Renormalization theory of Herman curves

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Diophantine assumption

Fix an irrational $\theta \in (0,1)$ and write

$$\theta = [a_1, a_2, a_3, \ldots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \ldots}}}.$$

θ is called

- **bounded** if $\sup a_n < \infty$.
- **periodic** with period p if $a_{n+p} = a_n$ for all n.

E.g. golden mean $= [1, 1, 1, \ldots] = \frac{\sqrt{5}-1}{2}$

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The n^{th} rational approximation of θ is

$$\frac{p_n}{q_n} = [a_1, \ldots, a_n].$$

Rotation curves

An invariant Jordan curve $\mathbf{H} \subset \hat{\mathbb{C}}$ of a holomorphic map f is

- a **rotation curve** if $f|_{\mathbf{H}}$ is conjugate to an irrational rotation $R_\theta : \mathbb{T} \to \mathbb{T}$;
- a **Herman curve** if additionally it isn't contained in the closure of a rotation domain.

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Trichotomy: When a rotation curve $\mathbf{H}$ has bounded rotation number θ, there are 3 cases:

a. $\mathbf{H} =$ an analytic curve inside a rotation domain,

b. $\mathbf{H} =$ the boundary of a rotation domain containing a critical point of f,

h. $\mathbf{H} =$ a Herman curve containing inner and outer critical points of f.

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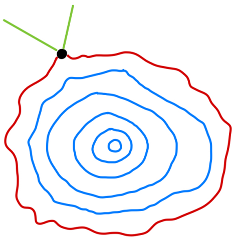
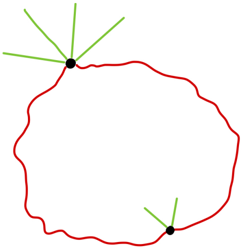




Motivation

Siegel disks and Herman rings have been pretty well studied by many many people.
On the other hand, not much is known about Herman curves.

Key questions

1. Regularity and smoothness of Herman curves?
2. Rigidity properties?
3. Regularity of conjugacy classes?
4. Structural instability?

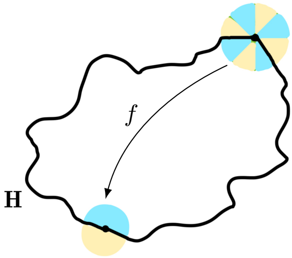
From now on, we will consider a Herman curve $f : \mathbf{H} \to \mathbf{H}$ with a single critical point c.

Inner & outer criticalities

Denote

$$d_0 = \text{ inner criticality of } c,$$
$$d_\infty = \text{ outer criticality of } c.$$

The total local degree of the critical point c is $d_0 + d_\infty - 1$.



Example of
$(d_0, d_\infty) = (2, 3)$

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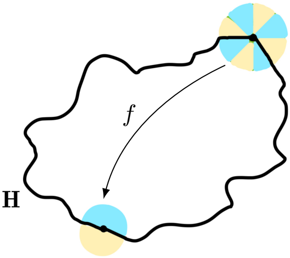


Example of
$(d_0, d_\infty) = (2, 3)$

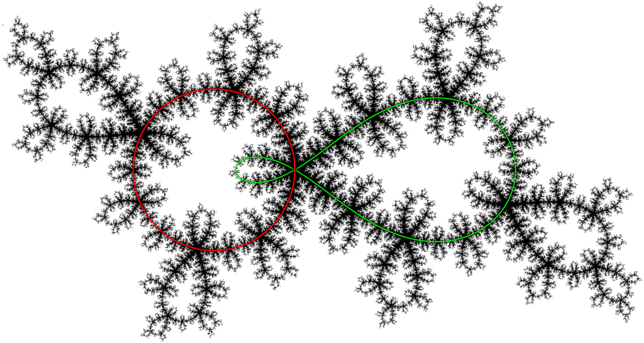
Critical circle maps (when $\mathbf{H} = S^1$) automatically have $d_0 = d_\infty$.
E.g. The Arnold family, $(d_0, d_\infty) = (2, 2)$:

$$A_t(x) = x + t - \frac{1}{2\pi}\sin(2\pi x), \quad x \in \mathbb{R}/\mathbb{Z}.$$

Blaschke product example, $d_0 = d_\infty = 2$

For any irrational θ, there is a unique $c_\theta \in \mathbb{T}$ such that the unit circle is a Herman curve of rotation number θ for the map

$$f_\theta(z) = c_\theta z^2 \frac{z-3}{1-3z}.$$



Can we generalize this?

Arbitrary criticalities

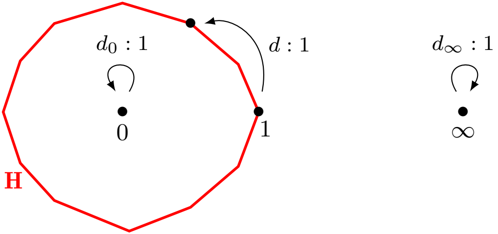
Fix a bounded irrational θ, a pair (d_0, d_∞), and $d := d_0 + d_\infty - 1$.

Realization+Uniqueness Theorem [wrl '23]

There exists a unique degree d rational map

$$F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

that has the critical portrait below and a Herman quasicircle $\mathbf{H}$ with rotation number θ.



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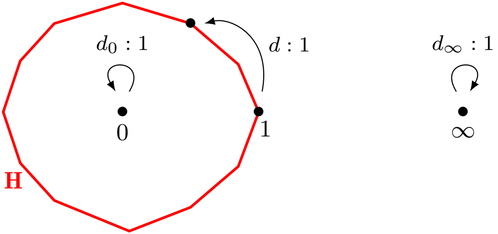
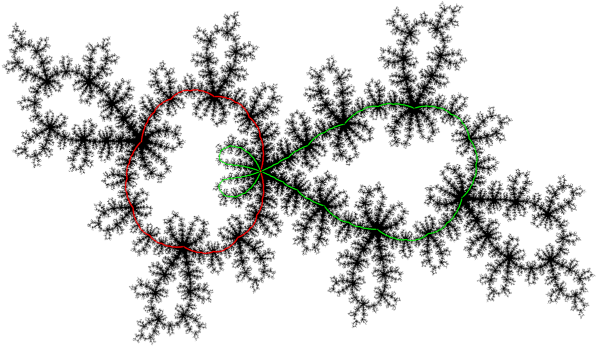


The realization follows from a priori bounds and degeneration of Herman rings.
The uniqueness follows from the absence of line fields in the Julia set.

Example for $(d_0, d_\infty) = (3, 2)$



$\theta =$ golden mean

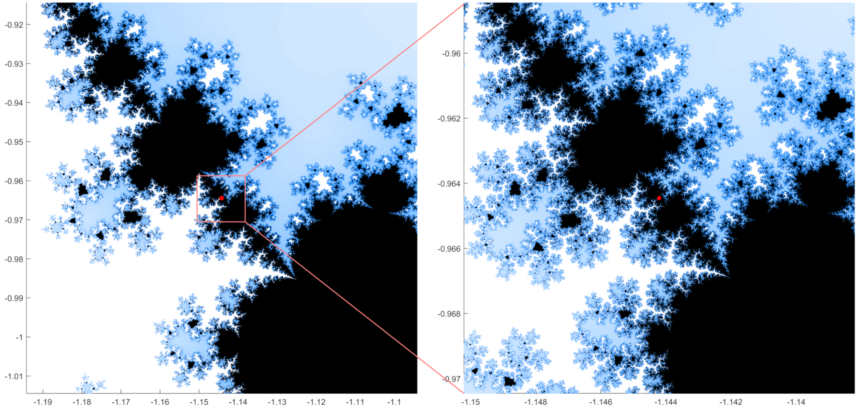
$$F_{c_*}(z) = c_* z^3 \; \frac{4 - z}{1 - 4z + 6z^2}$$

$$c_* \approx -1.14421 - 0.96445i$$

The map F_{c_*} naturally lives in the 1-parameter family

$$\left\{ F_c = cz^3 \; \frac{4 - z}{1 - 4z + 6z^2} \right\}_{c \in \mathbb{C}^*} .$$

The parameter space picture



Conjecture: The bifurcation locus of $\{F_c\}_{c\in\mathbb{C}^*}$ is asymptotically self-similar at $c_\star$.

From now on, fix integers $d_0, d_\infty \geq 2$. Denote the set

$$\mathsf{HC}_\theta = \left\{ (f, \mathbf{H}) \ : \ \begin{array}{l} f \text{ is a holomorphic map and } \mathbf{H} \text{ is a} \\ \text{unicritical Herman quasicircle of } f \text{ with} \\ \text{rotation number } \theta \text{ and criticalities } (d_0, d_\infty) \end{array} \right\}.$$

Going beyond rational maps

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Rigidity Theorem [wrl '23]

For any bounded type θ and any $(f, \mathbf{H})$ and $(\tilde{f}, \tilde{\mathbf{H}})$ in HC_θ,

1. there is a qc conjugacy ϕ between f and $\tilde{f}$ on a nbh of $\mathbf{H}$,
2. ϕ is $C^{1+\alpha}$-conformal on $\mathbf{H}$.

In the special case $\mathbf{H} = \tilde{\mathbf{H}} = S^1$, this was proven by de Faria-de Melo '00.

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Rigidity has many consequences, e.g.

1. $\dim(\mathbf{H})$ = universal constant;
2. $\dim(\mathbf{H}) = 1 \longleftrightarrow \mathbf{H}$ is C^1-smooth $\longleftrightarrow d_0 = d_\infty$;
3. if θ is pre-periodic, $\mathbf{H}$ is self-similar at the crit. pt. with universal scaling constant.

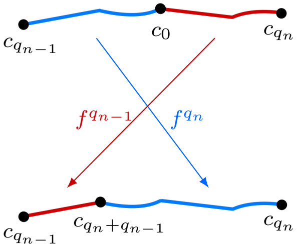
Denote $\{c_i := f^i(c)\}_{i\geq 0}$ = the critical orbit of f.

The **n^th pre-renormalization** $p\mathcal{R}^n f$ is the pair

$$\left(f^{q_n}|_{[c_{q_{n-1}},c_0]}, f^{q_{n-1}}|_{[c_0,c_{q_n}]}\right)$$

which is the first return map of f back to the interval $[c_{q_{n-1}}, c_{q_n}] \subset \mathbf{H}$.

The **n^th renormalization** $\mathcal{R}^n f$ is the normalized pair obtained by affine rescaling $c_{q_{n-1}} \mapsto -1$ and $c_0 \mapsto 0$.



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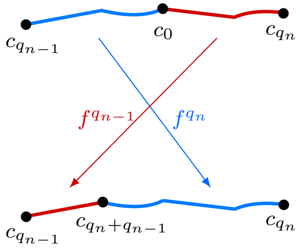
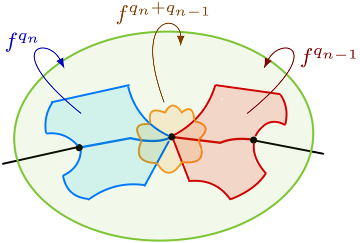


$\mathcal{R}$ acts on rotation number as the Gauss map:

$$\text{rot}(f) = \theta = [a_1, a_2, \ldots] \quad \Longrightarrow \quad \text{rot}(\mathcal{R}^n f) = G^n\theta = [a_{n+1}, a_{n+2}, \ldots].$$

1. Petersen '04: $f|_{\mathbf{H}} \sim_{\mathsf{qs}} \tilde{f}|_{\tilde{\mathbf{H}}}$
2. Prove uniform butterfly structure (complex bounds) for $p\mathcal{R}^n f$. $n \gg 1$.
3. Construct qc conjugacy between the butterflies of $p\mathcal{R}^n f$ and $p\mathcal{R}^n \tilde{f}$.
4. Spread this around to get a conjugacy ϕ on a nbh of $\mathbf{H}$.



Outline of proof of Rigidity

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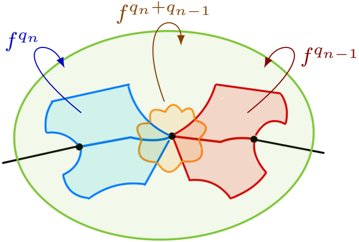
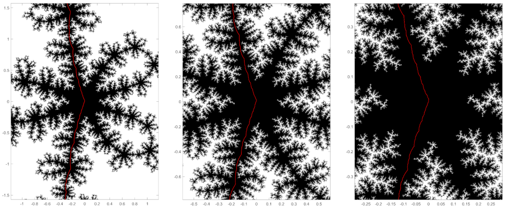


5. Show that $\bar{\partial}\phi = 0$ a.e. on $J_f = \overline{\cup_{k\geq 0} f^{-k}(\mathbf{H})}$.
6. Prove that points on $\mathbf{H}$ are "uniformly deep" in J_f:



As we zoom in near the critical pt, J_f converges to $\mathbb{C}$ exp. fast.

Renormalization fixed point

Fix a periodic irrational θ_* with some even period p.

Corollary

There is a unique normalized commuting pair ζ_* *with rot. no.* θ_* *satisfying* $\mathcal{R}^p\zeta_* = \zeta_*$*.*
For any $(f, \mathbf{H}) \in \mathsf{HC}_{\theta_*}$*,*

$$\mathcal{R}^{np} f \longrightarrow \zeta_* \quad \textit{exp. fast as } n \to \infty.$$

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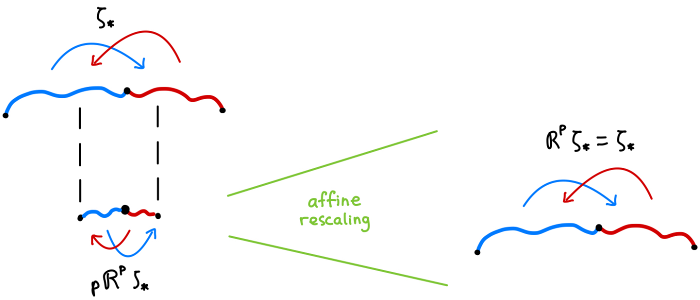

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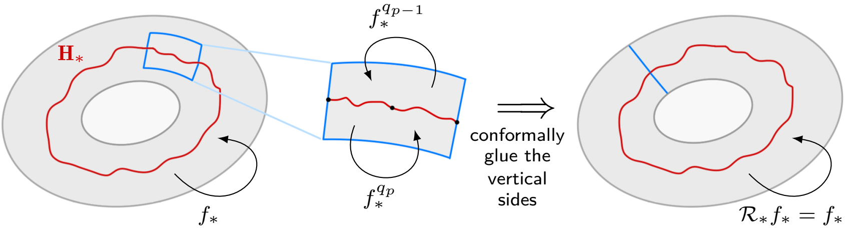
One can also glue the two ends of the commuting pair ζ_* to obtain a Herman quasicircle $f_* : \mathbf{H}_* \to \mathbf{H}_*$ fixed by a renormalization operator $\mathcal{R}_*$:



Hyperbolicity of renormalization

Fix a skinny annular nbh A of $\mathbf{H}_*$ and a small $\varepsilon > 0$. Define the Banach ball:

$$\mathcal{B}_\varepsilon(f_*) := \left\{ g \in \mathsf{Hol}(A, \mathbb{C}) \;\middle|\; g \text{ has a unique critical point and } \sup_{z \in A} |g(z) - f(z)| < \varepsilon \right\}.$$

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Hyperbolicity Theorem [wrl '24]

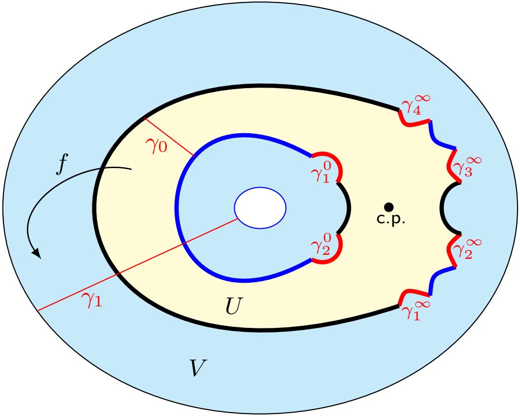
$\mathcal{R}_*$ can be naturally extended to a compact analytic operator on $\mathcal{B}_\varepsilon(f_*)$ such that:

1. f_* is the unique fixed point of $\mathcal{R}_*$.
2. f_* is hyperbolic with a single unstable direction.
3. $\mathcal{W}^s_{\mathrm{loc}}(f_*) = \{g \in \mathcal{B}_\varepsilon(f_*) \,|\, g \text{ has a Herman quasicircle with rot. no. } \theta_*\}$.

In the circle case ($d_0 = d_\infty$), the real version of this was proven by Yampolsky '03.

Key ingredient: Corona structure

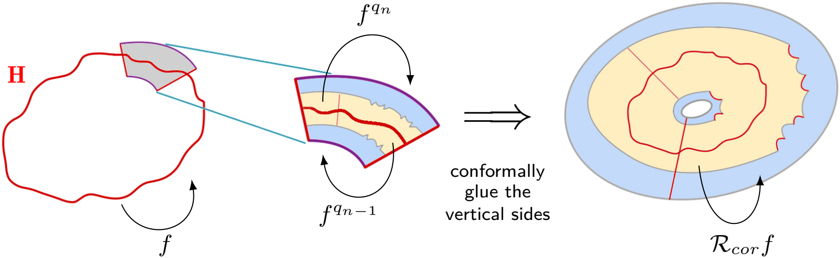
A **corona** is a holomorphic map $f : U \to V$ between nested annuli with radial arcs $\gamma_0 \subset U$ and $\gamma_1 \subset V$ such that $f : U \backslash \gamma_0 \to V \backslash \gamma_1$ is a covering map branched at a unique crit. pt.



A corona $f : (U, \gamma_0) \to (V, \gamma_1)$ with criticalities $(d_0, d_\infty) = (2, 3)$

Corona renormalization operator

Every unicritical Herman curve can be renormalized to a corona:



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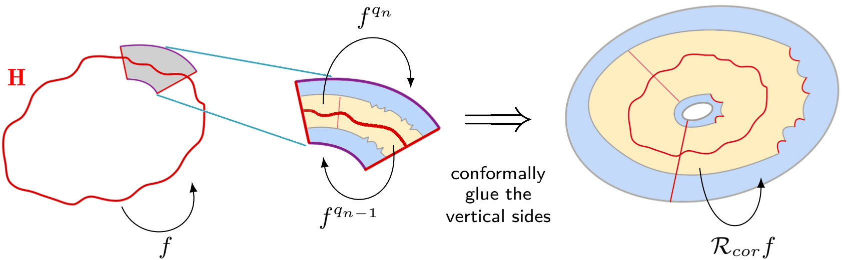

$\mathcal{R}_{cor}$ naturally extends to an analytic operator on a Banach ball $\mathcal{B}_\varepsilon(f)$.

Since $f_* : \mathbf{H}_* \to \mathbf{H}_*$ can be renormalized to itself, f_* admits a corona structure.
We extend $\mathcal{R}_* : f_* \mapsto f_*$ to an analytic renormalization operator on $\mathcal{B}_\varepsilon(f_*)$.

Most difficult part of the proof

With this corona framework, together with various soft methods[1], most of the hyperbolicity theorem can be proven.

Remaining obstacle: $\dim(\mathcal{W}^u_{\mathrm{loc}}) \leq 1$?

[1] holomorphic motions, renormalization tiling, Small Orbits Theorem, exponential convergence, etc

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Solution: $\mathcal{W}^u_{\mathrm{loc}} =$ a parameter space of transcendental maps of unknown dimension.

[1] holomorphic motions, renormalization tiling, Small Orbits Theorem, exponential convergence, etc

For the renormalization fixed point f_*,

$$\underset{\left\{ f_*^{q_{p(m+n)}} \right\}_{m \geq -n}}{\text{rescalings of}} \quad \xrightarrow[m\to\infty]{} \quad \underset{\left\{ \mathbf{F}_*^{Q_{pn}} : \mathsf{Dom}(\mathbf{F}_*^{Q_{pn}}) \to \mathbb{C} \right\}_{n\in\mathbb{Z}}}{\text{bi-infinite tower of commuting } \sigma\text{-proper maps}}$$

We can normalize this process so that 0 is the critical value.
$\left\{ \mathbf{F}_*^{Q_n} \right\}_{n\in\mathbb{Z}}$ generates a semigroup of transcendental maps $\mathbf{F}_* = (\mathbf{F}_*^P)_{P\in\mathbf{T}}$.

Transcendental dynamics

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For $f \in \mathcal{W}^u_{\mathrm{loc}}$,

$$\left\{\mathcal{R}^{-n} f\right\}_{n\leq 0} \xrightarrow[\text{rescaling}]{\text{appropriate}} \underset{\left\{\mathbf{F}^{Q_{pn}} : \mathrm{Dom}(\mathbf{F}^{Q_{pn}}) \to \mathbb{C}\right\}_{n\leq 0}}{\text{backward tower of commuting } \sigma\text{-proper maps}}$$

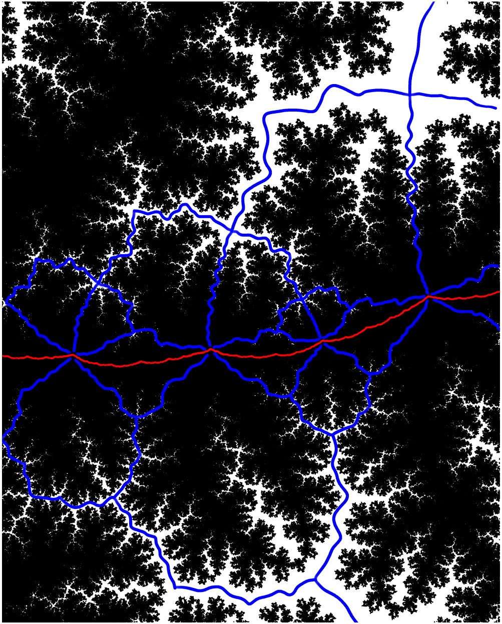
This still forms a semigroup of transcendental maps $\mathbf{F} = (\mathbf{F}^P)_{P\in\mathbf{T}}$.

Dynamical sets for cascades

For $f \in \mathcal{W}^u$, define...

- Fatou set:
$$\mathfrak{F}(\mathbf{F}) = \text{points of normality of } \left(\mathbf{F}^P\right)_{P \in \mathbf{T}}$$
- Julia set:
$$\mathfrak{J}(\mathbf{F}) = \mathbb{C} \backslash \mathfrak{F}(\mathbf{F})$$
- postcritical set:
$$\mathfrak{P}(\mathbf{F}) = \text{ closure of the critical orbit } \left(\mathbf{F}^P(0)\right)_{P \in \mathbf{T}}$$
- finite-time escaping set:
$$\mathbf{I}_{<\infty}(\mathbf{F}) = \bigcup_{P \in \mathbf{T}} \mathbb{C} \backslash \mathrm{Dom}\left(\mathbf{F}^P\right)$$
- infinite-time escaping set:
$$\mathbf{I}_{\infty}(\mathbf{F}) = \text{ points } x \text{ where } \mathbf{F}^P(x) \to \infty \text{ as } P \to \infty$$
- full escaping set:
$$\mathbf{I}(\mathbf{F}) = \mathbf{I}_{<\infty}(\mathbf{F}) \cup \mathbf{I}_{\infty}(\mathbf{F}).$$

Approximate dynamical picture of $\mathbf{F}_*$



In blue:
Some rays in $\mathbf{I}_{<\infty}(\mathbf{F}_*)$
landing at critical points of $\mathbf{F}_*$

$\mathfrak{P}(\mathbf{F}_*)$

$\mathcal{W}^u$ is one-dimensional

Rigidity Theorem for $\mathcal{W}^u$

For $f \in \mathcal{W}^n_{\text{loc}}$,

1. $\mathbf{I}(\mathbf{F})$ supports no invariant line field & moves conformally away from pre-critical pts.
2. If $\mathbf{F}$ is hyperbolic, then $\mathfrak{J}(\mathbf{F})$ also supports no invariant line field.

At last,

$$\begin{array}{c}\exists \text{ hyperbolic component} \\ O \subset \mathcal{W}^u_{\text{loc}} \text{ near } f_* \end{array} \xRightarrow[\text{above}]{\text{theorem}} \dim(O) \leq \# \text{ free critical orbits} = 1.$$

Structure of the local conjugacy class

Corollary

Consider any pre-periodic irrational θ' and any $(f, \mathbf{H}) \in \mathsf{HC}_{\theta'}$. The local conjugacy class

$$\left\{ g \in \mathcal{B}_\varepsilon(f) \;\middle|\; g \text{ has a Herman quasicircle } \mathbf{H}_g \text{ with rot. no. } \theta' \right\}$$

is an analytic submanifold of $\mathcal{B}_\varepsilon(f)$ of codim ≤ 1 on which $\mathbf{H}_g$ moves holomorphically.

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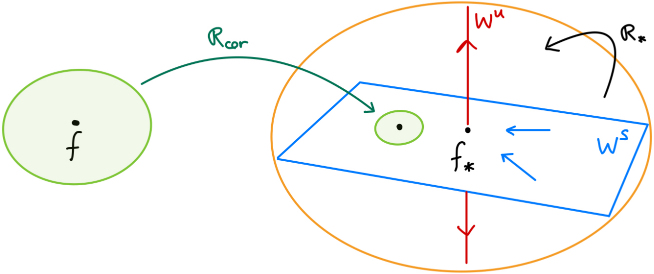

Development of renormalization theory in complex dynamics

real renorm. horseshoe	∞-renorm unimodal maps	critical circle maps
beau bounds	Sullivan, Lyubich-Yampolsky, Levin-van Strien	de Faria, Yampolsky, de Faria-de Melo
exp contraction	McMullen, Avila-Lyubich	de Faria-de Melo
hyperbolicity	Lyubich Avila-Lyubich	Yampolsky

renorm. fixed point	∞-renorm PL maps	Siegel disks	Herman curves
complex a priori bounds	Kahn, Dudko-Lyubich	McMullen, Avila-Lyubich	Lim
$C^{1+\alpha}$-rigidity/ exp contraction	McMullen, Lyubich	McMullen, Avila-Lyubich	Lim
hyperbolicity	Lyubich	Gaidashev-Yampolsky, Dudko-Lyubich-Selinger	Lim

The parameter picture

features	**∞-renorm. PL maps**	**Siegel disks**	**Herman curves**
nice family	$z^d + c$	$z^d + c$	$-c\,\dfrac{\sum_{j=d_0}^{d}\binom{d}{j}\cdot(-z)^j}{\sum_{j=0}^{d_0-1}\binom{d}{j}\cdot(-z)^j}$
$\mathcal{R}$-invariant lamination	hybrid classes	level sets of multiplier	?
structural instability	yes	yes	partially known
parameter self-similarity	complete	partially known	?

Thank you!