac{4z}{1 - 4z + 6z^2}$$

$$c = 1.14421 - 0.96445i$$

The map F_c naturally lives in the 1-parameter family

$$F_c = c z^3 \frac{4z}{1 - 4z + 6z^2} \quad ; \quad c \in \mathbb{C}$$

The parameter space picture



Conjecture: The bifurcation locus of $fF_{c_0}g_{c_2c}$ is self-similar at $c_?$.

Rmk: Self-similarity of the Mandelbrot set at the Feigenbaum param. was proven by Lyubich⁴.

⁴Mikhail Lyubich. Feigenbaum-Couillet-Tresser universality and Milnor's hairiness conjecture. *Annals*, 149(2):319–420, 1999.

Thank you!