A priori bounds via totally degenerate regime

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Quadratic polynomials

Every quadratic polynomial over $\mathbb{C}$ is affinely conjugate to a unique map of the form

$$f_c(z) = z^2 + c.$$

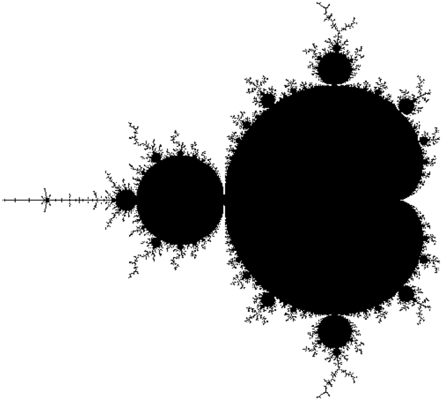
- Filled Julia (FJ) set of f_c:

$$K(f_c) = \{z \in \mathbb{C} \,:\, f_c^n(z) \not\to \infty \text{ as } n \to \infty\}$$

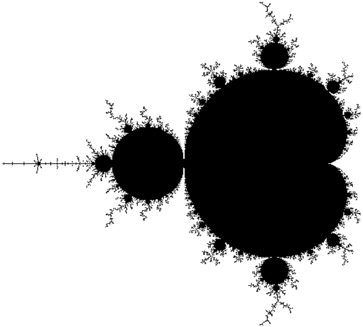
- Mandelbrot set:

$$\begin{aligned}\mathbb{M} &= \{c \in \mathbb{C} \,:\, 0 \in K(f_c)\} \\ &= \{c \in \mathbb{C} \,:\, K(f_c) \text{ is connected}\}\end{aligned}$$

The Mandelbrot set



MLC



MLC Conjecture: $\mathbb{M}$ is locally connected.

QL maps

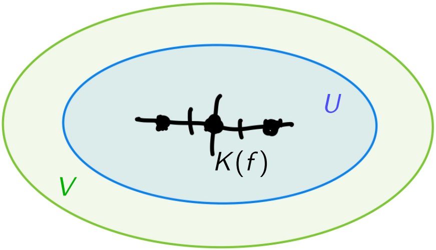
A **quadratic-like (QL) map** is a holomorphic double branched covering map

$$f : U \to V$$

between nested disks $U \Subset V$ such that its FJ set

$$K(f) := \{z \ : \ f^n(z) \in U \text{ for all } n \geq 1\},$$

is connected.



Straightening

Theorem (Douady-Hubbard '84)

For every QL map $f : U \to V$, there exists

- *a unique $c = c(f)$ in $\mathbb{M}$ and*
- *a quasiconformal map $\phi : U \to \mathbb{C}$ with $\bar{\partial}\phi = 0$ on $K(f)$*

such that

$$\phi \circ f = f_c \circ \phi.$$

This theorem defines the **straightening map**

$$S : \{\text{QL maps}\} \to \mathbb{M}, \quad f \mapsto c(f).$$

Renormalization

A quadratic(-like) map f is called **renormalizable** with period $p \geq 2$ if there exist disks $A \Subset B$ containing the critical point of f such that

$$f^p : A \to B$$

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Denote:
- $K(0) =$ FJ set of $f^p : A \to B$,
- $K(n) = f^n K(0)$.

$$K(0) \underset{f}{\longrightarrow} K(1) \underset{f}{\longrightarrow} K(2) \underset{f}{\longrightarrow} \dots \underset{f}{\longrightarrow} K(p-1) \underset{f}{\longrightarrow} K(0).$$

The renormalization is called

- **primitive** if $K(i)$'s are pairwise disjoint,
- **satellite** if $K(i)$ intersects $K(0)$ for some $i \neq 0$.

Baby Mandelbrot sets

Theorem (Douady-Hubbard '84)

If f_{c_*} is renormalizable with period p, then there is a subset $M \subset \mathbb{M}$ containing c_* such that

- *for all $c \in M$, f_c is renormalizable with period p,*
- *the straightening map is a homeomorphism onto $\mathbb{M}$:*

$$S_M : M \to \mathbb{M}, \quad c \mapsto S(f_c^p).$$

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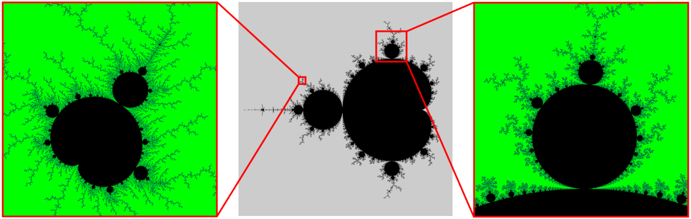
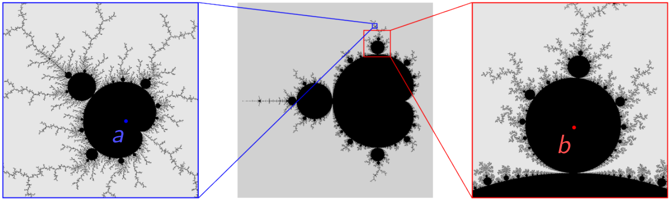
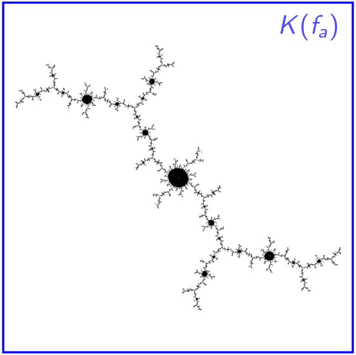
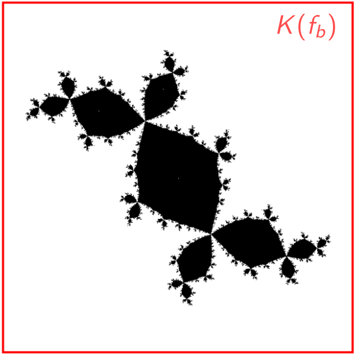


primitive copy　　　　satellite copy

Primitive copy vs Satellite copy







MLC becomes a renormalization problem

$c \in \mathbb{M}$ is **infinitely renormalizable** if it's contained in an infinite nest of baby Mandelbrot copies

$$c \in \ldots \subsetneq M_3 \subsetneq M_2 \subsetneq M_1 \subsetneq \mathbb{M}.$$

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Theorem (Yoccoz '90s)

If c is not infinitely renormalizable, then $\mathbb{M}$ is locally connected at c.

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If c is not infinitely renormalizable, then $\mathbb{M}$ is locally connected at c.

MLC is equivalent to Combinatorial Rigidity:

"Every infinite nest of baby Mandelbrot copies shrinks to a point."

A priori bounds

Suppose f_c is ∞ renormalizable with periods $p_1 < p_2 < p_3 < \dots$.

We say f_c has **a priori bounds** if for all $n \geq 1$, there exists an n^{th} renormalization $f_c^{p_n} : U_n \to V_n$ such that

$$\sup_{n \geq 1} W(V_n \setminus U_n) < \infty.$$

This means that up to affine rescaling, $\left\{ f_c^{p_n} : U_n \to V_n \right\}_n$ is pre-compact.

Main Theorem

Theorem

If $f_c \in \mathbb{M}$ *is* ∞ *renormalizable with bdd combinatorics* $\left(\sup_n \frac{p_{n+1}}{p_n} < \infty\right)$,

then (1) f_c *has a priori bounds,*

(2) $\mathbb{M}$ *is locally connected at* c.

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[Kahn '06] proved (1) assuming every renormalization is primitive.
[Dudko-Lyubich '23] proved the satellite and the general mixed case of (1).
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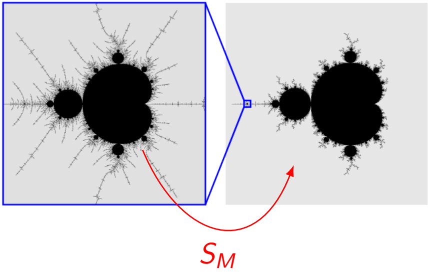
I will explain new alternative proofs of (1) using "totally degenerate regime" and (2) using Teichmüller's theorem. This is all joint work with Jeremy Kahn.

Example: stationary airplane combinatorics

Fix $M =$ maximal baby Mandelbrot copy containing -1.755.



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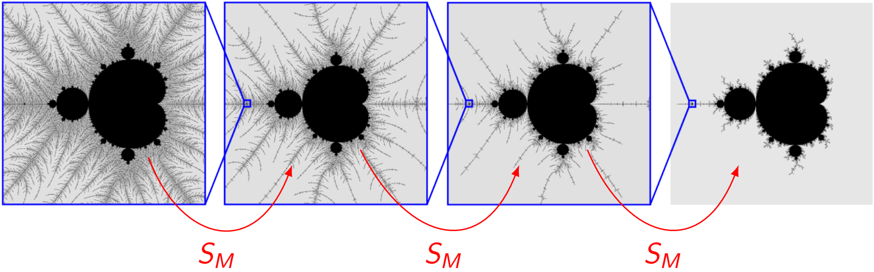
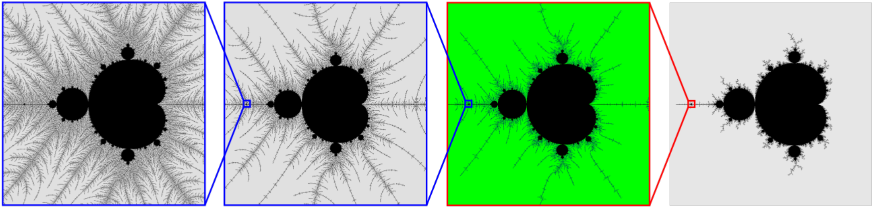
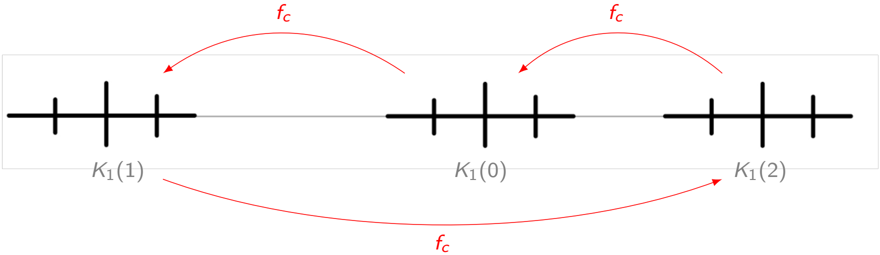
Suppose our infinitely renormalizable parameter c lies in

$$\bigcap_{n \geq 1} (S_M)^{-n}(\mathbb{M}).$$

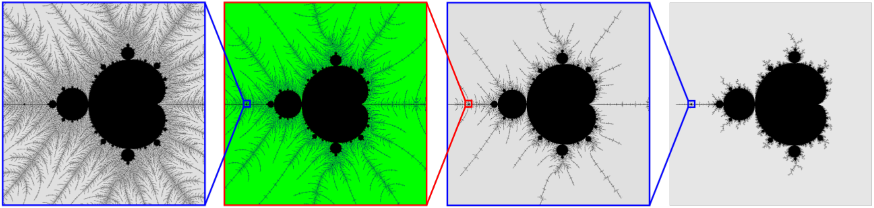


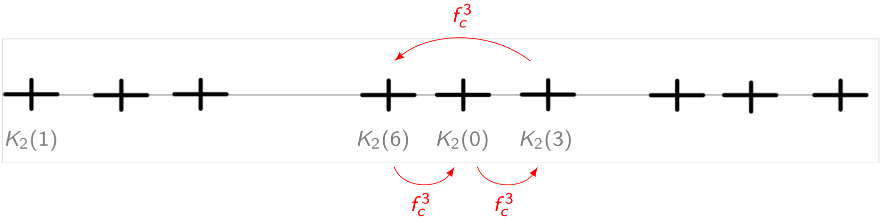
1^{st} renormalization, $p_1 = 3$



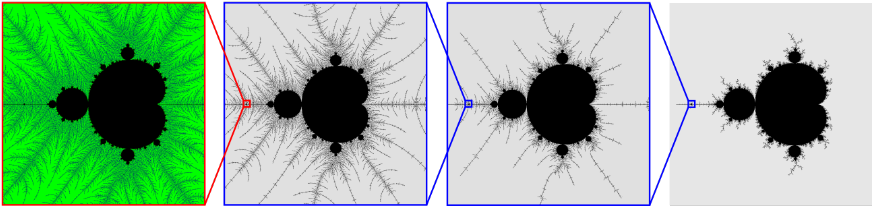


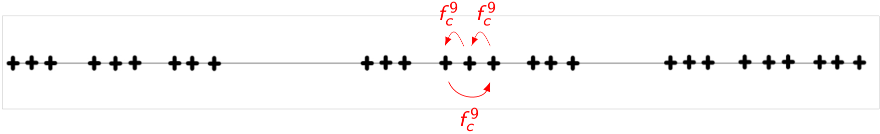
2^{nd} renormalization, $p_2 = 3^2$





3^{rd} renormalization, $p_3 = 3^3$





Hyperbolic geodesics

Denote:

$$\mathcal{K}_n := \bigcup_{i=0}^{3^n-1} K_n(i)$$

$$\gamma_n(i) := \text{ hyp geodesic of } \mathbb{C}\backslash\mathcal{K}_n \text{ going around } K_n(i)$$

$$\lambda_n := \pi \cdot \text{length}_{\text{hyp}}\gamma_n(0).$$

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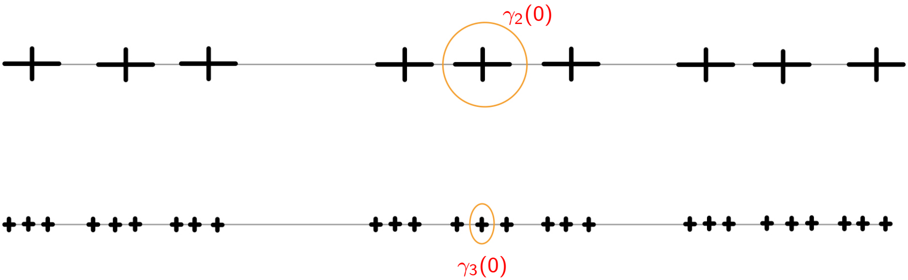

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Properties:

- comparability: for all i,

$$\frac{\lambda_n}{2} \leq \pi \cdot \text{length}_{\text{hyp}}\gamma_n(i) \leq \lambda_n.$$

- exp growth: $\exists\ C > 1$ such that for all n,

$$\lambda_{n+1} \leq C\lambda_n.$$

Degeneration

A priori bounds = the surfaces $\mathbb{C} \setminus \mathcal{K}_n$ have uniformly bdd geometry

Assume the contrary. There exist infinitely many **record highs**

$$n_1 < n_2 < n_3 < n_4 < \dots$$

where

$$\lambda_{n_k} = \max_{m \leq n_k} \lambda_m \quad \text{and} \quad \lim_{k \to \infty} \lambda_{n_k} = \infty.$$

Strategy: Study the limit of $\mathbb{C} \setminus \mathcal{K}_{n_k}$ as $k \to \infty$.

Control of far-reaching curves

V_n^s = disk bdd by $\gamma_{n-s}(0)$.

Lemma 1: For any $s \geq 1$, any suff. high $k \geq 1$, and any $K_{n_k}(i)$ within $V_{n_k}^s$,

$$W\left(V_{n_k}^s \backslash K_{n_k}(i)\right) \leq 24 \cdot 3^{-s/4} \lambda_{n_k}.$$

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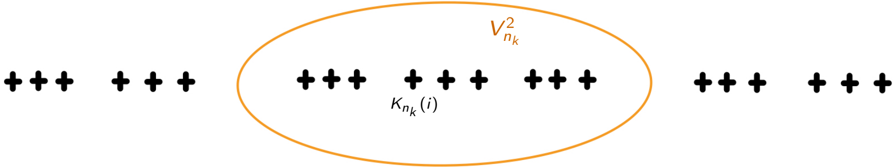

The proof is an application of Quasi-Additivity Law + Covering Lemma.

Control of waves

$\mathcal{H}_n$ = level n Hubbard continuum
(smallest compact connected forward invariant set containing $\mathcal{K}_n$)

Lemma 2: Let $\mathcal{F}$ be any proper lamination in $V^s_{n_k} \backslash \mathcal{K}_{n_k}$ such that the intersection number between any leaf and $\mathcal{H}_{n_k}$ is at least $m \geq 1$. Then,

$$W(\mathcal{F}) \leq C m^{-2} 3^s \lambda_{n_k}.$$

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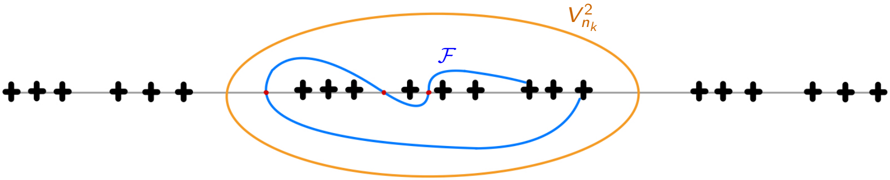

Presumed geometric limit

We can construct

- $\Delta :=$ countable discrete disjoint union of disks,
- $\Delta^k =$ disks in Δ contained in $\mathbb{D}_k$,
- $F : \mathbb{C} \to \mathbb{C}$, a topological σ-proper map fixing every comp. of Δ,
- $\psi_k : V^k_{n_k} \backslash \mathcal{K}_{n_k} \to \mathbb{D}_k \backslash \Delta^k$, "Thurston equivalence" between the degree 2^{3^k} QL map $f^{p_{n_k}} : U^k_{n_k} \to V^k_{n_k}$ and the map $F : \mathbb{D}_{k-0.1} \to \mathbb{D}_k$.

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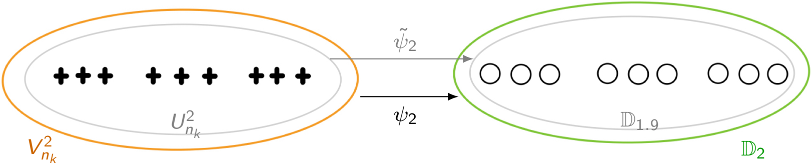

Limiting measured lamination

The surface $V_{n_k}^k \backslash \mathcal{K}_{n_k}$ induces a complex structure ρ_k on $\mathbb{D}_k \backslash \Delta^k$.

Up to rescaling by λ_{n_k},

$$\rho_k \xrightarrow[k\to\infty]{} \begin{array}{c}\text{proper measured lamination}\\ (\Xi, \mu) \text{ on } \mathbb{C}\backslash\Delta.\end{array}$$

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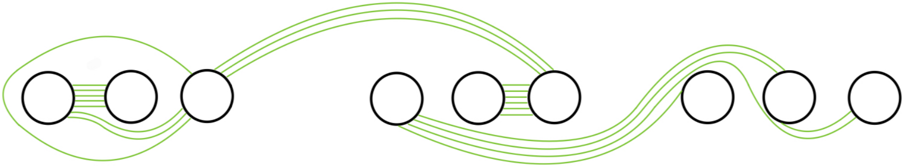

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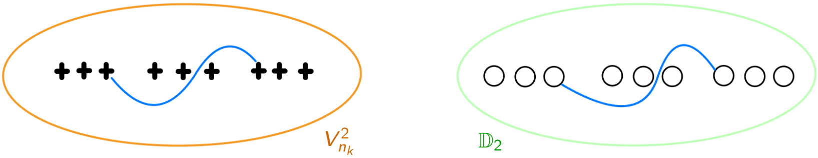
For any proper arc α,

$$\mu(\Xi \text{ along } \alpha) = \lim_{k\to\infty} \frac{W(\text{curves in } (\mathbb{D}_k \backslash \Delta^k, \rho_k) \text{ homotopic to } \alpha)}{\lambda_{n_k}}.$$



A compound lamination

Inside of Δ, we can fill in disk laminations Λ associated to $K_{n_k}(i)$'s. This gives us a compound lamination

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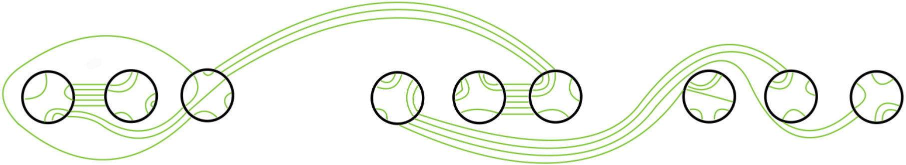
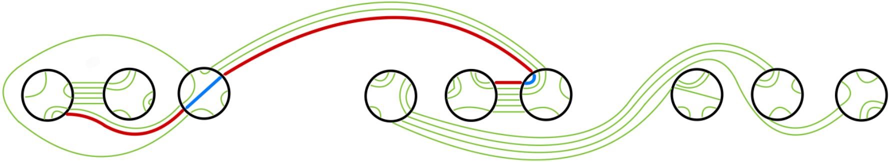


How to define modulus

A natural path γ along $\mathbf{X}$ is a concatenation

$$\xi_1 \# \delta_2 \# \xi_2 \# \delta_3 \# \xi_3 \ \ldots \ \# \delta_m \# \xi_m$$

of external segments $\xi_i \in \Xi$ and internal segments $\delta_i \in \Lambda$.



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For a function $\rho : \Xi \to [0, \infty)$, define

$$L_\rho(\gamma) = \sum_{i=1}^{m} \rho(\xi_i).$$

For any natural path family Γ along $\mathbf{X}$,

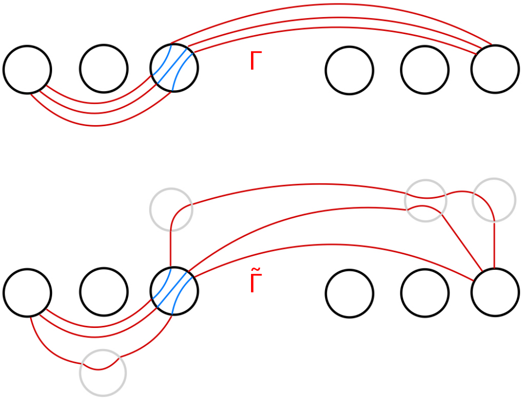
$$\mathrm{mod}_{\mathbf{X}}(\Gamma) := \inf \left\{ \int_\Xi \rho^2 d\mu \ : \ L_\rho(\gamma) \geq 1 \text{ for all } \gamma \in \Gamma \right\}.$$

Domination Property

For any natural path family Γ along $\mathbf{X}$,

$$\mathrm{mod}_{\mathbf{X}}(\Gamma) \leq \mathrm{mod}_{F^*\mathbf{X}}(\tilde{\Gamma}),$$

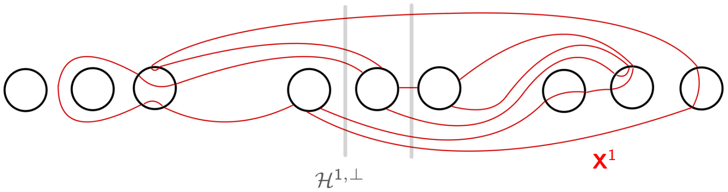
where $\tilde{\Gamma}$ is a natural path family along $F^*\mathbf{X}$ “homotopic to Γ rel Δ“.



Returning family

Disks in Δ^s are connected by a (homotopy) Hubbard tree $\mathcal{H}^s$.

- $\mathbf{X}^s$ = all natural paths along $\mathbf{X}$ from Δ^s to Δ^s
- $\mathcal{H}^{s,\perp}$ = system of infinite curves dual to $\mathcal{H}^k$
- $\left\langle \mathbf{X}^s, \mathcal{H}^{s,\perp} \right\rangle$ = weighted intersection number



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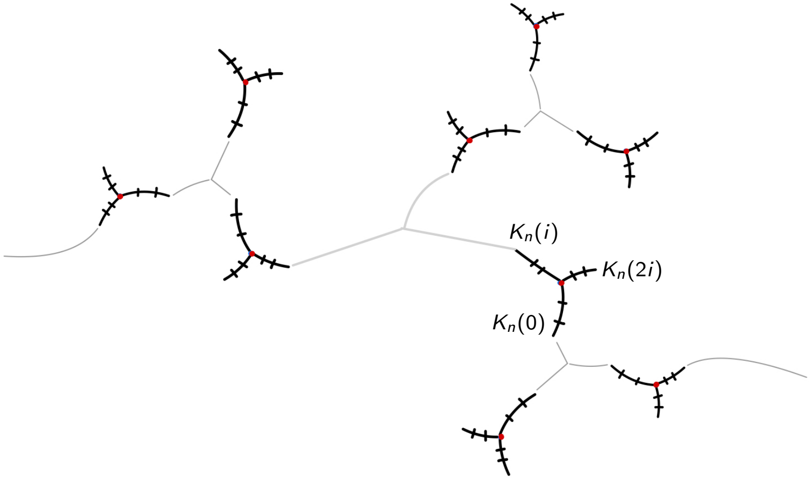
Three observations:

- For $s \geq 12$, $0 < \left\langle \mathbf{X}^s, \mathcal{H}^{s,\perp} \right\rangle < \infty$
- By the Domination Property, $\left\langle \mathbf{X}^s, \mathcal{H}^{s,\perp} \right\rangle \leq \left\langle (F^*\mathbf{X})^s, \mathcal{H}^{s,\perp} \right\rangle$
- By forward invariance of $\mathcal{H}^s$, $\left\langle \mathbf{X}^s, \mathcal{H}^{s,\perp} \right\rangle > \left\langle (F^*\mathbf{X})^s, \mathcal{H}^{s,\perp} \right\rangle$

Satellite case

We can perform similar analysis for bouquets of little Julia sets.



A priori bounds imply MLC?

Let f_c and $f_{\tilde{c}}$ be two combinatorially equivalent ∞ renormalizable maps. The goal is to show that $c = \tilde{c}$. The major step is to prove:

Goal: There exists a sequence of uniformly qc maps $\phi_n : \mathbb{C} \to \mathbb{C}$ that

- sends $f_c^j(0)$ to $f_{\tilde{c}}^j(0)$ for $1 \leq j \leq p_n$,
- lifts to a qc map ψ_n homotopic to ϕ_n rel $\{f_c^i(0)\}_{1 \leq i \leq p_n - 1}$.

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Consider the Teichmüller extremal map h_n satisfying the above. We want:

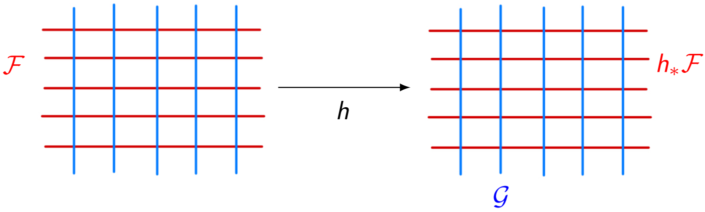
$$\sup_n \operatorname{Dil}(h_n) < \infty.$$

Via measured foliations

Fix n. Then $h = h_n$ sends a unit area quad. diff. Q to a quad. diff. $\tilde{Q}$.

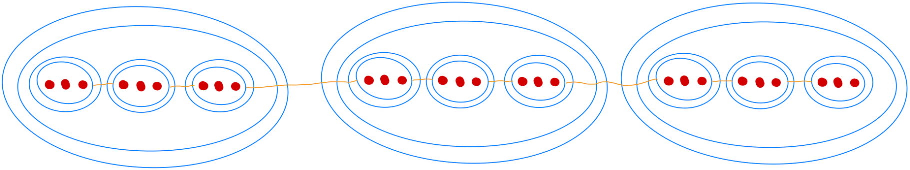
Consider measured foliations $\mathcal{F} = Hor(Q)$ and $\mathcal{G} = Ver(\tilde{Q})$. Then,

$$\text{Dil}(h) = \langle h_* \mathcal{F}, \mathcal{G} \rangle.$$



Splitting into blocks

On the dyn. plane of f_c, split the punctured plane into blocks by cutting along the level n geodesic multicurve collars and Hubbard continuum.



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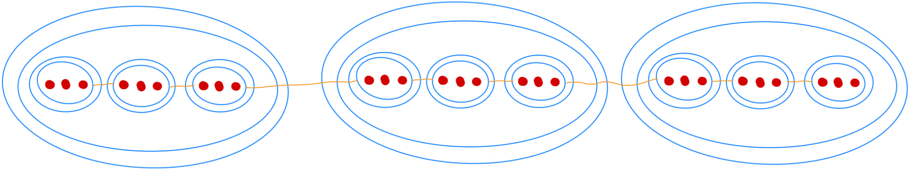

This induces proper laminations $\mathcal{F}|_b$ i.e. restrictions of $\mathcal{F}$ onto each block.

$$\langle h_* \mathcal{F}, \mathcal{G} \rangle^2 \leq \max_{\text{block } b} W(\mathcal{F}|_b) \cdot \max_{\text{block } \tilde{b}} W(\mathcal{G}|_{\tilde{b}}).$$

The final upper bound follows from a priori bounds.

Thank you!