The Full Attractor of Neutral Renormalization

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Complex dynamics: a century on from Fatou and Julia
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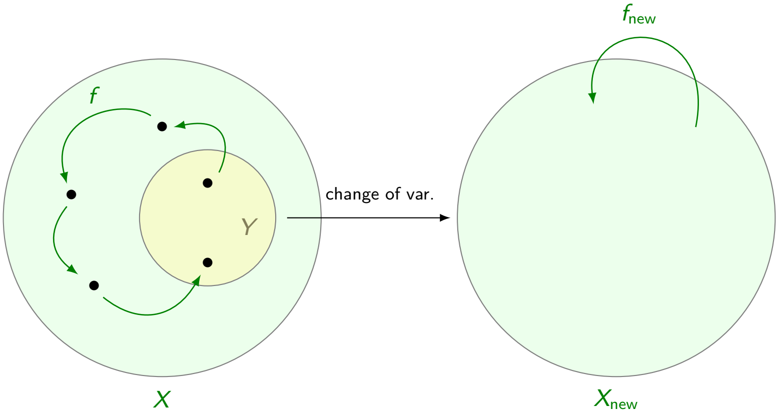
What is renormalization?



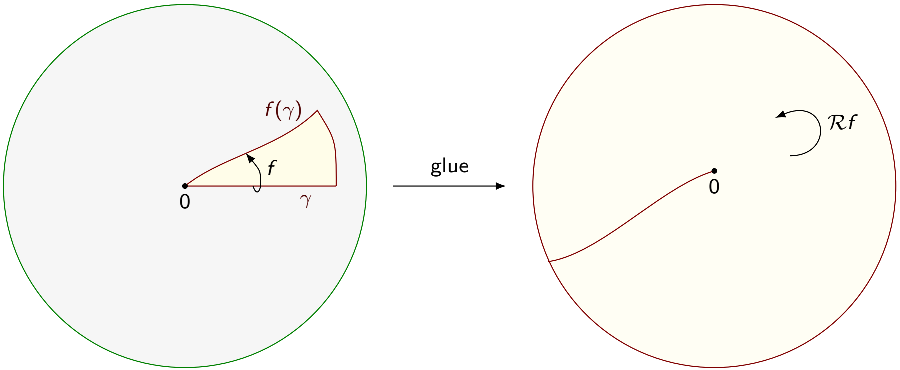
$f_{\text{new}} : X_{\text{new}} \dashrightarrow X_{\text{new}}$ is the rescaled 1^{st} return map of $f : X \dashrightarrow X$ back to $Y \subset X$.

f_{new} is called renormalization if it is of the same "class" as the original map $f : X \dashrightarrow X$.

Yoccoz's sector renormalization

Input: $f : (\mathbb{D}, 0) \to (\mathbb{C}, 0)$ is univalent and $f'(0) = e^{2\pi i\theta}$ with $0 < |\theta| < \frac{1}{2}$.

Output: $\mathcal{R}f : (\mathbb{D}, 0) \to (\mathbb{C}, 0)$ is univalent, $\mathcal{R}f'(0) = e^{2\pi i \mathfrak{g}(\theta)}$ where $\mathfrak{g}(\theta) \equiv -\dfrac{1}{\theta} \pmod 1$.



Application: Optimality of Brjuno condition

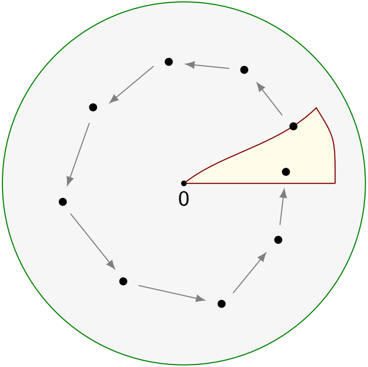
Sector renormalization combinatorics

- orientation: $\varepsilon(\theta) \in \{-, +\}$
- 1^{st} return time: $\bar{a}(\theta) \in \{2, 3, 4, \ldots\}$

Every irrational $\theta \in (-\frac{1}{2}, \frac{1}{2})$ is uniquely determined by the sequence

$$\langle(\varepsilon_n, \bar{a}_n)\rangle_{n \geq 1}$$

where $\varepsilon_{n+1} = \varepsilon(\mathfrak{g}^n(\theta))$ and $\bar{a}_{n+1} = \bar{a}(\mathfrak{g}^n(\theta))$.



E.g. $\varepsilon(\theta) = +$, $\bar{a}(\theta) = 8$

Question

For neutral quadratic polynomials

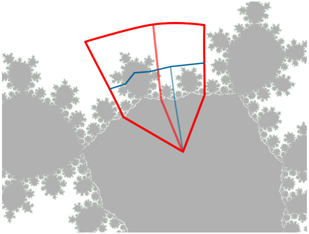
$$f_\theta(z) = e^{2\pi i\theta} z + z^2,$$

can the renormalization sector be constructed to capture the critical orbit well?

Two partial solutions

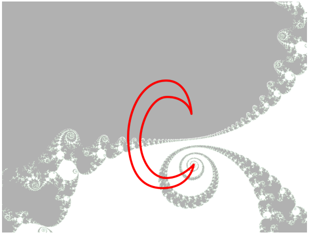
Siegel/Pacman Renorm.

- bounded type: $\sup \bar{a}_n < \infty$
- McMullen '98, Dudko-Lyubich-Selinger '20



Near-Parabolic Renorm.

- high type: $\inf \bar{a}_n \gg 1$
- Inou-Shishikura '08



Applications:
dynamical universality, parameter self-similarity, examples of positive area Julia sets, measure and topology of $P(f_\theta)$, MLC at some satellite parameters, etc

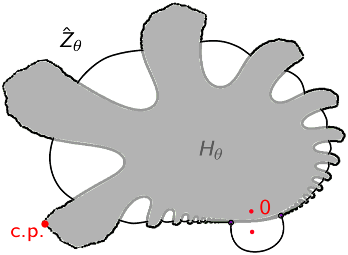
Pseudo-Siegel Bounds

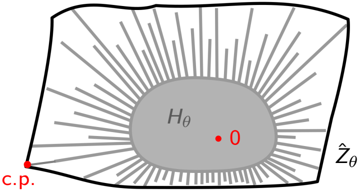
Theorem (Dudko-Lyubich '22)

For every irrational θ, f_θ admits

1. *a pseudo-Siegel disk $\hat{Z}_\theta$, where*
   - $(\hat{Z}_\theta, 0)$ *is a closed K-quasidisk,*
   - $\hat{Z}_\theta$ *contains $P(f_\theta)$,*
   - $f_\theta : \hat{Z}_\theta \to \mathbb{C}$ *is injective;*
2. *a Mother Hedgehog H_θ, where*
   - H_θ *is a full continuum containing* 0 *and c.p.,*
   - $f_\theta : H_\theta \to H_\theta$ *is a homeo.,*
   - $\partial H_\theta = P(f_\theta)$,
   - $\mathrm{int} H_\theta =$ *Siegel disk.*

$\hat{Z}_\theta$ can be constructed out of H_θ by adding hyperbolic geodesics outside (fjords).





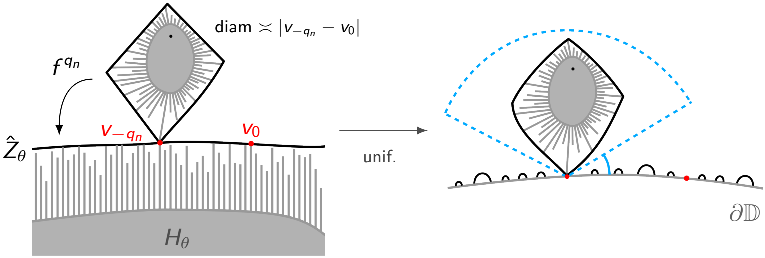
Pseudo-Bubble Bounds

Pseudo-bubbles $=$ preimages of $\hat{Z}_\theta$

Theorem (Dudko-Lim '26)

For every θ,

1. *all pseudo-bubbles are K-qc;*
2. *all primary pseudo-bubbles have uniformly controlled "angle" and "size".*



Sectorial Bounds

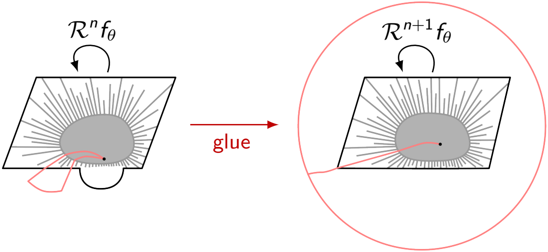
Theorem (Dudko-Lyubich '26)

For every θ, it is possible to construct sector renormalizations

$$\mathcal{R}^n f_\theta : (\mathbb{D}, 0) \dashrightarrow (\mathbb{D}, 0), \qquad n \geq 0,$$

such that

- *each $\mathcal{R}^n f_\theta$ has a Mother Hedgehog and a uniform pseudo-Siegel disk,*
- *$\{\mathcal{R}^n f_\theta\}_{\theta,n}$ is pre-compact.*



The renormalization attractor

There is a compact renormalization attractor $\mathcal{A}$ where

$$\mathcal{R}^n f_\theta \to \mathcal{A} \qquad \text{as } n \to \infty \text{ for all } \theta.$$

Every element of $\mathcal{A}$ is a bi-infinite tower $\underline{\boldsymbol{f}} = \langle f_n \rangle_{n\in\mathbb{Z}}$ where

$$\ldots \xrightarrow{\mathcal{R}} f_{n-1} \xrightarrow{\mathcal{R}} f_n \xrightarrow{\mathcal{R}} f_{n+1} \xrightarrow{\mathcal{R}} \ldots$$

The combinatorics of $\underline{\boldsymbol{f}}$ is encoded by an element $\langle (\varepsilon_n, \bar{a}_n) \rangle_{n\in\mathbb{Z}}$ of

$$\overleftarrow{\overline{\Theta}} = \left(\{-,+\} \times \{2,3,\ldots,\infty\} \right)^{\mathbb{Z}},$$

where $\bar{a}_{n+1} = \infty$ if $f_n'(0) = 1$.

Main theorems [DLL '26]

Combinatorial Rigidity

If $\underline{f}, \underline{g} \in \mathcal{A}$ are combinatorially equivalent, then for all $n \in \mathbb{Z}$,

$$f_n \underset{conf}{\sim} g_n \qquad \text{on definite nbhd of their Mother Hedgehogs.}$$

Corollary: Set Hors $:= \mathcal{A}/\underset{conf}{\sim}$. Then, $\mathcal{R}$: Hors $\circlearrowleft$ is conjugate to the shift map on $\underleftarrow{\overline{\Theta}}$.

Rigidity of Backward Parabolic Towers

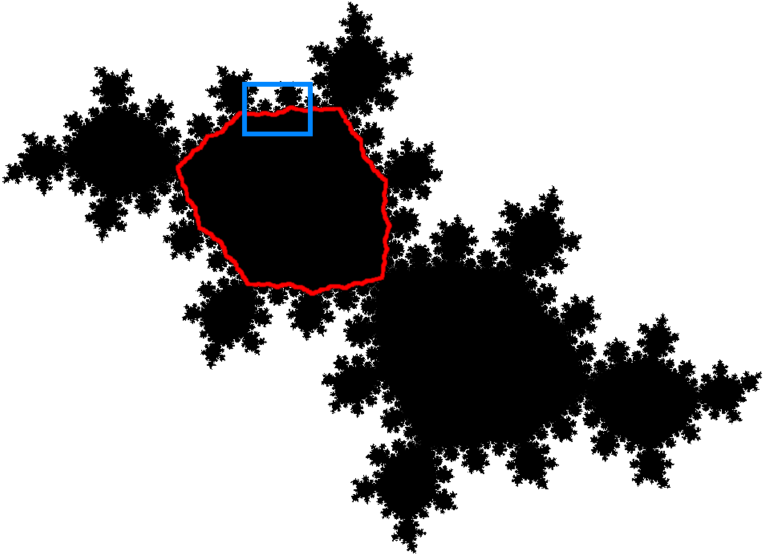
If $\underline{f}, \underline{g} \in \mathcal{A}$ have combinatorics $\langle(\varepsilon_n, \bar{a}_n)\rangle_{n\in\mathbb{Z}}$ and $\langle(\varepsilon'_n, \bar{a}'_n)\rangle_{n\in\mathbb{Z}}$ respectively and

$$\bar{a}_1 = \bar{a}'_1 = \infty \qquad \text{and} \qquad (\varepsilon'_n, \bar{a}_n) = (\varepsilon_n, \bar{a}_n) \quad \text{for } n \leq 1,$$

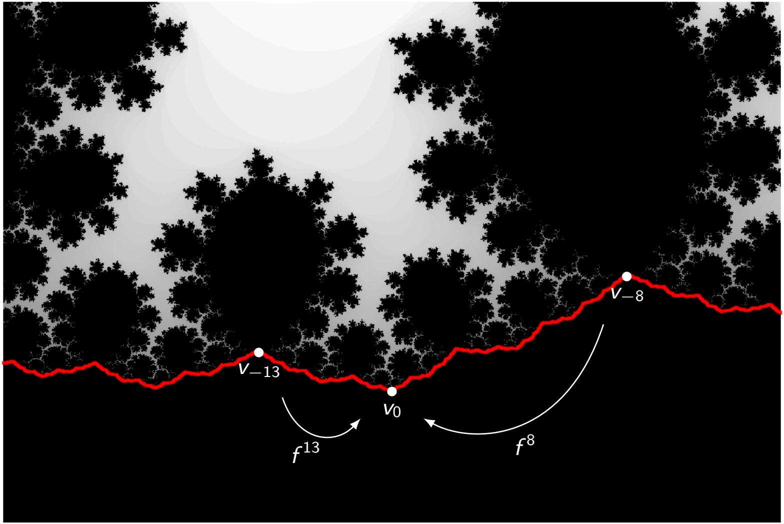
then for all $n \leq 0$,

$$f_n \underset{conf}{\sim} g_n \qquad \text{on definite nbhd of their Mother Hedgehogs.}$$

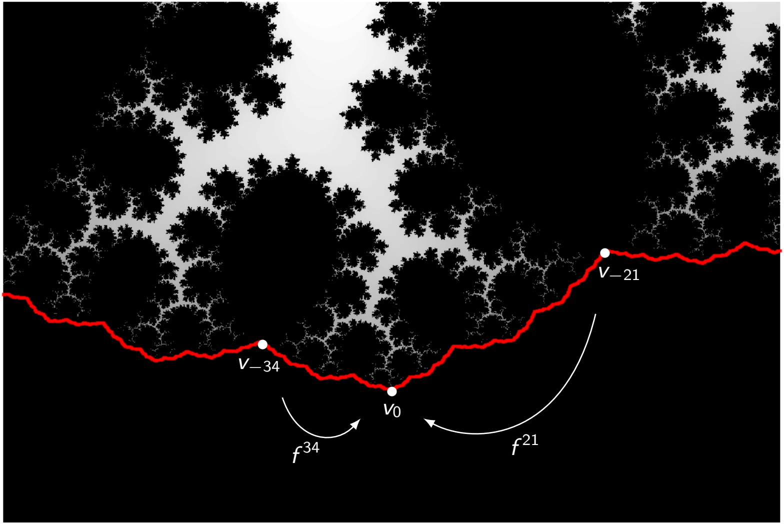
The dynamical plane of f_θ



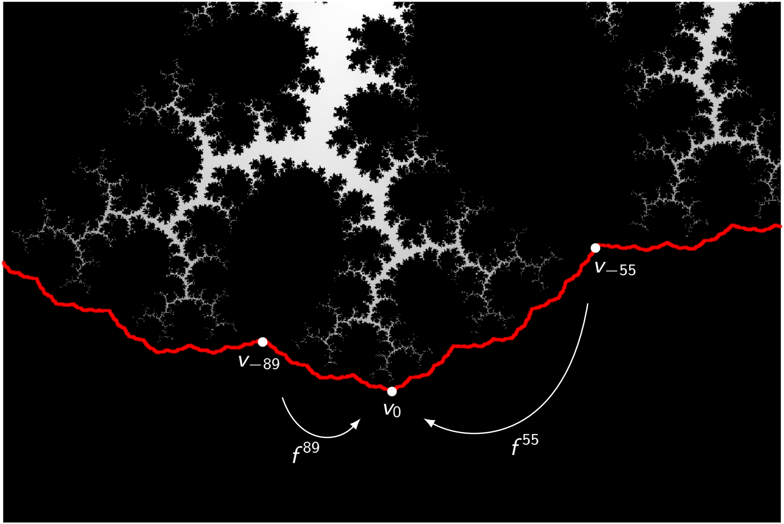
The dynamical plane of f_θ, scale: 0.2



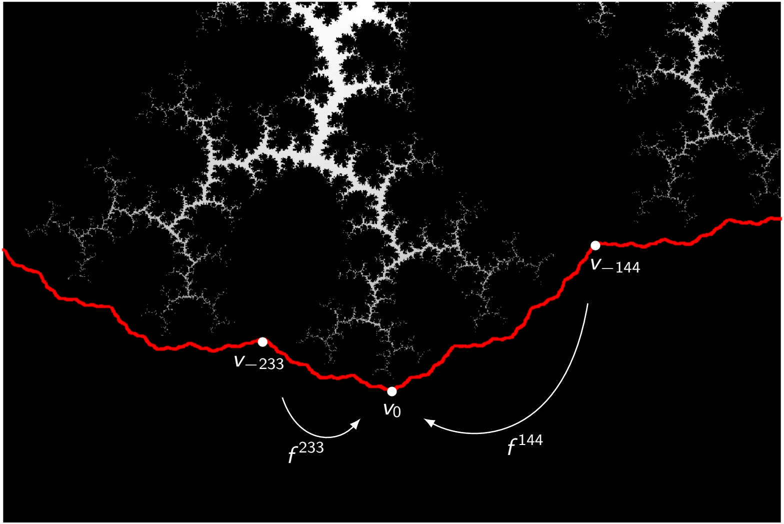
The dynamical plane of f_θ, scale: 0.06



The dynamical plane of f_θ, scale: 0.018



The dynamical plane of f_θ, scale: 0.0055



Limiting transcendental dynamics

For all θ, the rescaling limit(s) of $\{f_\theta^m\}_{m\geq 1}$ at the critical value is a transcendental dynamical system

$$\mathbf{F} = \left(\mathbf{F}^P : \mathrm{Dom}(\mathbf{F}^P) \to \mathbb{C}\right)_{P\in\mathbf{T}}$$

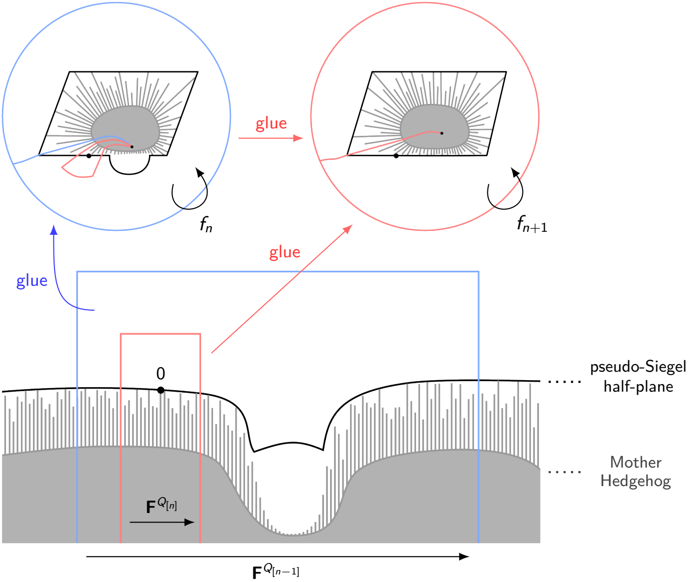
called a neutral cascade.

F comes with some combinatorics

$$\langle(\varepsilon_n, \bar{a}_n)\rangle_{n\in\mathbb{Z}} \in \underleftarrow{\overline{\Theta}}.$$

The time semigroup **T** has a generating set $\{Q_{[n]}\}_{n\in\mathbb{Z}}$ satisfying recurrence relations depending on the combinatorics.

Neutral cascade F $\Leftrightarrow$ bi-infinite tower $\underline{f} = \langle f_n \rangle_{n\in\mathbb{Z}}$



Reformulation

Here is another version of combinatorial rigidity.

Rigidity of neutral cascades

Given two combinatorially equivalent neutral cascades $\mathbf{F} = (\mathbf{F}^P)_{P \in \mathbf{T}}$ *and* $\mathbf{G} = (\mathbf{G}^P)_{P \in \mathbf{T}}$, *there is an affine map* $\Phi : \mathbb{C} \to \mathbb{C}$ *such that*

$$\Phi \circ \mathbf{F}^P = \mathbf{G}^P \circ \Phi \qquad \text{for all } P \in \mathbf{T}.$$

This theorem gives a complete dynamical universality for neutral quadratic polynomials.

There is a similar reformulation for parabolic backward rigidity.

Summary of proof

For a neutral cascade $\mathbf{F} = (\mathbf{F}^P)_{P \in \mathbf{T}}$ with combinatorics $\underline{\theta}$,

1. $\mathbf{F}$ has a Mother Hedgehog $\mathbf{H_F}$ and a pseudo-Siegel half-plane $\hat{\mathbf{Z}}_\mathbf{F}$.
2. Lim '26: $\mathbf{H_F}$ has
   - zero area boundary,
   - non-empty interior iff $\underline{\theta}$ is eventually Brjuno.
3. Dudko-Lim '26: Preimages of $\hat{\mathbf{Z}}_\mathbf{F}$ satisfy Pseudo-Bubble Bounds.

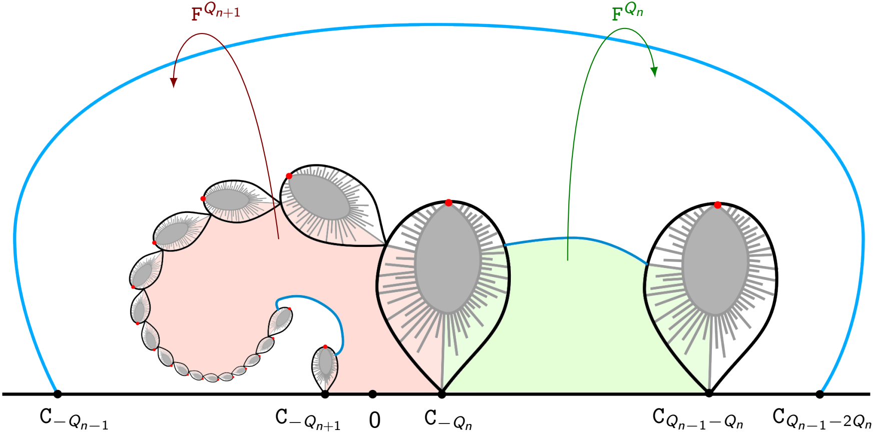
Consider a Riemann map

$$\Psi_\mathbf{F} : \mathbb{C} \backslash \mathbf{H_F} \to \mathbb{H},$$

transforming $\mathbf{F}$ into the "external cascade" $\mathtt{F} = (\mathtt{F}^P)_{P \in \mathbf{T}}$.

1. $\mathtt{F} : \mathbb{R} \to \mathbb{R}$ is well-defined and has Real Bounds.
2. $\mathtt{F}$ has Complex Butterfly Bounds at all scales.
3. $\mathtt{F}$ has dynamical hairiness & no invariant line field.
4. Construct qc conjugacy using renormalization tiling + butterflies.

Complex Butterfly Bounds



What's next?

By working with the transcendental dynamics, we can promote:

$$\text{sector renorm. } \mathcal{R} \quad \Longrightarrow \quad \text{cylinder renorm. } \mathcal{R}_{\mathrm{cyl}}$$

<u>Goal:</u> $\mathcal{R}_{\mathrm{cyl}} : \mathrm{Hors} \circlearrowleft$ is uniformly hyperbolic with exactly 1 unstable direction.

<u>Current strategy:</u>

Unexpected ≥ 0 Lyapunov exponent:

$\Longrightarrow$ Adapt Oseledets-Pesin Theory for Banach analytic bundle maps.

$\Longrightarrow$ $\mathcal{R}_{\mathrm{cyl}}$ has a small backward orbit $\underline{g} = \langle g_n \rangle_{n \leq 0}$ shadowing some $\langle f_n \rangle_{n \in \mathbb{Z}} \in \mathrm{Hors}$.

$\Longrightarrow$ $\underline{g}$ has transcendental dynamics.

$\Longrightarrow$ Derive contradiction through parabolic backward rigidity.

<u>Potential applications:</u>

Universal scaling law, ∂Molecule has zero measure, MMY conjecture, some MLC, etc