Area of the postcritical set of neutral quadratic polynomials

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Consider the quadratic polynomial

$$f_\theta(z) = e^{2\pi i\theta} z + z^2.$$

It has:

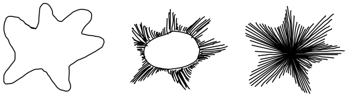
- a neutral fixed point at 0,
- a unique finite critical point at

$$c_0 := -e^{2\pi i\theta}/2.$$

The **postcritical set** is the closure of the critical orbit

$$P(f_\theta) := \overline{\{f_\theta^n(c_0)\}_{n\geq 1}}.$$

Depending on the arithmetics of θ, $P(f_\theta)$ can have rather complicated topology:



Theorem (wrl '25)

For any $\theta \in [0, 1)$, area $P(f_\theta) = 0$.

Corollary

Almost every point in $J(f_\theta)$ is non-recurrent.

Theorem (wrl '25)

For any $\theta \in [0, 1)$, area $P(f_\theta) = 0$.

If $\theta \in \mathbb{Q}$, this theorem is trivial.
Else, we can write

$$\theta := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \ldots}}}, \quad a_n \geq 1.$$

Previously, this theorem was known under different arithmetic conditions on θ.

Bounded type: $\sup_n a_n < \infty$

By Douady-Ghys surgery, $P(f_\theta)$ is a quasicircle and it's the boundary of the Siegel disk of f_θ.

Petersen-Zakeri Class: $\log a_n = O(\sqrt{n})$

By trans-qc surgery, $P(f_\theta)$ is a David Jordan curve and it's the boundary of the Siegel disk of f_θ.

High type: $\inf_n a_n \gg 1$

Cheraghi proved that area $P(f_\theta) = 0$ using Inou-Shishikura's near-parabolic theory.

Sector Renormalization

Pick an irrational $\theta \in (-\frac{1}{2}, \frac{1}{2})$. Consider

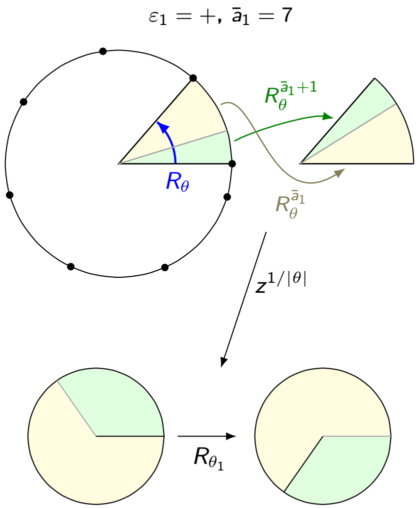
$$R_\theta(z) = e^{2\pi i\theta} z.$$

Define

- $\varepsilon_1 = \frac{\theta}{|\theta|}$, the orientation of R_θ,
- $\bar{a}_1 = \left\lfloor \frac{1}{|\theta|} \right\rfloor$, the first return time.
- $\theta_1 =$ the irrational in $(-\frac{1}{2}, \frac{1}{2})$ such that $\frac{1}{\theta} + \theta_1 \in \mathbb{Z}$.

The power map $z \mapsto z^{1/|\theta|}$ projects the 1$^{\text{st}}$ return map of R_θ on the shaded sector to a new rotation

$$R_{\theta_1} = \mathcal{R}_{\text{sec}}(R_\theta).$$



Sector Renormalization

Inductively, θ induces sequences $\{\varepsilon_n\}_{n\geq 1}$, $\{\bar{a}_n\}_{n\geq 1}$, and $\{\theta_n\}_{n\geq 1}$ where

- $\varepsilon_{n+1} =$ the orientation of R_{θ_n},
- $\bar{a}_{n+1} =$ the first return time of R_{θ_n},
- $R_{\theta_{n+1}} = \mathcal{R}_{\text{sec}}(R_{\theta_n})$.

This map is a homeomorphism

$$\left(-\frac{1}{2}, \frac{1}{2}\right) \setminus \mathbb{Q} \longleftrightarrow \Big(\{-,+\} \times \mathbb{N}_{\geq 2}\Big)^{\mathbb{N}}$$
$$\theta \longleftrightarrow \langle (\varepsilon_n, \bar{a}_n) \rangle_{n\geq 1}.$$

$\mathcal{R}_{\text{sec}}$ is conjugate to the shift map.

We can study the sector renormalizations of local neutral germs $f(z) = e^{2\pi i\theta} z + O(z^2)$.

Applications: lower bound on the size of the Siegel disk of f,
non-Brjuno = Cremer for quadratic polynomials [Yoccoz '95]

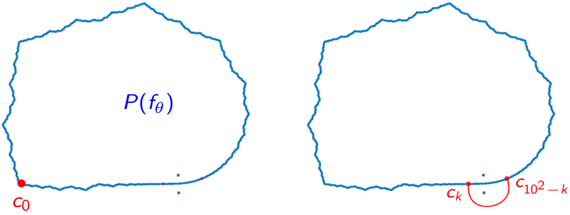
Can we define $\mathcal{R}_{\text{sec}}$ on quadratic polynomials so that it captures the critical orbit well?

Previous partial answers:

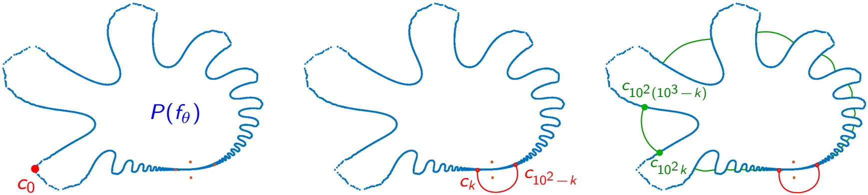
- Bounded type: Siegel-pacman renormalization [McMullen, Dudko-Lyubich-Selinger]
- High type: near-parabolic / cylinder renormalization [Inou-Shishikura]

There's a new unified approach: **pseudo-Siegel disks**

$$\theta = \langle (+, \mathbf{10^2}), (+, 2), (+, 2), (+, 2), \ldots \rangle$$



$$\theta = \langle (+, \mathbf{10^2}), (+, \mathbf{10^3}), (+, 2), (+, 2), \ldots \rangle$$



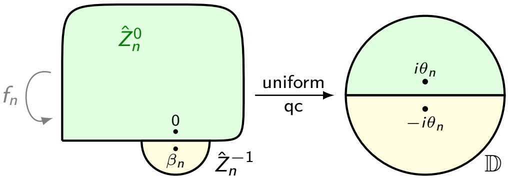
Dudko-Lyubich's Pseudo-Siegel Theory

For every $f = f_\theta$, it is possible to construct renormalizations

$$f_n = (\mathcal{R}_{\text{sec}})^n f : (\mathbb{D}, 0) \dashrightarrow (\mathbb{D}, 0)$$

such that for every $n \geq 0$,

1. f_n has two closed uniform quasidisks $\hat{Z}_n^0$ and $\hat{Z}_n^{-1}$ where
   - $P(f_n) \subset \hat{Z}_n^0 \subset \hat{Z}_n^{-1}$,
   - $\hat{Z}_n^{-1}$ is almost f_n–invariant,
   - $\hat{Z}_n^0$ is almost $f_n^{\bar{a}_{n+1}}$–invariant.
2. There's a universal constant $0 < r < \frac{1}{2}$ such that $\mathbb{D}_r \subset \hat{Z}_n^{-1} \subset \mathbb{D}_{1-r}$.
3. There's a universal constant $\mathbf{M} \in \mathbb{N}$ such that
   - if $\bar{a}_{n+1} \leq \mathbf{M}$, $\hat{Z}_n^0 = \hat{Z}_n^{-1}$,
   - if $\bar{a}_{n+1} > \mathbf{M}$, we can qc uniformize:

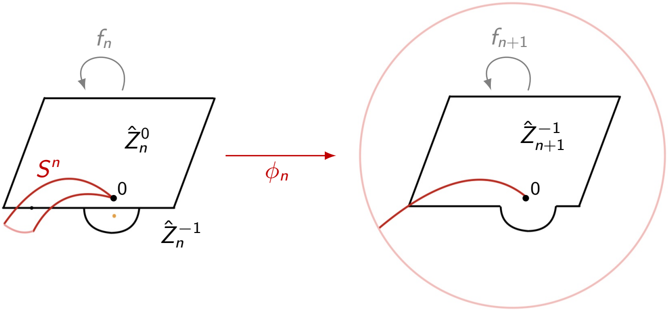


Dudko-Lyubich's Pseudo-Siegel Theory

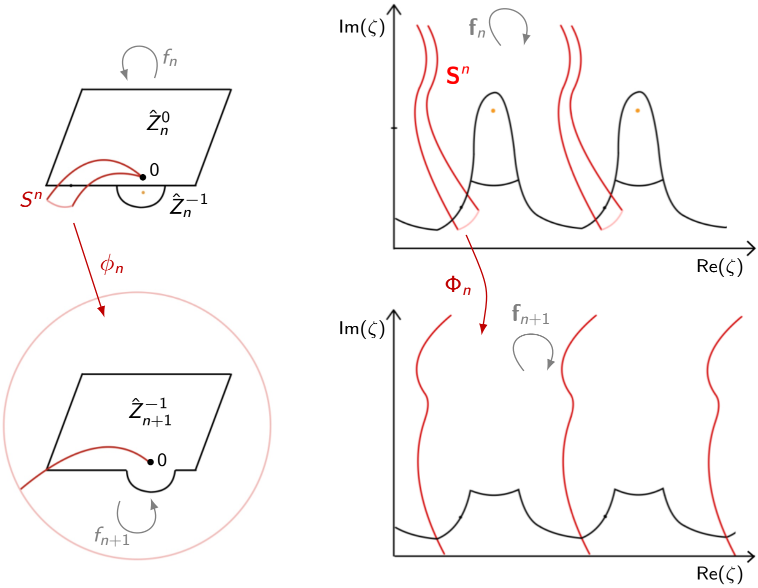
4. There's a sector S^n with vertex at 0 and adjacent edges γ_n and $f_n(\gamma_n)$. S^n contains the critical value of f_n. Gluing $\gamma_n \sim f_n(\gamma_n)$ gives a conformal map

$$\phi_n : S_n \to \mathbb{D}^*$$

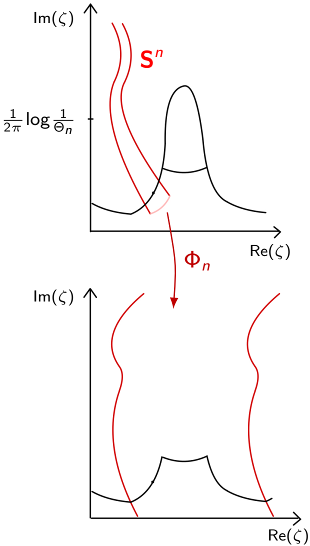
which projects the 1st return map of f_n into f_{n+1}, and $\hat{Z}_n^0$ into $\hat{Z}_{n+1}^{-1}$.



Lift via $\exp(\zeta) = e^{2\pi i \zeta}$ and bring it to logarithmic coordinates.



Main estimate in log coordinates



Write $\Theta_n = |\theta_n|$.

For any sufficiently high $y > 0$ and any $\zeta \in \mathbf{S}^n$,

$$\operatorname{Im}(\zeta) \geq \frac{1}{2\pi}\log\frac{1}{\Theta_n} + y$$

$$\Downarrow$$

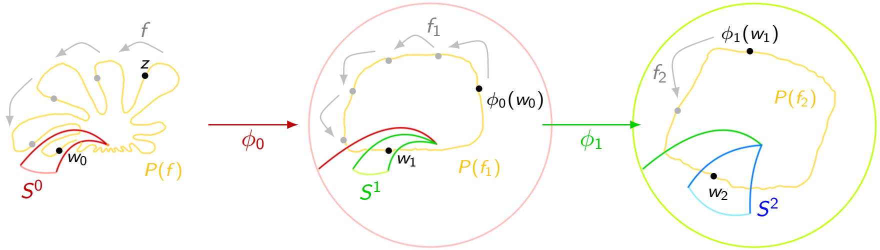
$$\left|\Theta_n\Phi_n'(\zeta) - 1\right| = O(y^{-2})$$

$$\left|\Theta_n\Phi_n(\zeta) - \zeta + \kappa\right| = O(y^{-1})$$

where $\kappa = \kappa(f_n) = \frac{i}{2\pi}\log\frac{1}{\Theta_n} + O(1)$.

Key Lemma

Every point $z \in P(f)$ induces a sequence of points $\{w_n \in S^n\}_{n \geq 0}$:



There's a universal constant $C > 0$ such that the set

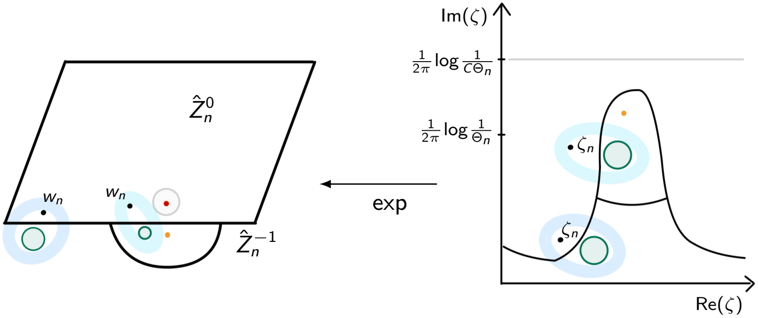
$$\{z \in P(f) \ : \ |w_n| \leq C\Theta_n \text{ for all } n\}$$

has zero area. If non-Brjuno, this set is the singleton $\{0\}$.

Non-uniform porosity elsewhere

Assume $|w_n| > C\Theta_n$ for some n. Bring it to log coordinates: $\exp(\zeta_n) = w_n$.

Use the qc uniformization of $(\hat{Z}_n^{-1}, \hat{Z}_n^0)$ to catch a hole in $\mathbb{D}\backslash\hat{Z}_n^0$ near w_n of definite size with annular buffer of definite modulus.



Then, lift to a hole of $P(f)$ near z.

What's next?

The $\mathcal{R}_{\text{sec}}$-orbits $(\mathcal{R}_{\text{sec}})^n(f_\theta)$ are precompact. They converge to an attractor $\mathcal{A}$.

Theorem (Dud-Lim-Lyu, in prep.)

For any two bi-infinite $\mathcal{R}_{\text{sec}}$-orbits $(f_n)_{n\in\mathbb{Z}}$ and $(g_n)_{n\in\mathbb{Z}}$ in $\mathcal{A}$,

$$f_n'(0) = g_n'(0) \text{ for all } n \quad \Longrightarrow \quad f_n \text{ and } g_n \text{ are conformally conjugate for all } n.$$

Hence, $\mathcal{A}/\sim$ is conjugate to a shift map on $\left(\{-,+\} \times \{2,3,\ldots,\infty\}\right)^{\mathbb{Z}}$.

The proof will use

- some transcendental dynamics,
- ideas from critical circle maps,
- some near-degenerate analysis (construct qc conjugacy),
- the zero area theorem (rule out invariant line field).

In progress: $\mathcal{A}$ is uniformly hyperbolic?

Some related questions

1. For all $c \in \mathbb{C}$, area $P(z^2 + c) = 0$?

   This is now reduced to infinitely ql-renormalizable without a priori bounds.

2. Hubbard's conjecture: $\inf_\theta$ area $K(f_\theta) > 0$.

   To this date, there are no examples of Cremer θ with area $K(f_\theta) = 0$?

Gràcies!

Gracias!

Thank you!

Terima kasih!